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The Glauber-Sudarshan and Kirkwood-Rihaczec functions

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Abstract

It is shown how to write the Kirkwood-Rihaczec quasiprobability distribution as an expectation value of the vacuum state. We do this, by writing the position eigenstates as a "displacement" of the vacuum. We also give a relation between the Glauber-Sudarshan and Kirkwood-Rihaczec quasiprobability distributions.

PACS numbers:

I. INTRODUCTION

Quasiprobability distribution functions are widely used in quantum mechanics [1, 2] and optical physics [4]. One of the best known quasiprobability distribution functions is the Wigner function [1, 3] with applications in reconstruction of signals [4] in the classical regime and reconstruction of quantum states of different systems such as ions [5] or quantized fields [6, 7] in the quantum regime. Quasiprobability distribution functions are also useful to identify non-classical states of light [8, 9]. In this contribution we would like to re-introduce a lesser known quasiprobability distribution function, namely the Kirkwood-Rihaczek function [10–13], and use its relation to the Wigner function to show how it may be related to the Glauber-Sudarshan P -function. This may be done as we express the Kirkwood-Rihaczek function as an expectation value in term of the vacuum.

A. Wigner function

We start by introducing the Wigner function, probably the best known. It may be written in two forms: series representation (see for instance [14]), and integral representation

$$W(q, p) = \frac{1}{2\pi} \int du e^{iup} \langle q + \frac{u}{2} | \rho | q - \frac{u}{2} \rangle, \quad (1)$$

where ρ is the density matrix. In 1932, Wigner introduced this function $W(q, p)$, known now as his distribution function [1, 3] and contains complete information about the state of the system as the density matrix for a pure state is given by $\rho = |\psi\rangle\langle\psi|$.

The Wigner function may be written also as in terms of the (double) Fourier transform of the characteristic function

$$W(\alpha) = \frac{1}{4\pi^2} \int \exp(\alpha\beta^* - \alpha^*\beta) C(\beta) d^2\beta, \quad (2)$$

with $\alpha = (q + ip)/\sqrt{2}$ and where $C(\beta)$ in terms of annihilation and creation operators is given by

$$C(\beta) = Tr\{\rho \exp(\beta a^\dagger - \beta^* a)\}, \quad (3)$$

also known as ambiguity function in classical optics [15]. a and a^\dagger are the annihilation and creation operators for the harmonic oscillator.

B. Cohen-class distribution functions

A function of the Cohen class is described by the general formula [16]

$$W_C = \frac{1}{2\pi} \int \int \int \phi(y + \frac{1}{2}x')\phi(y - \frac{1}{2}x')k(x, u, x', u')e^{-i(ux' - u'x + u'y)} dx dx' du' \quad (4)$$

and the choice of the kernel $k(x, u, x', u')$ selects one particular function of the Cohen class. The Wigner function, for instance arises for $k(x, u, x', u') = 1$, whereas the ambiguity function is obtained for $k(x, u, x', u') = 2\pi\delta(x - x')\delta(u - u')$.

II. THE KIRKWOOD-RIHACZEK QUASIDISTRIBUTION FUNCTION

Now we turn our attention to a lesser known distribution, the Kirkwood-Rihaczek function, that may be written using the notation above as [17]

$$K(\beta) = \int d^2\alpha e^{\beta\alpha^* - \beta^*\alpha} e^{\frac{\alpha^2 - \alpha^{*2}}{4}} C(\alpha), \quad (5)$$

may also be expressed as the double Fourier transform

$$K(q, p) = \int dudv e^{-iup} e^{ivq} Tr\{\rho e^{ivq} e^{iup}\}, \quad (6)$$

and the trace is to be taken in the form

$$Tr\{\rho A\} = \langle\psi|A|\psi\rangle = \int_{-\infty}^{\infty} dq \langle q|\psi\rangle \langle\psi|A|q\rangle. \quad (7)$$

We will now do an analysis similar to the one done in reference [14]. We relate the Kirkwood-Rihaczek function to the Wigner function by using (5), via the following exponential of derivatives

$$K(\beta) = e^{-\frac{1}{4}\frac{\partial^2}{\partial^2\beta}} e^{\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}} W(\beta). \quad (8)$$

In the above equation we will use a non-integral expression for the Wigner function [14]

$$W(\beta) = Tr \left[(-1)^{a^\dagger a} D^\dagger(\beta) \rho D(\beta) \right], \quad (9)$$

with $D(\beta) = e^{\beta a^\dagger - \beta^* a}$, the so-called Glauber displacement operator. Rearranging the displacement operators and the parity operator, we obtain

$$W(\beta) = Tr \left[(-1)^{a^\dagger a} \rho D(2\beta) \right], \quad (10)$$

where we have used the trace property $Tr(AB) = Tr(BA)$ and the following identities $(-1)^{a^\dagger a} D^\dagger(\beta) = D(\beta) (-1)^{a^\dagger a}$.

Now we use the factorized form of the Glauber displacement operator [18] $D(2\beta) = e^{-2|\beta|^2} e^{2\beta a^\dagger} e^{-2\beta^* a}$ to obtain

$$W(\beta) = Tr \left[(-1)^{a^\dagger a} \rho e^{-2|\beta|^2} e^{2\beta a^\dagger} e^{-2\beta^* a} \right]. \quad (11)$$

Therefore, we have that the Kirkwood-Rihaczek function may be written as

$$\begin{aligned} K(\beta, \beta^*) &= e^{-\frac{1}{4} \frac{\partial^2}{\partial^2 \beta}} e^{\frac{1}{4} \frac{\partial^2}{\partial^2 \beta^*}} W(\beta, \beta^*) \\ &= Tr \left[(-1)^{a^\dagger a} \rho e^{-\frac{1}{4} \frac{\partial^2}{\partial^2 \beta}} e^{\frac{1}{4} \frac{\partial^2}{\partial^2 \beta^*}} D(2\beta) \right]. \end{aligned} \quad (12)$$

The calculation of the exponential of derivatives of the Glauber operator will be tedious but straightforward. We note that

$$e^{\frac{1}{4} \frac{\partial^2}{\partial^2 \beta^*}} D(2\beta) = e^{-\beta^2} e^{2\beta(a^\dagger + a - \beta^*)} e^{a^2} e^{-2\beta^* a}, \quad (13)$$

and by using the expression for the generating function for Hermite polynomials [19]

$$e^{-t^2 + 2tx} = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!} \quad (14)$$

we can express the above equation as

$$e^{\frac{1}{4} \frac{\partial^2}{\partial^2 \beta^*}} D(2\beta) = \sum_{k=0}^{\infty} H_k(a^\dagger + a - \beta^*) \frac{\beta^k}{k!} e^{a^2} e^{-2\beta^* a}. \quad (15)$$

From the above equation, it is easy to note that

$$\frac{\partial^{2n}}{\partial \beta^{2n}} \sum_{k=0}^{\infty} H_k(x) \frac{\beta^k}{k!} = \sum_{k=0}^{\infty} H_{k+2n}(x) \frac{\beta^k}{k!} \quad (16)$$

such that

$$e^{-\frac{1}{4} \frac{\partial^2}{\partial^2 \beta}} e^{\frac{1}{4} \frac{\partial^2}{\partial^2 \beta^*}} D(2\beta) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-\frac{1}{4})^n}{n!} H_{k+2n}(a^\dagger + a - \beta^*) \frac{(\beta)^k}{k!} e^{a^2} e^{-2\beta^* a}. \quad (17)$$

Now we use the integral form of the Hermite polynomials [19]

$$H_p(x) = \frac{2^p}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + it)^p e^{-t^2} dt \quad (18)$$

to obtain

$$K(\beta, \beta^*) = \frac{e^{-\beta^{*2}} e^{-2\beta\beta^*}}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt e^{(-2\sqrt{2}x + 2\beta^* + 2\beta)it} e^{-2x^2} e^{2\sqrt{2}x(\beta^* + \beta)} \times \langle x | e^{a^2} e^{-2\beta^* a} (-1)^{a^\dagger a} \rho | x \rangle \quad (19)$$

by using

$$\int_{-\infty}^{\infty} e^{-iyt} dt = 2\pi\delta(y). \quad (20)$$

If we take $y = 2\sqrt{2}x - 2\beta^* - 2\beta$ we have

$$K(\beta, \beta^*) = 2\sqrt{\pi} e^{-\beta^{*2}} e^{-2\beta\beta^*} \int_{-\infty}^{\infty} dx \delta(2\sqrt{2}x - 2\beta^* - 2\beta) e^{-2x^2} e^{2\sqrt{2}x(\beta^* + \beta)} \times \langle x | e^{a^2} e^{-2\beta^* a} (-1)^{a^\dagger a} \rho | x \rangle. \quad (21)$$

Making use of the identity $\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$ we finally obtain

$$K(\beta, \beta^*) = \sqrt{\frac{\pi}{2}} e^{\beta^2 - \beta^{*2}} \langle X | e^{(a - \beta^*)^2} (-1)^{a^\dagger a} \rho | X \rangle \quad (22)$$

with $\beta = \frac{X + iY}{\sqrt{2}}$.

The position eigenstate $|X\rangle$ may be written as

$$|X\rangle = \sum_{n=0}^{\infty} \psi_n(X) |n\rangle \quad (23)$$

with $\psi_n(X) = \frac{e^{-X^2/2} H_n(X)}{\sqrt{2^n} \sqrt{\pi n!}}$ such that the position eigenstate may be re-written as

$$|X\rangle = \frac{e^{-X^2/2}}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{H_n(X)}{2^{n/2} n!} a^{\dagger n} |0\rangle \quad (24)$$

that may be added via the generating function for Hermite polynomials(14) to give

$$|X\rangle = \frac{e^{-X^2/2}}{\pi^{1/4}} e^{-\frac{a^{\dagger 2}}{2} + \sqrt{2} a^\dagger X} |0\rangle. \quad (25)$$

In the above equation the application of an operator to the vacuum produces the position eigenstate.

By using the above expression in equation (22) we obtain

$$K(\beta, \beta^*) = \frac{e^{\beta^2 - X^2}}{\sqrt{2}} \langle 0 | e^{\frac{a^2}{2} - \sqrt{2}iYa} (-1)^{a^\dagger a} \rho e^{-\frac{a^{\dagger 2}}{2} + \sqrt{2}a^\dagger X} | 0 \rangle, \quad (26)$$

or

$$K(\beta, \beta^*) = \frac{e^{\beta^2 - X^2}}{\sqrt{2}} \langle 0 | e^{\frac{a^2}{2} + \sqrt{2}iYa} \rho e^{-\frac{a^{\dagger 2}}{2} + \sqrt{2}a^\dagger X} | 0 \rangle, \quad (27)$$

that may be finally written as an expectation value in terms of coherent states

$$K(\beta, \beta^*) = \frac{e^{\beta^2 + Y^2}}{\sqrt{2}} \langle -\sqrt{2}iY | e^{\frac{a^2}{2}} \rho e^{-\frac{a^{\dagger 2}}{2}} | \sqrt{2}X \rangle. \quad (28)$$

We can relate the Kirkwood function to the Glauber-Sudarshan P -function [18, 20] by using the relation $\rho = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha|$, i.e.,

$$K(\beta, \beta^*) = \frac{e^{\beta^2 + Y^2}}{\sqrt{2}} \int d^2\alpha P(\alpha) e^{\frac{\alpha^2 - \alpha^{*2}}{2}} \langle -\sqrt{2} | Y | \alpha \rangle \langle \alpha | \sqrt{2} X \rangle, \quad (29)$$

or

$$K(\beta, \beta^*) = \frac{e^{iXY}}{\sqrt{2}} \int d^2\alpha P(\alpha) e^{\frac{\alpha^2 - \alpha^{*2}}{2} - |\alpha|^2} e^{\sqrt{2}(X\alpha^* - iY\alpha)}. \quad (30)$$

Therefore we have written the Kirkwood-Rihaczek function as an expectation value in terms of the vacuum state, just as the Q -function may be written as a coherent states expectation value, the Wigner and Glauber-Sudarshan functions in terms of a series of displaced number states expectation values [14], and relate it to the Glauber-Sudarshan P -function.

III. CONCLUSIONS

We have written the position eigenstates as a "displacement" of the vacuum state, which has allowed us to use a former expression for the Kirkwood-Rihaczek distribution function to write it as an expectation value in terms of the vacuum state. This made easy to relate this function to the Glauber-Sudarshan P -function [22].

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