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Useful transformations: from ion-laser interactions to master equations

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We show set of transformations which allow to obtain analytic solutions in several quantum-optical problems. We start with the ion-laser (time dependent) interaction, continue with the problem of a slow atom interacting with a quantized field to end with a master equation that describes losses. In all cases it is shown that one may find useful transformations that simplify the problems.

I. INTRODUCTION

The purpose of this contribution is to show some transformations that simplify Hamiltonians such that they may be treated in an analytic form. We will study various problems, starting with the ion-laser interaction, where we have shown that there exists a time dependent transformation that linearizes the Hamiltonian with no approximations [1]. We follow with the interaction of a slow atom with a quantized field, the main result of the paper. In this case the atom is affected by the mode shape of the field [2] and we show that the interaction may be simplified as we pass from a three-body system to a two-body effective interaction, again with no approximations. Finally we analyze the case of the master equation (ME) that describes losses for an anharmonic oscillator [3], one can transform such a ME to obtain a simpler equation where all the superoperators commute, allowing its simple integration [4].

II. ION-LASER INTERACTION

We consider the Hamiltonian of a single ion (with unity mass trapped in a harmonic potential in interaction with laser light in the (optical) rotating wave approximation [5])

$$\hat{H} = \frac{1}{2}\left[\hat{p}^2 + \nu^2(t)\hat{x}^2\right] + \hbar\omega_{21}\hat{A}_{22} + \hbar\lambda(t)[E^{(-)}(\hat{x},t)\hat{A}_{12} + H.c.], \tag{1}$$

$\hat{A}_{ab}$ are the operators relating the different electronic transitions (two-level) flip operator for the $|b\rangle \rightarrow |a\rangle$ transition of frequency ω_{21}, respectively. $\nu(t)$ is the trap (time dependent) frequency, λ the electronic coupling matrix element, and $E^{(-)}(\hat{x},t)$ the negative part of the classical electric field of the driving field. The operators $\hat{x}$ and $\hat{p}$ are the position and momentum of the centre of mass of the ion. We assume the ion driven by a laser field $E^{(-)}(\hat{x},t)$

$$E^{(-)}(\hat{x},t) = E_0 e^{-i(k\hat{x}-\omega t)}. \tag{2}$$

We want to solve the Schrödinger equation

$$i\hbar\frac{\partial|\xi(t)\rangle}{\partial t} = \hat{H}|\xi(t)\rangle, \tag{3}$$

in order to do this, we make the transformation $|\phi\rangle = \hat{T}(t)|\xi\rangle$, with [6]

$$\hat{T}(t) = e^{i\frac{\ln\{\rho(t)\sqrt{\nu_0}\}}{2\hbar}(\hat{x}\hat{p}+\hat{p}\hat{x})}e^{-i\frac{\dot{\rho}(t)}{2\hbar\rho(t)}\hat{x}^2} \tag{4}$$

with $\rho(t)$ a function that obeys the Ermakov equation

$$\ddot{\rho} + \nu^2(t)\rho = \frac{1}{\rho^3}. \tag{5}$$

such that we obtain the equation for $|\phi\rangle$

$$i\hbar\frac{\partial|\phi(t)\rangle}{\partial t} = \hat{\mathcal{H}}|\phi(t)\rangle, \tag{6}$$

with the transformed Hamiltonian given by

$$\hat{\mathcal{H}} = \frac{1}{2\nu_0\rho^2}\left(\hat{p}^2 + \nu_0^2\hat{x}^2\right) + \hbar\omega_{21}\hat{A}_{22} + \hbar\Omega(t)[e^{-i(k\hat{x}\rho(t)\sqrt{\nu_0}-\omega t)}\hat{A}_{12} + H.c.], \tag{7}$$

with $\Omega = \lambda E_0$. We consider that $\omega_{21} = \omega + \delta$ where δ is the so-called detuning. We transform to a frame rotating at ω by means of the transformation $\hat{T}_\omega = e^{-i\omega t\hat{A}_{22}}$ to obtain the Hamiltonian $\hat{\mathcal{H}}_\omega = \hat{T}_\omega\hat{\mathcal{H}}\hat{T}_\omega^\dagger$ $(|\phi\rangle \to |\phi_\omega\rangle)$

$$\hat{\mathcal{H}}_\omega = \hbar\tilde{\omega}(t)\left(\hat{n} + \frac{1}{2}\right) + \hbar\delta\hat{A}_{22} + \hbar\Omega(t)[e^{-i(\hat{a}+\hat{a}^\dagger)\eta(t)}\hat{A}_{12} + H.c.]. \tag{8}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ with

$$\hat{a} = \sqrt{\frac{\nu_0}{2\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{2\hbar\nu_0}}, \qquad \hat{a}^\dagger = \sqrt{\frac{\nu_0}{2\hbar}}\hat{x} - i\frac{\hat{p}}{\sqrt{2\hbar\nu_0}} \tag{9}$$

the annihilation and creation operators respectively. $\tilde{\omega}(t) = 1/\rho^2$ is the characteristic frequency of the time dependent harmonic oscillator. The time dependent Lamb-Dicke parameter is written as $\eta(t) = \eta_0\rho(t)\sqrt{\nu_0}$ with $\eta_0 = k\sqrt{\frac{\hbar}{2\nu_0}}$, where k is the magnitude of the wave vector of the driving field.

We will consider now the resonant interaction ($\delta = 0$). Passing to a frame rotating at the frequency $\tilde{\omega}(t)$ we may get rid off the harmonic oscillator term in (8) to end up with the (time dependent) interaction Hamiltonian

$$\hat{\mathcal{H}}(t) = \hbar\Omega(t)[e^{-i(\hat{a}e^{-i\int\tilde{\omega}(t')dt'}+\hat{a}^\dagger e^{i\int\tilde{\omega}(t')dt'})\eta(t)}\hat{A}_{12} + H.c.]. \tag{10}$$

A. Linearizing the system

Finally we make the transformation $|\psi\rangle = \hat{R}(t)|\phi\rangle$ with (see [7] for the time independent case)

$$\hat{R}(t) = e^{\frac{\pi}{4}(\hat{A}_{21}-\hat{A}_{12})}e^{-i\frac{\eta(t)}{2}(\hat{a}+\hat{a}^\dagger)(\hat{A}_{22}-\hat{A}_{11})} \tag{11}$$

to obtain

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = \hbar\left\{\tilde{\omega}(t)\hat{n} + \Omega(\hat{A}_{22}-\hat{A}_{11}) + \left(\frac{\delta}{2} + i[\hat{a}\beta(t) - \hat{a}^\dagger\beta^*(t)]\right)(\hat{A}_{12}+\hat{A}_{21})\right\}|\psi\rangle, \tag{12}$$

with $\beta(t) = \frac{\eta(t)\tilde{\omega}}{2} - i\dot{\eta}(t)/2$ and we have disregarded the term $\tilde{\omega}/2$ as it would add an overall phase. A method to solve JC-like interactions with time dependent parameters has been published by Shen *et al.* [8].

B. Many ions

We generalize the transformation that allows to linearize the ion-laser Hamiltonian [7] in an exact form for two different interactions, namely, for the case of many ions in interaction with laser fields [9], and for an ion vibrating in two dimensions interacting with a laser field. This linearization has been shown to be important for instance in the implementation of fast gates in ion-laser interactions [10]. This is possible, as more regimes may be studied with such a linearization: high intensity, low intensity and middle intensity regimes.

In particular we will show that in the case of many ions, the transformation produces a term which correspond to a dipole-dipole interaction.

Ions in a linear trap interacting with a laser field may be described by the Hamiltonian [9]

$$H_M = \nu a^\dagger a + \frac{\delta}{2}\sum_j \sigma_{zj} + \sum_j \Omega_j(\sigma_{+j}e^{i\eta_j(a^\dagger+a)} + H.c.) \tag{13}$$

where ν is the frequency of the vibration, $a^\dagger$ and a are the creation and annihilation operators of the quantized oscillator, δ is the detuning between the transition frequency of the internal states of the ion, ω_{eg}, and the laser frequency ω_L, and Ω_i is the resonant Rabi frequency of the i'th ion in the laser field. The exponentials account for the position dependence laser-field and the recoil of the ions upon absorption of a photon. The positions of the ions x_i

are replaced by ladder operators $kx_i = \eta_i(a^\dagger + a)$, where the Lamb-Dicke parameter η_i represents the ration between the ionic excursions within the vibrational ground state wavefunction and the wavelenght of the exciting radiation. We can linearize the Hamiltonian (13) via the transformation [7]

$$T_M = \prod_j e^{\frac{\pi}{4}(\sigma_{+j} - \sigma_{-j})} e^{-i\eta_j \sigma_{zj}(a^\dagger + a)/2} \tag{14}$$

which gives the Hamiltonian

$$\begin{aligned} \mathcal{H}_M &= THT^\dagger \\ &= \nu a^\dagger a - \frac{\delta}{2} \sum_j \sigma_{xj} + \sum_j \Omega_j \sigma_{zj} + i \sum_j \frac{\eta_j \nu(a - a^\dagger)}{2} \sigma_{xj} + \sum_{j,k} \frac{\eta_j \eta_k}{4} \sigma_{xj} \sigma_{xk} \end{aligned} \tag{15}$$

Here $\sigma_{xj} = \sigma_{+j} + \sigma_{-j}$. Note that the transformation, besides linearizing the Hamiltonian, produces an ion-ion (dipole) interaction.

C.   Two-dimensional vibration

An ion vibrating in two-dimensions has the Hamiltonian

$$H_{2d} = \nu_x a_x^\dagger a_x + \nu_y a_y^\dagger a_y + \frac{\delta}{2}\sigma_z + \Omega(\sigma_+ e^{i\eta_x(a_x^\dagger + a_x)} e^{i\eta_y(a_y^\dagger + a_y)} + H.c.) \tag{16}$$

with the transformation

$$T_{2d} = e^{\frac{\pi}{4}(\sigma_+ - \sigma_-)} e^{-i[\eta_x(a_x^\dagger + a_x) + \eta_y(a_y^\dagger + a_y)]\sigma_z/2} \tag{17}$$

we can cast the Hamiltonian (16) into the linearized Hamiltonian

$$\mathcal{H}_{2d} = \nu_x a_x^\dagger a_x + \nu_y a_y^\dagger a_y - \frac{\delta}{2}\sigma_x + \Omega\sigma_x + \frac{i\nu}{2}\sigma_x[\eta_x(a_x - a_x^\dagger) + \eta_y(a_y - a_y^\dagger)] \tag{18}$$

where we have disregarded a constant term. The Hamiltonian above looks like a two-mode quantized-field-atom interaction.

III.   SLOW ATOM INTERACTING WITH A QUANTIZED FIELD

Here we show how a three body problem may be reduced to a two body problem via a transformation, we treat the problem of a slow atom interacting with a quantized field. Because the slowness of the atom, the field mode-shape affects the interaction. We can write down the Hamiltonian describing a single two-level atom passing an electromagnetic field confined to a cavity. Inn addition to the Jaynes-Cummings Hamiltonian, we have to add the energy of the free atom and the spatial variation it feels from the cavity, the Hamiltonian reads

$$H = \frac{p^2}{2} + \omega\hat{n} + \frac{\omega_0}{2}\sigma_z + g(x)(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-), \tag{19}$$

on resonance, we can pass to the interaction picture Hamiltonian

$$H_I = \frac{p^2}{2} + g(x)(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-), \tag{20}$$

we use the 2×2 notation for the Pauli spin matrices and write the interaction Hamiltonian as (see [4])

$$\hat{H}_I = \frac{p^2}{2} + g(x)\hat{T}^\dagger \begin{pmatrix} 0 & \sqrt{\hat{n}+1} \\ \sqrt{\hat{n}+1} & 0 \end{pmatrix} \hat{T} \tag{21}$$

where a non unitary transformation $\hat{T}$ has been used. We define $\hat{T}$ as

$$\hat{T} = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V} \end{pmatrix} \tag{22}$$

Note that $\hat{T}\hat{T}^\dagger = 1$ but $\hat{T}^\dagger\hat{T} = 1 - \rho_{g,v}$ with

$$\rho_{g,v} = \begin{pmatrix} 0 & 0 \\ 0 & |0\rangle\langle 0| \end{pmatrix} \tag{23}$$

We can use the definitions above to rewrite (19) as

$$\hat{H}_I = (\hat{T}^\dagger\hat{T} + \rho_{g,v})\frac{p^2}{2}(\hat{T}^\dagger\hat{T} + \rho_{g,v}) + g(x)\hat{T}^\dagger \begin{pmatrix} 0 & \sqrt{\hat{n}+1} \\ \sqrt{\hat{n}+1} & 0 \end{pmatrix} \hat{T} \tag{24}$$

By noting that $\hat{T}\rho_{g,v} = 0$ we rewrite the above equation as

$$\hat{H}_I = \hat{T}^\dagger\frac{p^2}{2}\hat{T} + \frac{p^2}{2}\rho_{g,v} + g(x)\hat{T}^\dagger \begin{pmatrix} 0 & \sqrt{\hat{n}+1} \\ \sqrt{\hat{n}+1} & 0 \end{pmatrix} \hat{T} \tag{25}$$

where we have used that $\rho_{g,v}^2 = \rho_{g,v}$. Finally we factorize the transformation operators in the Hamiltonian above to obtain

$$\hat{H}_I = \hat{T}^\dagger(\frac{p^2}{2} + g(x)\sigma_x\sqrt{\hat{n}+1})\hat{T} + \frac{p^2}{2}\rho_{g,v} \tag{26}$$

Note that $[\hat{T}^\dagger(\frac{p^2}{2} + g(x)\sigma_x\sqrt{\hat{n}+1})\hat{T}, \frac{p^2}{2}\rho_{g,v}] = 0$ so that the evolution operator for the Hamiltonian above is given by

$$\hat{U}_I(t) = e^{-i\hat{T}^\dagger(\frac{p^2}{2} + g(x)\sigma_x\sqrt{\hat{n}+1})\hat{T}t}e^{-i\frac{p^2}{2}\rho_{g,v}t} \tag{27}$$

to obtain the first exponential we can do Taylor series, and we note that the powers of the argument are simply

$$[\hat{T}^\dagger(\frac{p^2}{2} + g(x)\sigma_x\sqrt{\hat{n}+1})\hat{T}]^k = \hat{T}^\dagger(\frac{p^2}{2} + g(x)\sigma_x\sqrt{\hat{n}+1})^k\hat{T}, \qquad k \geq 1 \tag{28}$$

such that

$$e^{-i\hat{T}^\dagger(\frac{p^2}{2} + g(x)\sigma_x\sqrt{\hat{n}+1})\hat{T}t} = \hat{T}^\dagger e^{-i(\frac{p^2}{2} + g(x)\sigma_x\sqrt{\hat{n}+1})t}\hat{T} + \rho_{g,v}. \tag{29}$$

Note that the evolution operator in (29) is effectively the interaction of two systems, as it is written in a form in which the filed operators commute, unlike the Hamiltonian (19), where all the operator involved do not commute.

IV. MASTER EQUATIONS

Now we turn our attention to the superoperator solution of master equations for more quantum optical systems, namely a dissipative cavity filed with a Kerr medium [3], master equation describing phase sensitive processes [11] and parametric down conversion [12]. Usually these equations are solved by transforming them to Fokker-Planck equations [13] which are partial differential equations for quasiprobability distribution functions typically the Glauber-Sudarshan P-function and the Husimi Q-function. Another usual approach is to solve system-environment problems is through the use of Langevin equations, this is stochastic differential equations that are equivalent to the Fokker-Planck equation [14].

These approaches to the problem makes it usually difficult to apply the solutions to an arbitrary initial field in contrast with the superoperator techniques where it is direct the application to an initial wave function. We have used this feature in the former Section where we have exploited this fact to obtain reconstruction mechanisms that allowed us to obtain information on the state of the quantized electromagnetic field via quasiprobability distribution functions.

A. Kerr medium

Before applying superoperator methods in the solution of the above equation, let us show how it may be casted into a Fokker-Planck equation. In order to do this one writes the density matrix in terms of the Glauber-Sudarshan

P-function, $\hat{\rho} = 1/\pi \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha$. Noting that the creation and annihilation operators have the following relations with the coherent state density matrix [15]

$$\hat{a}^\dagger|\alpha\rangle\langle\alpha| = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)|\alpha\rangle\langle\alpha|, \tag{30}$$

$$|\alpha\rangle\langle\alpha|\hat{a} = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|, \tag{31}$$

we can obtain the following correspondence

$$\hat{a}\hat{\rho} \to \alpha P(\alpha), \qquad \hat{a}^\dagger\hat{\rho} \to \left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P(\alpha), \tag{32}$$

and

$$\hat{\rho}\hat{a}^\dagger \to \alpha^* P(\alpha), \qquad \hat{\rho}\hat{a} \to \left(\alpha - \frac{\partial}{\partial\alpha^*}\right)P(\alpha). \tag{33}$$

In this form, whenever a creation or annihilation operator occurs in the master equation, we can translate this into a corresponding operation on the Glauber-Sudarshan *P*-function. The equation that results is a Fokker-Planck equation [13]

$$\frac{\partial P(\alpha,t)}{\partial t} = \left[\gamma\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right) + 2\gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P(\alpha,t). \tag{34}$$

This equation is equivalent to the stochastic differential equation [14]

$$\frac{d\alpha}{dt} = \gamma\alpha + \sqrt{2\gamma\bar{n}}\xi(t), \tag{35}$$

and the corresponding complex conjugate equation. The quantity $\xi(t)$ is a white noise fluctuating force with the following correlation properties

$$\langle\xi(t)\rangle = 0, \qquad \langle\xi(t)\xi^*(t)\rangle = \delta(t-t') \qquad \langle\xi(t)\xi(t)\rangle = \langle\xi^*(t)\xi^*(t)\rangle = 0 \tag{36}$$

Equation (35) is usually called a Langevin equation, and also as the "Stratonovich form" of the Fokker-Plank equation (see for instance [16, 17]). The Fokker-Planck equation can be also obtained from the Kramers-Moyal [18, 19] expansion, a Langevin equation that does not stop after second order derivatives (see for instance [20]). The master equation for a Kerr medium in the Markov approximation and interaction picture has the form [3]

$$\frac{d\hat{\rho}}{dt} = -i\chi[\hat{n}^2, \hat{\rho}] + 2\gamma\hat{a}\hat{\rho}\hat{a}^\dagger - \gamma\hat{a}^\dagger\hat{a}\hat{\rho} - \gamma\hat{\rho}\hat{a}^\dagger\hat{a}. \tag{37}$$

Milburn and Holmes [3] solved this equation by changing it to a partial differential equation for the *Q*-function and for an initial coherent state. We can have a different approach to the solution by again using superoperators. If we define

$$\hat{Y}\hat{\rho} = -i\chi[\hat{n}^2, \hat{\rho}] \tag{38}$$

we rewrite (37) as

$$\frac{d\hat{\rho}}{dt} = (\hat{Y} + \hat{J} + \hat{L})\hat{\rho}, \tag{39}$$

where the superoperators $\hat{J}$ and $\hat{L}$ are defined as

$$\hat{J}\hat{\rho} = 2\gamma\hat{a}\hat{\rho}\hat{a}^\dagger, \qquad \hat{L}\hat{\rho} = -\gamma\hat{a}^\dagger\hat{a}\hat{\rho} - \gamma\hat{\rho}\hat{a}^\dagger\hat{a}. \tag{40}$$

Now we use the transformation

$$\tilde{\hat{\rho}} = \exp[(\hat{Y} + \hat{L})t]\hat{\rho} \tag{41}$$

to obtain

$$\frac{d\hat{\tilde{\rho}}}{dt} = \exp[-i\chi\hat{R}t - 2\gamma t]\hat{J}\hat{\tilde{\rho}}, \tag{42}$$

with

$$\hat{R}\hat{\tilde{\rho}} = 2(\hat{n}\hat{\tilde{\rho}} - \hat{\tilde{\rho}}\hat{n}). \tag{43}$$

In arriving to equation (42) we have used the formula $e^{y\hat{A}}\hat{B}e^{-y\hat{A}} = \hat{B}+y[\hat{A},\hat{B}]+y^2/2![\hat{A},[\hat{A},\hat{B}]]+\dots$ for y a parameter and $\hat{A}$ and $\hat{B}$ operators. We have also used the commutation relation

$$[\hat{Y},\hat{J}]\hat{\rho} = 2i\chi\hat{R}\hat{J}\hat{\rho}. \tag{44}$$

Now it is easy to show that $\hat{R}$ and $\hat{J}$ commute, so that we can finally find the solution to equation (37) as

$$\hat{\rho}(t) = e^{\hat{Y}t}e^{\hat{L}t}\exp[e^{\frac{-i\chi\hat{R}t-2\gamma t-1}{-i\chi\hat{R}-2\gamma}}\hat{J}]\hat{\rho}(0). \tag{45}$$

Note that the above solution may be applied easily to any initial density matrix:

$$\hat{\rho}(t) = \sum_{k,n,m=0}^{\infty}\hat{\rho}_{n+k,m+k}(0)e^{-i\chi t(n^2-m^2)-\gamma t(n+m)}\sqrt{\frac{(n+k)!(m+k)!}{n!m!}}$$
$$\left[\frac{1-e^{-2i\chi t(n-m)-2\gamma t}}{2i\chi(n-m)+2\gamma}\right]^k\frac{(2\gamma)^k}{k!}|n\rangle\langle n|, \tag{46}$$

where $\hat{\rho}_{n,m}(0)$ are the (Fock) matrix elements of the initial density matrix.

V. CONCLUSIONS

We have shown the importance of looking for transformations that simplify Hamiltonians before trying to solve a given system. In particular, we have shown how a 3-body system can be reduced to a 2-body system, in the case of a slow atom interacting with a quantized field. We have given the most complete solutions for the ion-laser interaction in several cases: time dependent case, several ions, ions vibrating in two dimensions.

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