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Review

Variational iteration algorithm-II for solving linear and non-linear ODEs

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Variational iteration method is widely used for solving linear and non-linear ODE. In this paper, the variational iteration algorithm-II is constructed and some examples are given to show its simple solution procedure and high accuracy of the obtained results.

Key words: Variational iteration method, series solution, ordinary differential equations.

INTRODUCTION

In recent years, much attention has been paid to the application of the Variational iteration method (He, 1999) to various problems due to its simplicity. The variational iteration method was proposed first by the Chinese researcher He (2006) and was further developed by him (He, 2006, 2008, 2012; He and Wu, 2007). Recently, Wazwaz (2009) applied a universal variational iteration algorithm for linear and non-linear ODEs which caught an immediate attention in the mathematics community. The highly entertaining work, related to fractional calculus, shows that the variational iteration method (He, 2011) is a highly useful mathematical tool for discovery of real-life problems. This method has wider application because it reduces the size of computation and is a very powerful mathematical tool for various kinds of linear and nonlinear problems arising in differential equations/ fractional differential equations in a recent publication by He et al. (2010). "The variational iteration method should be followed", for the problem to be completely eliminated, and a new iteration algorithm was suggested, and the algorithm was termed as the variational iteration algorithm-II (He, 2007; Faraz et al., 2010; Wu and Li, 2011). It is an alternative approach to linear and non-linear differential equations using the variational iteration method. By a careful insight into the iteration formulas, we found some unnecessary repeated calculation in each iteration step. To overcome the shortcoming, the variational iteration algorithm-II was used, which was suggested by He et al. (2010). In general, the solutions produced by the VIM-II are as accurate as the solutions given by the other classical methods (Usman et al., 2011; Mohyud-Din et al., 2011; Khan et al., 2011; Khan and Wu, 2011; Akbarzade et al., 2012).

VARIATIONAL ITERATION ALGORITHM-II FOR LINEAR AND NON-LINEAR ODES

In a recent review article, He et al. (2010) summarized three variational iteration algorithms. Here, the variational iteration algorithm-II was used for the study. We will briefly elucidate Wazwaz's iteration formulations for linear and non-linear ODEs:

Example 1: Solve the following first order homogeneous ODE:

$$\frac{\partial u}{\partial x} - 2xu = 0, n \geq 0 \qquad (1)$$
$$u(0) = 1.$$

According to the variational iteration method (He, 1999,

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2006, 2008, 2012; He and Wu, 2007). Wazwaz (2009) constructed the following iteration formulation:

$$u_{n+1}(x)=u_n(x)-\int_0^x\left(\frac{\partial u_n}{\partial t}-2tu_n(t)\right)dt,n\geq 0 \tag{2}$$

In this iteration formulation, unnecessary repeated calculation occurred in each of the iteration. Many modifications appeared in the literature to overcome the shortcoming, among which the variational iteration algorithm-II is the best candidate (He et al., 2010; He, 2007; Faraz et al., 2010; Wu and Li, 2011). According to He et al. (2010), we can construct a more concise iteration formulation, which reads:

$$u_{n+1}(x)=u_0(x)+\int_0^x 2tu_n(t)dt,n\geq 0 \tag{3}$$

Where $u_0=1.$

It is obvious that our iteration algorithm is much simpler. To precede the solution procedure, we wrote down the following successive approximations:

$$\begin{aligned}
u_1(x)&=u_0(x)+\int_0^x 2tu_0(t)dt=1+x^2\\
u_2(x)&=u_0(x)+\int_0^x 2tu_1(t)dt=1+x^2+\frac{x^4}{4!}\\
u_3(x)&=u_0(x)+\int_0^x 2tu_2(t)dt=1+x^2+\frac{x^4}{4!}+\frac{x^6}{6!}\\
u_4(x)&=u_0(x)+\int_0^x 2tu_3(t)dt=1+x^2+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}\\
&\vdots\\
u_n(x)&=u_0(x)+\int_0^x 2tu_{n-1}(t)dt=1+x^2+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}+\ldots\frac{x^{2n}}{2n!}
\end{aligned} \tag{4}$$

Which converges to the exact solution $u(x)=e^{x^2}$. The result is exactly same with that obtained by Equation 2.

Example 2: Solve the following second order non-homogeneous ODE:

$$\begin{aligned}
&u^{//}-3u^{/}+2u=2x-3,\\
&u(0)=1,u^{/}(0)=2
\end{aligned} \tag{5}$$

Using the variational iteration method, Wazwaz (2009) suggested the following iteration formulation:

$$u_{n+1}(x)=u_n(x)+\int_0^x (t-x)(u_n^{//}(t)-3u_n^{/}(t)+2u_n(t)-2t+3)dt,n\geq 0 \tag{6}$$

Using the variational iteration algorithm-II [8], we give an alternative iteration formulation, which reads:

$$u_{n+1}(x)=u_0(x)-\int_0^x (t-x)(3u_n^{/}(t)-2u_n(t)+2t-3)dt,n\geq 0 \tag{7}$$

Where $u_0=1+2x,$ satisfying the initial conditions. Thus, the consecutive approximations can be obtained as follows:

$$\begin{aligned}
u_1(x)&=u_0(x)+\int_0^x (t-x)(-3u_0^{/}(t)+2u_0(t)-2t+3)dt=1+2x+\frac{x^2}{2!}-\frac{x^3}{3},\\
u_2(x)&=u_0(x)+\int_0^x (t-x)(-3u_1^{/}(t)+2u_1(t)-2t+3)dt=1+2x+\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{3}+\frac{x^5}{30},\\
u_3(x)&=u_0(x)+\int_0^x (t-x)(-3u_2^{/}(t)+2u_2(t)-2t+3)dt=1+2x+\frac{x^2}{2!}+\frac{x^3}{3!}\\
&+\frac{x^4}{4!}-\frac{13x^5}{30}+\ldots,\\
&\vdots\\
u_n(x)&=x+(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots).
\end{aligned} \tag{8}$$

This in turn gives the exact solution:

$$u(x)=x+e^x.$$

Example 3: Solve the following second order homogenous ODE:

$$\begin{aligned}
&u^{///}-2u^{//}+u^{/}=1,\\
&u(0)=0,u^{/}(0)=2,u^{//}(0)=2
\end{aligned} \tag{9}$$

The iteration formula used by Wazwaz (2009) is given as:

$$u_{n+1}(x)=u_0(x)-\frac{1}{2!}\int_0^x t-x\ ^2(u_n^{///}(t)-2u_n^{//}(t)+u_n^{/}(t)-1)dt,n\geq 0 \tag{10}$$

Instead of Equation 9, the following iteration formulation was suggested according to He et al. (2010):

$$u_{n+1}(x)=u_0(x)+\frac{1}{2!}\int_0^x t-x\ ^2(2u_n^{//}(t)-u_n^{/}(t)+1)dt,n\geq 0 \tag{11}$$

We begin with $u_0=2x+2x^2$, and obtain the following approximations:

$$
\begin{aligned}
u_1(x) &= u_0(x) + \frac{1}{2!}\int_0^x t-x\ ^2 (2u_0^{//}(t) - u_0^{/}(t) + 1)dt = 2x + x^2 + \frac{x^3}{2} - \frac{x^4}{12}, \\
u_2(x) &= u_0(x) + \frac{1}{2!}\int_0^x t-x\ ^2 (2u_1^{//}(t) - u_1^{/}(t) + 1)dt = 2x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} - \frac{7x^5}{120}, \\
u_3(x) &= u_0(x) + \frac{1}{2!}\int_0^x t-x\ ^2 (2u_2^{//}(t) - u_2^{/}(t) + 1)dt = 2x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \frac{x^5}{4!} + \dots, \\
&\vdots \\
u_n(x) &= x + x(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots).
\end{aligned} \tag{12}
$$

Finally the exact solution is $u(x) = x(1+e^x)$.

Example 4: Find a series solution near $x = 0$ for:

$$
u^{//} - xu^{/} = 0, \tag{13}
$$

$$
u(0) = a_0, u^{/}(0) = a_1
$$

The variational iteration formulation for Equation 13 was proposed by Wazwaz (2009):

$$
u_{n+1}(x) = u_n(x) + \int_0^x (t-x)(u_n^{//}(t) - tu_n^{/}(t))dt, n \geq 0 \tag{14}
$$

Similarly the iteration formulation can be updated by the following one:

$$
u_{n+1}(x) = u_0(x) - \int_0^x (t-x)(tu_n^{/}(t))dt, n \geq 0 \tag{15}
$$

Where $u_0 = a_0 + a_1 t$.

By Equation 15, we have:

$$
\begin{aligned}
u_1(x) &= u_0(x) - \int_0^x (t-x)(tu_0^{/}(t))dt = a_0(1 + \frac{x^3}{6}) + a_1(x + \frac{x^4}{12}), \\
u_2(x) &= u_0(x) - \int_0^x (t-x)(tu_1^{/}(t))dt = a_0(1 + \frac{x^3}{6} + \frac{x^6}{180}) + a_1(x + \frac{x^4}{12} + \frac{x^7}{504}), \\
u_3(x) &= u_0(x) - \int_0^x (t-x)(tu_2^{/}(t))dt = a_0(1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960}) + a_1(x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360}), \\
&\vdots \\
u_n(x) &= a_0(1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{3n(3n-1)(3n-3)(3n-4)\dots 3.2}) + a_1(x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3n+1)(3n)(3n-2)(3n-3)\dots 4.2}).
\end{aligned} \tag{16}
$$

The solution of the Airy equation in Wazwaz (2009) required a lot of unnecessary repeated calculation; however, our iteration formulation results are a much simpler solution procedure.

Example 5: Solve the second order Euler equation:

$$
x^2 y^{//}(x) - 2xy^{/} + 2y = 0,
$$

$$
y(1) = 2, y^{/}(1) = 3, x \geq 0 \tag{17}
$$

The problem has singularity at $x = 0$. Using the following transformation:

$$
z = \ln x, x = e^z, \tag{18}
$$

We obtain the following equation:

$$
\begin{aligned}
&\frac{d^2 y}{dz^2} - 3\frac{dy}{dz} + 2y = 0, \\
&y(0) = 2, y^{/}(0) = 3.
\end{aligned} \tag{19}
$$

Its iteration formulation was:

$$
y_{n+1}(z) = y_n(z) + \int_0^z (t-z)(y_n^{//}(t) - 3y_n^{/}(t) + 2y_n(t))dt, \tag{20}
$$

Alternative form is giving by:

$$
y_{n+1}(z) = y_0(z) - \int_0^z (t-z)(3y_n^{/}(t) - 2y_n(t))dt, \tag{21}
$$

We begin with $y_o(z) = 2 + 3z$ and obtain with ease the following approximates:

$$
\begin{aligned}
y_1(z) &= y_0(z) - \int_0^z (t-z)(3y_0^{/}(t) - 2y_0(t))dt = 2 + 3z + \frac{5z^2}{2} - z^3, \\
y_2(z) &= y_0(z) - \int_0^z (t-z)(3y_1^{/}(t) - 2y_1(t))dt = 2 + 3z + \frac{5z^2}{2} + \frac{3z^3}{2} - \frac{7z^4}{6} + \frac{z^5}{10}, \\
y_3(z) &= y_0(z) - \int_0^z (t-z)(3y_2^{/}(t) - 2y_2(t))dt = 2 + 3z + \frac{5z^2}{2} + \frac{3z^3}{2} + \frac{17z^4}{24} - \frac{17z^5}{20} + \dots, \\
y_4(z) &= y_0(z) - \int_0^z (t-z)(3y_3^{/}(t) - 2y_3(t))dt = 2 + 3z + \frac{5z^2}{2} + \frac{3z^3}{2} + \frac{17z^4}{24} + \frac{11z^5}{40} + \dots, \\
&\vdots \\
y(z) &= y_0(z) + y_1(z) + y_2(z) + y_3(z) + y_4(z) = e^z + e^{2z}
\end{aligned} \tag{22}
$$

After using $z = \ln x$, the exact solution becomes $y(x) = x + x^2$.

Example 6: The logistic differential equation:

$$
u^{/} = \mu u(1-u), u(0) = \frac{1}{2}, \mu > 0, \tag{23}
$$

The VIM admits the use of the iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x (u_n'(t) - \mu u_n(t)(1 - u_n(t))dt, n \geq 0 \qquad (24)$$

However, using the new concept of VIM [8], we suggest an alternative iteration formula:

$$u_{n+1}(x) = u_0(x) + \int_0^x (\mu u_n(t)(1 - u_n(t))dt, n \geq 0 \qquad (25)$$

This gives the successive approximations:

$$\begin{aligned}
&u_0 = \frac{1}{2},\\
&u_1(x) = u_0(x) + \int_0^x (\mu u_0(t)(1-u_0(t))dt = \frac{1}{2} + \frac{\mu x}{4},\\
&u_2(x) = u_0(x) + \int_0^x (\mu u_1(t)(1-u_1(t))dt = \frac{1}{2} + \frac{\mu x}{4} - \frac{\mu^3 x^3}{48},\\
&u_3(x) = u_0(x) + \int_0^x (\mu u_2(t)(1-u_2(t))dt = \frac{1}{2} + \frac{\mu x}{4} - \frac{\mu^3 x^3}{48} + \frac{\mu^5 x^5}{480} - \frac{\mu^7 x^7}{16{,}128},\\
&u_4(x) = u_0(x) + \int_0^x (\mu u_3(t)(1-u_3(t))dt = \frac{1}{2} + \frac{\mu x}{4} - \frac{\mu^3 x^3}{48} + \frac{\mu^5 x^5}{480} - \frac{17\mu^7 x^7}{80{,}640} + \frac{19\mu^9 x^9}{1{,}451{,}520},\\
&u_5(x) = u_0(x) + \int_0^x (\mu u_4(t)(1-u_4(t))dt = \frac{1}{2} + \frac{\mu x}{4} - \frac{\mu^3 x^3}{48} + \frac{\mu^5 x^5}{480} - \frac{17\mu^7 x^7}{80{,}640} + \frac{31\mu^9 x^9}{1{,}451{,}520} + \ldots,
\end{aligned} \qquad (26)$$

This in turn gives the exact solution:

$$\frac{e^{\mu x}}{1 + e^{\mu x}}.$$

CONCLUSIONS

In this paper, we suggest an effective variational iteration algorithm for ODEs. The obtained results are same as those obtained by Wazwaz (2009), while our iteration procedure based on the method as described by He et al. (2010) is much simpler. The aim of this paper is in two folds. Firstly, we revealed that the new iteration formulations are much more effective; secondly, the new algorithm is of mathematical significance and of application features.

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