HPM Applied to Solve Nonlinear Circuits:
A Study Case

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Abstract

In this paper, the homotopy perturbation method HPM, is employed to obtain an approximate solution for the nonlinear differential equation which describes an alternating current RC circuit, taking into account the effect of a nonlinear resistance on the current. In order to compare this solution, the problem is solved using two homotopy approaches: the standard HPM method and a modified version of HPM. From these methods, two different approximate solutions are obtained. Comparison with exact solution shows that the HPM method is extremely efficient if the initial guess is chosen adequately.

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1   Introduction

Industrial competition constantly pushes the area of electronics circuits to the limits of technology. This has caused a rapid growth in the levels of integration for integrated circuits (IC) and the emergence of novel devices such as single electron transistors and memristors. Because of this, it is important the development and improvement of mathematical and numerical tool, applied to circuit simulation at the static and dynamic domain. In the dynamic domain (AC or transient), the circuit analysis is carried out only numerically (by methods such as Runge Kutta), because the resulting differential equations are highly nonlinear. Nevertheless, there are several methods used to find approximate solutions for nonlinear problems such as variational approximations [2, 15], Tanh method [7], exp function [21], Adomian decomposition method [1, 3], parametric expansion [22], HPM [12, 11, 14, 8, 16, 4, 13, 6, 10, 9, 20, 19], and so on. Among all the above methods, the HPM method is one of the most employed because is powerful and its practical application is simpler than other techniques.

Homotopy perturbation method [12, 11] was first proposed by Ji Huang He, and it was introduced as a powerful tool to approach various kinds of nonlinear problems. As it is well known, nonlinear phenomena appear in a broad variety of scientific fields, such as, applied mathematics, physics, engineering. Scientists in those disciplines are constantly faced with the task of discovering solutions for nonlinear ordinary differential equations, partial differential equations and nonlinear differential equations systems. HPM method can be considered as a combination of the classic perturbation technique and the homotopy (whose origin is in topology) but limitations from traditional perturbation methods have been eliminated. For instance HPM method does not require a small parameter or linearisation, in fact, it only requires few iterations to obtain highly precise solutions. Furthermore, this method has been successfully used to solve integral equations, for example the Volterra integral equation case [6].

The homotopy perturbation method can be used to solve, analytically, nonlinear problems from electronic circuits. This paper presents two versions of this method in order to calculate the effects that a nonlinear resistance has on the current for the case of an RC circuit. Will be seen that, in this case, one version is clearly better and discuss the reasons.

2   HPM Methods

2.1   Standard Homotopy Perturbation Method

To get an idea of how HPM works consider a general nonlinear equation of the form

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{1}$$

with the following boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right), \qquad r \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, and Γ is the domain boundary for Ω.

A can be divided into two parts, L and N, where L is linear and N nonlinear; from this last statement, (1) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Generally, a homotopy can be constructed in the form [11]

$$H(v, p) = (1 - p)(L(v) - L(u_0)) + p[A(v) - f(r)] = 0, \qquad p \in [0, 1], \qquad r \in \Omega, \tag{4}$$

or

$$H(v, p) = (L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \qquad p \in [0, 1], \qquad r \in \Omega, \tag{5}$$

where p is an homotopy parameter and u_o is the first approach to the approximation for the solution of (1) which satisfies boundary conditions.

Assuming that solution for (4) or (5) can be written as a power series of p

$$v = v_0 + v_1 p^1 + v_2 p^2 + \ldots \tag{6}$$

Substituting (6) into (5) and equating identical powers of p terms, there can be found values for the succession u_0, u_1, u_2, ... When $p \to 1$, it yields in the approximate solution for (1), in the form

$$v = v_0 + v_1 + v_2 + \cdots \tag{7}$$

Another way to develop an homotopy, which is relevant for this article, is by considering the following general equation

$$L(u) + N(u) = 0, \tag{8}$$

where $L(u)$ and $N(u)$ are linear and nonlinear operators, respectively; solution for $L = 0$, appropriately describes the original nonlinear system. Employing the homotopy technique it is possible to construct an homotopy such as [8]

$$(1 - p)L(v) + p(L(v) + N(v)) = 0, \tag{9}$$

assuming, again, that solution for (9) can be written in the form (6) and taking the limit $p \to 1$, results an approximate solution for (8).

2.2   A modified version of HPM Method

In [10] was introduced a modified HPM method to solve the problem of a non-damped, nonlinear oscillator. This method consists in adding and subtracting an adequate linear term from the equation, in such way that the equation itself remains unaltered; however, one of the additional terms need to be added to the linear operator, this is done to avoid a divergent solution and, at the same time, results in a quick and precise convergent solution. If this is compared with the stardard HPM method, the only difference between this version of HPM and the standard method is the linear operator, while the rest of the procedures are the same for both cases. In this work is employed this modified version of HPM method in order to improve the obtained results from standard HPM method.

3   Study case

In order to show how to obtain approximate analytical solutions in some non-linear circuits, we consider the case of dependence of the resistance R with the current $i(t)$. We assume, for simplicity, the following first order expression [5]

$$R(t) = R_0 + Ki(t), \tag{10}$$

where R_0 and K are constant parameters.

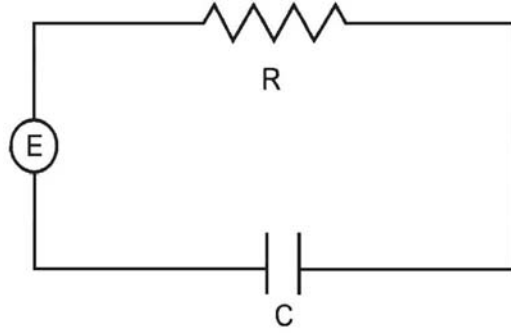Figure 1: Basic RC circuit

A basic RC alternate current circuit may consists of a constant resistor R_0 and a capacitor c, connected in series with a source $V = V_0 \sin(wt)$, as shown in Fig. 1.

Equation for electric current, is obtained from Kirchoff's second Law

$$R_0 i + \frac{1}{c} \int i\,dt = V_0 \sin(wt), \tag{11}$$

differentiating (11), we obtain

$$R_0 \frac{di}{dt} + \frac{i}{c} = V_0 w \cos(wt). \tag{12}$$

To exemplify, assigning the following quantities: $w = 1, R_0 = 2, V_0 = 1$, and $c = 1$, in such way that (12) adopts the simple form

$$\frac{di}{dt} + \frac{1}{2}i = \frac{1}{2}\cos(t). \tag{13}$$

Assuming, for instance, that $i(0) = 0$ it is possible to solve (13), and obtain

$$i(t) = \frac{1}{5}\left(\cos(t) + 2\sin(t)\right) - \frac{1}{5}\exp\left(\tfrac{-t}{2}\right). \tag{14}$$

Fig. 2 shows that there is a short transitory period where the exponential term of equation (14) is notorious, and the graphic for $i(t)$ is not symmetrical with respect to t axis since the beginning. Next, the value of the current approaches the stationary solution, which is given by the trigonometrical part of (14). At this stage, current oscillates at the frequency given by the driving source $w = 1$. It is clear that the graphic quickly becomes symmetrical with respect to t axis. Rewriting the stationary part $i_s(t)$ of (14) in the form
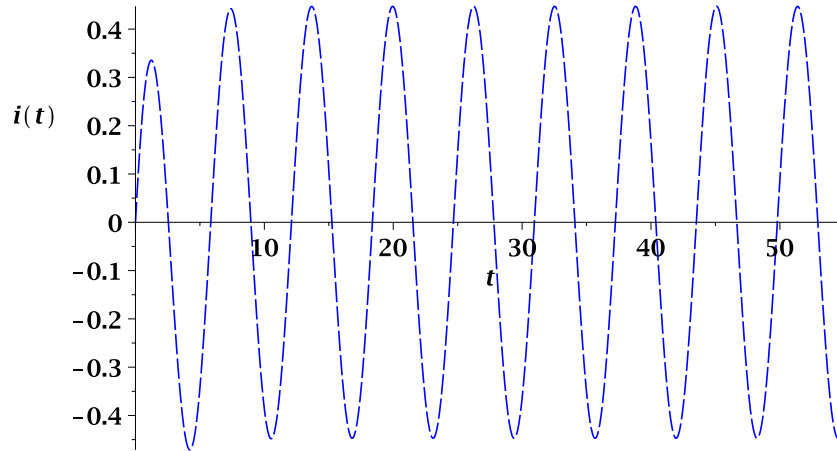
Figure 2: Numerical solution RK (14).

$$i_s(t) = \frac{1}{\sqrt{5}} \sin\left(t + \frac{23}{50}\right), \tag{15}$$

it is observed another characteristic in this kind of circuits, current leads voltage in capacitive circuits [17, 18].

Nonlinear effects are taking into account replacing R_0 by $R(t)$, given in (10), into (11) and differentiating to obtain

$$(R_0 + 2Ki)\frac{di}{dt} + \frac{i}{c} = V_0 w \cos(wt). \tag{16}$$

Substituting values: $w = 1, R_0 = 2, V_0 = 1, c = 1$, and $K = \frac{1}{2}$, the equation results in a generalization of (13)

$$\left(1 + \frac{1}{2}i\right)\frac{di}{dt} + \frac{1}{2}i = \frac{1}{2}\cos(t), \qquad i(0) = 0. \tag{17}$$

However, unlike (13), this is a nonlinear equation.

Next, approximate solutions for (17) are found utilizing HPM techniques showed in sections 2A and 2B.

3.1   Method 1 Standard HPM Method

Following (4), a homotopy is constructed in the form

$$(1-p)\left[\frac{dv}{dt} + \left(\frac{1}{2}+\alpha\right)v - \beta - \frac{di_o}{dt} - \left(\frac{1}{2}+\alpha\right)i_o + \beta\right]$$
$$+p\left[\frac{dv}{dt} + \frac{1}{2}v\frac{dv}{dt} + \frac{1}{2}v - \frac{1}{2}\cos(t)\right] = 0 \qquad (18)$$

where α and β are adjusting constants. Assuming that the initial approximation for (17) that satisfies the initial condition has the form

$$i_0(t) = \frac{2\beta}{1+2\alpha}\left[1 - \exp\left(-(\frac{1}{2}+\alpha)t\right)\right]. \qquad (19)$$

Substituting (6) into (18), and grouping coefficients of similar powers of p, yields

$$p^0) \qquad \frac{dv_o}{dt} + \left[\frac{1}{2}+\alpha\right]v_0 - \beta - \frac{di_0}{dt} - \left(\frac{1}{2}+\alpha\right)i_0 + \beta = 0, \qquad v_0 = 0,$$
$$(20)$$

$$p^1) \qquad \frac{dv_1}{dt} + \left[\frac{1}{2}+\alpha\right]v_1 + \left[\frac{dv_o}{dt} + \frac{1}{2}v_o\frac{dv_o}{dt} + \frac{1}{2}v_o - \frac{1}{2}\cos(t)\right] = 0, \qquad v_1 = 0,$$
$$(21)$$

from (20) is obtained

$$v_o = i_o = \frac{2\beta}{1+2\alpha}\left[1 - \exp\left(-(\frac{1}{2}+\alpha)t\right)\right]. \qquad (22)$$

Substituting (22) into (21), yields

$$\frac{dv_1}{dt} + \left[\frac{1}{2}+\alpha\right]v_1$$
$$+ \exp\left(-\left(\frac{1}{2}+\alpha\right)t\right)\left[\beta + \frac{2\beta^2}{1+2\alpha}\left[1 - \exp\left(-\left(\frac{1}{2}+\alpha\right)t\right)\right] - \frac{\beta}{1+2\alpha}\right]$$
$$(23)$$
$$+ \frac{\beta}{1+2\alpha} - \frac{1}{2}\cos(t) = 0,$$

integrating this equation using an integration factor, yields

$$v_1(t) = \frac{\sin(t) + \alpha(\cos(t) - 1)}{2(1 + \alpha^2)}$$
$$+ \frac{2\beta \left[2\beta \left(1 - \exp\left(-(1 + 2\alpha)t\right)\right) - (\alpha + \beta)(1 + 2\alpha)t \exp\left(-(\tfrac{1}{2} + \alpha)t\right)\right]}{(1 + 2\alpha)^2}. \tag{24}$$

According to (7) an approximate solution for (17) would be

$$i(t) = v_0 + v_1 + \dots$$

For this case, we choose the first order approximation

$$i(t) = v_0 + v_1, \tag{25}$$

substituting (22) and (24) into (25) this results in,

$$i(t) = \frac{\sin(t) + \alpha(\cos(t) - 1)}{2(1 + \alpha^2)^2}$$
$$+ \frac{2\beta(1 + 2\alpha)(1 - \exp(-(\tfrac{1}{2} + \alpha)t))}{(1 + 2\alpha)^2}$$
$$+ \frac{4\beta^2 \left(1 - \exp(-(1 + 2\alpha)t)\right)}{(1 + 2\alpha)^2}$$
$$- \frac{2\beta(\alpha + \beta)t \exp(-(\tfrac{1}{2} + \alpha)t)}{(1 + 2\alpha)} \tag{26}$$

Constants α and β are calculated using the non linear fit command followed by the "convert" command (using the "rational" option) in maple 15; which results in

$$i(t) = \frac{24}{59}\sin(t) - \frac{22}{113}\cos(t) + \frac{46}{277} + \frac{14}{475}\exp\left(-\frac{2}{93}t\right)$$
$$- \frac{1}{1151}\exp\left(-\frac{4}{93}t\right) - \frac{5}{354}t \exp\left(-\frac{2}{93}t\right) \tag{27}$$

Equation (27) is an approximation to the solution of differential equation (17), (see Fig. 3).

It is evident that the proposed standard HPM method is not adequate to provide a first order approximation in this case. Higher order approximations, although possible, would give place to much larger and difficult expressions to handle. Instead, we solve (17) using the modified HPM method.
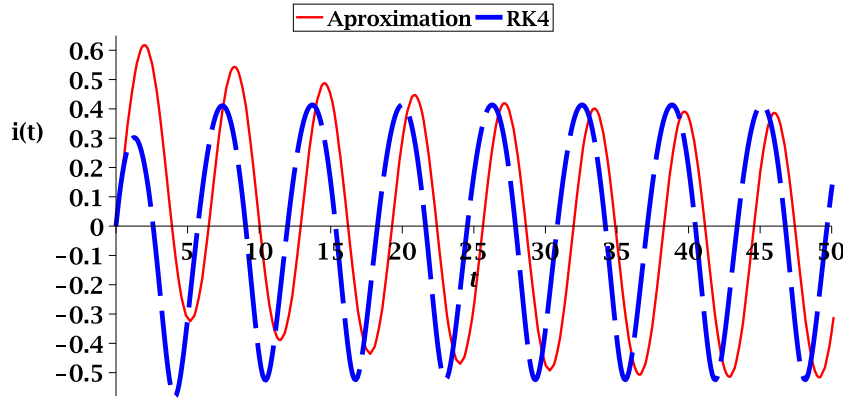
Figure 3: Numerical solution RK (17) (dashed) and its approximate solution (27) (solid line).

3.2 Method 2 Modified HPM Method

In order to obtain a better approximation with less effort; we differentiate (17) with respect to time and obtain the second order equation

$$\frac{d^2 i}{dt^2} + \frac{1}{2}\frac{di}{dt} + \frac{1}{2}\sin(t) + \frac{1}{2}\left(\frac{di}{dt}\right)^2 + \frac{1}{2}i\frac{d^2 i}{dt^2} = 0, \qquad i(0) = 0, \qquad i'(0) = \frac{1}{2}, \tag{28}$$

where the initial conditions are deduced from (17).

Linear part from last equation is identified as

$$L(t) = \frac{d^2 i}{dt^2} + \frac{1}{2}\frac{di}{dt} + \frac{1}{2}\sin(t), \tag{29}$$

and the nonlinear

$$N(t) = \frac{1}{2}\left(\frac{di}{dt}\right)^2 + \frac{1}{2}i\frac{d^2 i}{dt^2}. \tag{30}$$

Following references [8] and [10], adding and subtracting $a\frac{d^2 i}{dt^2} + b\frac{di}{dt}$, as shown, in such way that equation (28) can be rewritten as

$$(1+a)\frac{d^2 i}{dt^2} + \left(\frac{1}{2}+b\right)\frac{di}{dt} + \frac{1}{2}\sin(t) + \frac{1}{2}\left(\frac{di}{dt}\right)^2 + \left(\frac{1}{2}i - a\right)\frac{d^2 i}{dt^2} - b\frac{di}{dt} = 0$$

where a and b are adjustment parameters.

In this manner, it is possible to identify the new $L(t)$ and $N(t)$ as follows

$$L(t) = (1+a)\frac{d^2 i}{dt^2} + \left(\frac{1}{2}+b\right)\frac{di}{dt} + \frac{1}{2}\sin(t), \qquad (31)$$

$$N(t) = \frac{1}{2}\left(\frac{di}{dt}\right)^2 + (\frac{1}{2}i - a)\frac{d^2 i}{dt^2} - b\frac{di}{dt}. \qquad (32)$$

Substituting (32), (31), and (6) into (9) and equating identical powers of p we obtain

$$(1+a)\frac{d^2 v_0}{dt^2} + \left(\frac{1}{2}+b\right)\frac{dv_0}{dt} + \frac{1}{2}\sin(t) = 0, \qquad v_0(0) = 0, \qquad v_0'(0) = \frac{1}{2}$$
$$(33)$$

$$(1+a)\frac{d^2 v_1}{dt^2} + \left(\frac{1}{2}+b\right)\frac{dv_1}{dt} + \frac{1}{2}\left(\frac{dv_0}{dt}\right)^2 + (\frac{1}{2}v_0 - a)\frac{d^2 v_0}{dt^2} - b\frac{dv_0}{dt} = 0, v_1(0) = 0, v_1'(0) = 0,$$
$$(34)$$

solving equation (33), we obtain

$$v_0(t) = -\frac{1363}{8912}\exp\left(-\frac{24t}{47}\right) + \frac{235}{557}\sin(t) + \frac{120}{557}\cos(t) - \frac{1}{16}, \qquad (35)$$

where a and b were calculated using again the non linear fit command.

According to (7), an approximate solution would be

$$i(t) = v_0(t) + v_1(t) + v_2(t) + \cdots$$

For this case, we choose the lowest order approximation

$$i(t) = v_0(t), \qquad (36)$$

in such way that

$$i(t) = -\frac{1363}{8912}\exp\left(-\frac{24t}{47}\right) + \frac{235}{557}\sin(t) + \frac{120}{557}\cos(t) - \frac{1}{16}, \qquad (37)$$

or rewriting (37)

$$i(t) = -\frac{1363}{8912} \exp\left(-\frac{24t}{47}\right) + \frac{\sqrt{69625}}{557} \sin\left(t + \frac{47}{100}\right) - \frac{1}{16}. \qquad (38)$$

Fig. 4, shows the comparison between the approximate solution (37) and the exact solution. It can be seen that figures are very similar. In fact, from Table 1 we can see the accuracy of (37) as an approximate solution to (28) and therefore to (17).
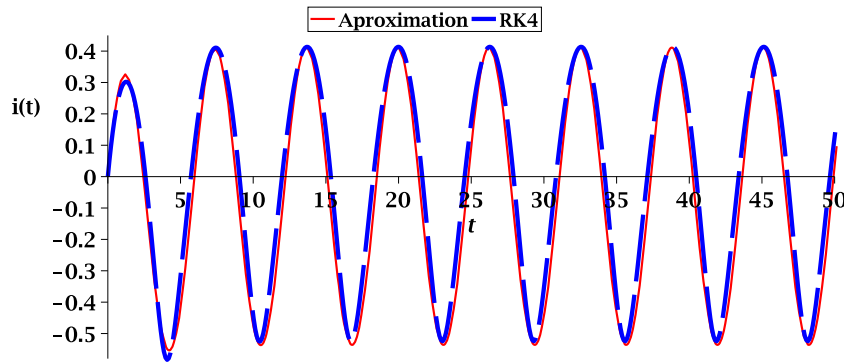

Figure 4: Numerical solution RK (17) (dashed) and its approximate solution (37) (solid line).

4   Discussion

This study presents the results of applying the HPM method to an nonlinear alternating current (AC) RC circuit. In the first case, a standard HPM method was employed; for the second case, a modified HPM version proposed in [8] and [10].

Two approximate solutions of different precision were found for the problem (27) and (37). (37) shows that if the initial guess is chosen adequately, it is possible to obtain an accurate approximation to the solution (see Fig. 4). This is not unusual because the second solution method allows to introduce the oscillatory character of the source since the beginning (see (33)), and the standard method does not allow this (see (20)). Therefore, in this case, the standard HPM requires higher order approximations, but from (27) is clear that it would lead to a not so handy and cumbersome approximation. This shows the convenience to use the modified HPM for this and similar cases.

Table 1 and Fig. 4 shows that (37) has good accuracy when compared to the Runge-Kutta numerical method . Also (37) can be used to make a

t	MHPM	RK4	RK4-MHPM
0.00	0.00000000	0.00000000	0.00000000
1.00	0.31714072	0.28977089	-0.02736983
2.00	0.17640176	0.205765	0.02936324
3.00	-0.24929846	-0.18927459	0.06002387
4.00	-0.54245403	-0.57991548	-0.03746145
5.00	-0.41786469	-0.31727632	0.10058837
6.00	0.01932911	0.1143201	0.09499099
7.00	0.37281829	0.38184233	0.00902404
8.00	0.32099410	0.35829268	0.03729858
9.00	-0.08646361	0.025988337	0.11245194
10.00	-0.47372012	-0.44207991	0.03164021
20.00	0.41058592	0.41384532	0.00325941
30.00	-0.44612168	-0.35463937	0.09148232
40.00	0.10818048	0.19933296	0.09115248
50.00	0.03469539	0.12681008	0.09211469

Table 1: Comparison between the obtained solution against Modified HPM.

qualitative and quantitative analysis of the solution, unlike Runge method, which is only useful for qualitative analysis. For instance, HPM method allows to quantitatively deduce that the current that satisfies the nonlinear equation (17) is no longer symmetrical respect to t axis, in contrast of what is occurring to the RC linear circuit described by (13) (see Figs. 2 and 4). Also, (38) shows that the current leads the voltage; as in the case of linear RC circuit (see (15)).

5   Conclusion

In this work we have shown the usefulness of HPM method to quantitatively describe the effect that a nonlinear resistance has on an alternating current RC circuit. For instance, we employed the simple mathematical model (10); the problem was solved using two versions of HPM method. A relevant fact of the HPM modified version is that, even utilizing the lowest order approximation (36) a highly precise solution for (17) can be obtained (see Fig. 4). In contrast to Runge-Kutta numerical solution, HPM method allows both quantitatively and qualitatively analyse the solution.

It is expected that other problems in the field of electronics described by nonlinear differential equations can be solved in a similar way, following the techniques employed in this work.

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