Transverse Travelling Paraxial Weber Beams

por el

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1 Introduction

Light is something that has fascinated the people on the world who have made a study of it, it is interesting because it presents properties of wave and particle, and changes depending of the experiment being performed, in this work we try to light as an electromagnetic wave, for that we study two kinds of waves, travelling waves ans standing waves [1, 2].

1.1.1 Waves

The waves are in many places, for example in the sound like our voice or musical instruments, another kind are the water waves, which consisting of long swells that we see coming in to the shore, or the smaller water waves consisting of surface tension ripples. Another example, there are two kinds of elastic waves in solids; a compressional (or longitudinal) wave in which the particles of solid oscillate back and forth along the direction of propagation of the wave (sound waves in a gas are of this kind), and a transverse wave in which the particles of the solid oscillate in a direction perpendicular to the direction of propagation. Earthquake waves contain elastic waves of both kinds, generated by a motion at some place in the earth's crust.

Still another example of waves is found in modern physics. These are waves which give the probability amplitude of finding a particle at a given place, the "matter waves". Their frequency is proportional to the energy and their wave number is proportional to the momentum. They are the waves of quantum mechanics.

We are interested in a particular waves, the light, and even more, beams, they are obtained by cavities composed of mirrors, this is like a string attached at the ends, and it is known that standing waves are formed due to the superposition of two travelling waves, which travel in opposite directions and these are the two independent solutions of the wave equation associated with string, it is why we briefly introduce the description of travelling waves and standing waves on a string.

When we studied light, in learning about properties of waves in that subject, we paid particular attention to the interference in space of waves from several sources at different locations and all at the same frequency. There are two important wave phenomena. The first of these is the phenomenon of interference in time rather than interference in space. This is more easily understand in sound, if we have two sources of sound which have slightly different frequencies and if we listen to both at the same time, then sometimes the waves come with the crests together and sometimes with the crest and trough together. The rising and falling of the sound that results is the phenomenon of beats or, in other words, of interference in time. The second phenomenon involves the wave patterns which result when the waves are confined within a given volume and reflect back and forth from walls.

we shall consider only waves fro which the velocity is independent of wavelength. This is for example for light in a vacumm. The speed of light is then the same for radiowaves, blue light, green light, or for any other wavelength.

1.1.2 Travelling Waves

If we were to picture the electric field in space at some instant of time, as in Fig. 1.1, the electric field at time t later would have moved the distance ct, as indicated in the figure.

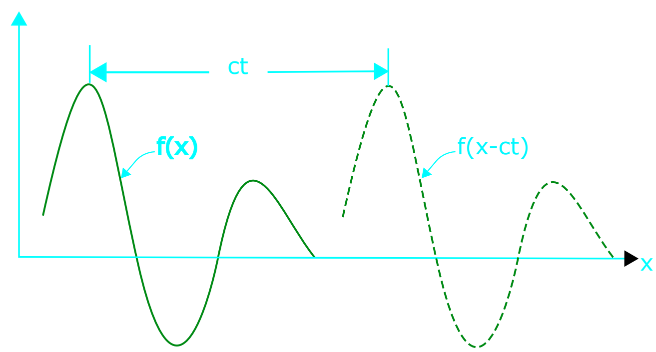


Figure 1.1: Fig:travellingwave.

Mathematically, we can say that in the one dimensional example we are taking, the electric field is a function, we need only to increase x somewhat yo get the same value of the electric field. For example, if the maximum field occurred at $x = 3$ at time zero, then to find the new position of the maximum field at time t we need

$$x - ct = 3 \quad \text{or} \quad x = 3 + ct, \tag{1.1}$$

We see that kind of function represents the propagation of wave.

Such a function, $f(x - ct)$, then represents a wave. We may summarize this description of a wave by saying simply that

$$f(x - ct) = f(x + \triangle x - c(t + \triangle t)), \tag{1.2}$$

when $\triangle x = c\triangle t$. There is, of course, another possibility, i.e., that instead of a source to the left as indicated in Fig. 1.1, we have a source on the right, so that the wave propagates toward negative x. Then the wave would be described by $g(x+ct)$.

There is the additional possibility that more than one wave exists in space at the same time, and so the electric field is the sum of the two fields, each one propagating independently. This behaviour of electric fields may be described by saying that if $f_1(x-ct)$ is a wave, and if $f_2(x+ct)$ is another wave, then their sum is also a waves. This is called the principle of superposition.

1.1.3 Standing Waves

1.3.1.3.1 The reflection of waves

we will consider some of the remarkable phenomena which are a result of confining waves in a some finite region. We will be led first to discover a few particular facts about vibrating strings, for example, and then the generalization of these facts will give us a principle which is probably the most far reaching principle of mathematical physics.

Our first example of confining waves will be to confine a wave at one boundary. Let us take the simple example of a one dimensional wave on a string. One could equally well consider sound in one dimension against a wall, or other situations of a similar nature, but the example of a string will be sufficient for our present purposes. Suppose that the string is held at one end, for example by fastening it to an "infinitely solid" wall. This can be expressed mathematically by saying that the displacement y of the string at the position $x=0$ must be zero, because the end does not move. Now if it were not for the wall, we know that the general solution for the motion is the sum of two function, $F(x-ct)$ and $G(x+ct)$, the first representing a ave travelling one way in the string, and the second a wave travelling the other way in the string:

$$y(x,t) = F(x-ct) + G(x+ct), \tag{1.3}$$

is the genereal solution for any string. But we have next to satisfy the condition that the string does not move at on end. If we put $x=0$ in Eq. (1.3) and examine $y(x,t)$ for any value o t, we get $y(x=0,t) = F(-ct)+G(+ct)$. Now if this is to be zero for all times, is means that the function $G(ct)$ must be $-F(-ct)$. In other words, G of anything must be $-F$ of minus that same thing. If this result is put back into Eq. (1.3), we find that the solution for the problem is

$$y(x,t) = F(x-ct) - F(-x-ct), \tag{1.4}$$

It is easy to check that we will get $y(x,t) = 0$ if we set $x = 0$.

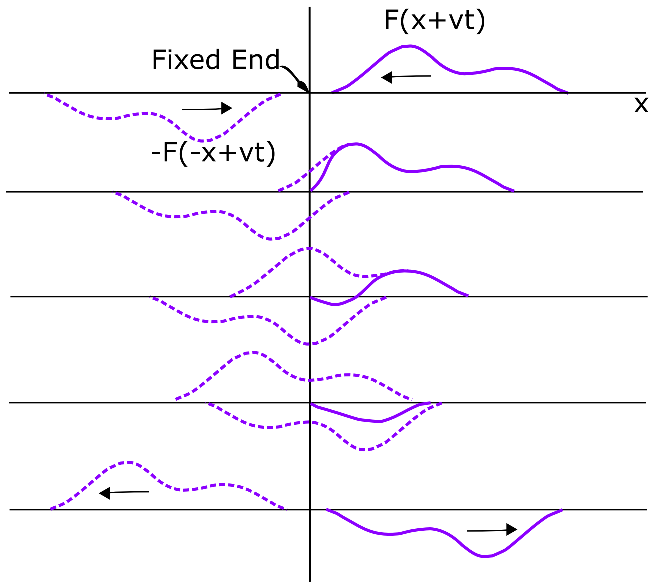


Figure 1.2: Reflection wave.

Fig. 1.2 show a wave travelling in the negative x-direction near $x = 0$, and a hypothetical wave travelling in the other direction reversed in sign and on the other direction reversed in sign and on the other side of the origin. We say hypothetical because, of course, there is no string to vibrate in that side of the origin. The total motion of the string is to be regarded as the sum of these two waves in the region of positive x. As they reach the origin, the y will always cancel at $x = 0$, and finally the second (reflected) wave will be the only one to exist for positive x and it will, of course, be travelling in the oppositive direction. These results are equivalent to the following statement: if a wave reaches the clamped end of a string, it will be reflected with a change in sign. Such a reflection can always be understood by imagining the wall. In short, if we assume that the string is infinite and that whenever we have going one way we have another one going the other way with the stated symmetry, the displacement at $x = 0$ will always be zero and it would make no difference if we clamped the string there.

The next point to be discussed is the reflection of a periodic wave. Suppose that the wave represented by $F(x-ct)$ is a sine wave and has been reflected; then the reflected wave $-F(-x-ct)$ is also a sine wave of the same frequency, but travelling in the opposite direction. This situation can be most simply described by using the complex function notation: $F(x-ct) = \mathrm{e}^{iw(t-x/c)}$ and $F(-x-ct) = \mathrm{e}^{iw(t+x/c)}$, It can be seen that if these are substituted in Eq. (1.4) and if x is set equal to 0, then $y(x=0,t) = 0$ for all values of t, so satisfies the necessary condition. Because of the properties of exponentials, this can be written in a simple form:

$$y(x,t) = e^{iwt}\left(e^{-iwx/c} - e^{iwx/c}\right) = -2ie^{iwt}\sin(wx/c) \tag{1.5}$$

There is something interesting and new here, in that this solution tells us that if we look at any fixed x, the string oscillates at frequency ω. NO matter where this point is, the frequency is the same. But there are some places, in particular wherever $\sin(\omega x/c)$, where there is no displacement at all. Furthermore, if at any time t we take a snapshot of the vibrating string, the picture will be a sine wave is equal to the wavelength of either of either of the superimposed waves:

$$\lambda = \frac{2\pi c}{\omega}. \tag{1.6}$$

The points where there is no motion satisfy the condition $\sin(\omega x/c) = 0, \pi, 2\pi, \cdots, n\pi, \cdots$. These points are called nodes. Between any two successive nodes, every point moves up ad down sinusoidally, but the pattern of motion stays fixed in space. This is the fundamental characteristic o what we call a mode. If one can find a pattern of motion which has the property that at any point the object moves perfectly sinusoidally, and that all points move at the same frequency (though some will move more that others), then we have what is called a mode.

1.3.1.3.2 Confined waves, with natural frequencies

The next interesting problem i to consider what happens if the string is held at both ends, say at $x = 0$ and $x = L$. We can begin with the idea of the reflection of waves, starting with some kind of a bump moving in one direction. As tie goes on. We would expect the bump to get near one end, and as time goes still further it will become a kid of little wobble, because it is combining with the reversed image bump which is coming from the other side. Finally the original bump will disappear and the image bump will move in the other direction to repeat the process at the other end. This problem has an easy solution, but an interesting question is whether we can have a sinusoidal motion (the solution just described is periodic, but of course it is not sinusoidally periodic). Let us try to put a sinusoidally periodic wave on string. If the string is tied at one end, it has to look the same at the other end. So the only possibility for periodic sinusoidal motion is that the sine wave must neatly fit into the string length. If it does not fit into the string length, then it is not a natural frequency at which the string can continue to oscillate. In short, if the string is started with a sine wave shape that just in, then it will continue to keep that perfect shape of a sine wave and will oscillate harmonically at some frequency.

Mathematically, we can write $\sin(kx)$ for the shape, where k is equal to the factor (ω/c) in Eq. (1.5) and (1.6), and this function will be zero at $x = 0$. However, It must also be zero at the other end. The significance of this is that k is no longer arbitrary, as was the case for the half.open string.

With the string closed at both ends, the only possibility is than $\sin(kL) = 0$, because this is the only condition that will keep both ends fixed. Now in order or a sine to be zero, the angle must be either $0, \pi, 2\pi$ or some other integral multiple of pi. The equation

$$kL = n\pi, \tag{1.7}$$

will, therefore, give any one of the possible k's depending on what integer is put in. For each of the k's there is a certain frequency ω, which, acoording to Eq. (1.5) is simply

$$\omega = kc = n\pi c/L. \tag{1.8}$$

So we have found the following: that a string has a property that it can have sinusoidal motions, but only at certain frequencies. This is the most important characteristic of confined waves. No matter how complicated the system is, it always turns out that there are some patterns of motions which have a perfect sinusoidal time dependence, but with frequencies that are a property of the particular system and the nature of its boundaries. In the case of the string we have many different possible frequencies, each one, by definition, corresponding to a mode, because a mode is a pattern of motion which repeats itself sinusoidally.

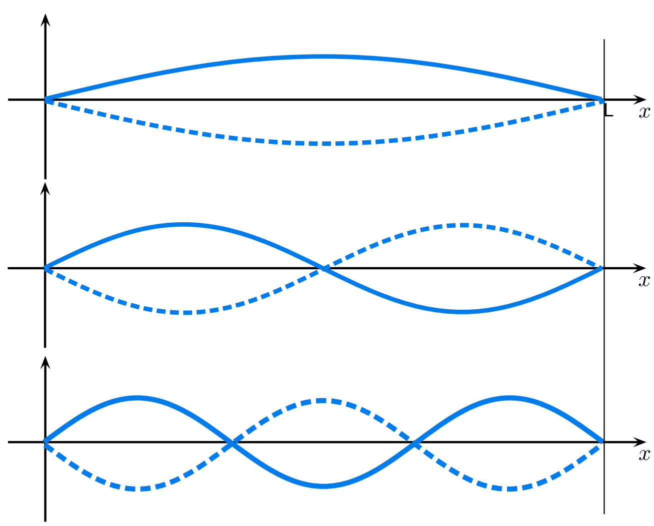


Figure 1.3: The first three modes of a vibrating string.

Fig. 1.3 show the first three modes for a string. For the first mode the wavelength λ is $2L$. This can be seen if one continues the wave out to $x = 2L$ to obtain one complete cycle of the sine wave. The angular frequency ω is $2\pi c$ divided by the wavelength, in general and in this case, since λ is $2L$, the frequency is $\pi c/L$ which is in agreement with Eq. (1.8) with $n = 1$. Let us call the first mode frequency ω_1. Now the next mode show two loops with one mode in the middle. For this mode the wavelength, then, is simply L. The corresponding value of k is twice as great and the frequency is twice as large; it is $2\omega_1$. For the third mode it is $3\omega_1$, and so on. So all the different frequencies of the string are multiples, $1,2,3,4$, and so on, of the lowest frequency w_1.

Returning now to the general motion of the string, it turns out that any possible motion can always be analyzed by asserting that more than one mode is operating at the same time. In fact, for general motion an infinite number of modes must be excited at the same time. To get some idea of this, let us illustrate what happens when there are two modes oscillating at the same time: Suppose that we have the first mode oscillating as shown by the sequence of pictures in Fig. 1.4, which illustrates the deflection of the string for equally spaced time intervals extending through half a cycle of the lowest frequency.

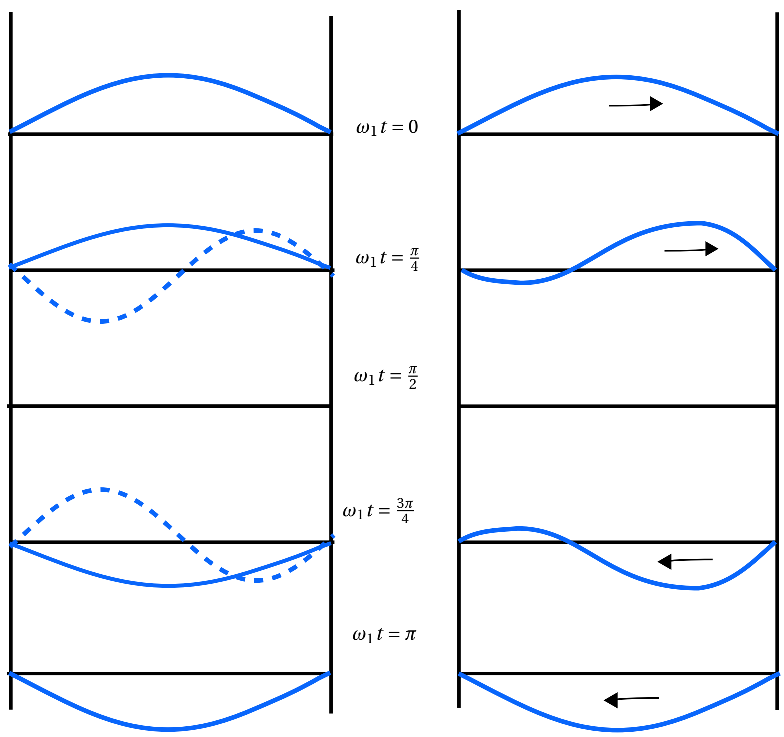


Figure 1.4: Travelling wave.

Now, at the same time, we suppose that there is an oscillation of the second mode also. Fig. 1.4 also shows a sequence of pictures of this mode, which at the start is 90° out of phase with the first mode. This means that at the star it has no displacement, but the two halves of the string have oppositely directed velocities. Now we recall a general principle relating to linear systems: if there are any two solutions, then their sum is also a solution. Therefore a third possible motion of the string would be a displacement obtained by adding the two solutions shown in Fig. 1.4. The result, also shown in the figure, begins to suggest the idea of a bump running back and forth between the ends of the string, although with only two modes we cannot make a very good picture of it; more odes are needed. This result is, in fact, a special case of a great principle for linear systems:

Any motion at all can be analyzed by assuming that it is the sum of the motions of all the different modes, combined with appropriate amplitudes and phases.

The importance of the principle derives from the fact that each mode is very simple, it is nothing but a sinusoidal motion in time. It is true that even the general motion of a string is not really very complicated, but there are other systems, for example the whipping of an airplane wing, in which the motion is much more complicated. Nevertheless, even with an airplane, we find there is a certain particular way of twisting which has one frequency and other ways f twisting that have other frequencies. If these modes can be found, then the complete motion can always be analyzed as a superposition of harmonic oscillations (except when the whipping is of such degree that the system can no longer be considered as linear).

2.2.1 Maxwell equations

Light is a electromagnetic wave and the equations governing the space-temporal dynamics of these waves are the Maxwell equations in the most general case and in International System of Units (SI) [3] are

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}. \end{aligned} \tag{2.1}$$

Where the four vector quantities **D**, **B**, **H**, y **E** depend of space and time, ie $\mathbf{D}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$, $\mathbf{H}(\mathbf{r}, t)$, and $\mathbf{E}(\mathbf{r}, t)$.

ρ is the density of charges existing in space, **J** is the current density, it measures the flow of charges per unit time and surface, and it is equal to $\mathbf{J} = \rho\mathbf{v}$; **D** is the electric field that suppresses the electrical effects of matter, and **E** is the electric field in space; **B** is the magnetic field or magnetic induction, finally **H** is the magnetic excitation.

The Fields **D** y **E**, as well as **B** y **H** are related through the following constitutive equations

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{2.2}$$

$$\mathbf{B} = \mu \mathbf{H}, \tag{2.3}$$

where ε is the electrical permittivity, y μ is the magnetic permeability.

Therefore, if these amounts (ε y μ) are constant, Maxwell's equations in terms of the fields **E** and **B** can be written as

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon},\\
\nabla \cdot \mathbf{B} &= 0,\\
\nabla \times \mathbf{B} &= \mu \left(\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \right),\\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}.
\end{aligned} \tag{2.4}$$

2.2.2 Wave equations derived from Maxwell's equations for a medium free of charges and currents

Now we see from Eqs. (2.4) are involved wave equations for the fields **E** and **B**.

To determine the wave equation for the field **E** apply the curl operator to the last Maxwell equation [Eqs. (2.4)], i.e.

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t}, \tag{2.5}$$

on the right side we have exchanged the temporal and spatial derivatives. Using vector properties on the left side and substituting the value of $\nabla \times \mathbf{B}$ of the third Maxwell equation [Ecs. (2.4)] on the right, and considering that medium is free of charges ($\rho = 0$) and currents ($\mathbf{J} = 0$) we have

$$-\nabla^2 \mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) = -\frac{\partial}{\partial t} \left(\mu \epsilon \frac{\partial \mathbf{E}}{\partial t} \right), \tag{2.6}$$

in this case ($\rho = 0$) the divergence of the field **E** is zero, so

$$\nabla^2 \mathbf{E} = \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{2.7}$$

and as $\mu\epsilon = 1/v^2$, where v is the phase velocity of the wave, we obtain the wave equation for **E**.

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \leftarrow \text{Wave equation for } \mathbf{E}. \tag{2.8}$$

When the wave propagates in a vacuum must be $v = c$ where c is the speed of light.

Now to find the equation of wave field **B** we apply the curl operator to the third Maxwell equation [Eqs. (2.4)], considering that before the medium is free of charges and currents ($\rho = 0$, $\mathbf{J} = 0$), we have

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \times \left(\mu\epsilon \frac{\partial \mathbf{E}}{\partial t}\right), \tag{2.9}$$

i.e.

$$-\nabla^2 \mathbf{B} = \frac{1}{v^2}\frac{\partial}{\partial t}(\nabla \times \mathbf{E}), \tag{2.10}$$

replacing $\nabla \times \mathbf{E}$ of the fourth equation of Maxwell [Ecs. (2.4)], we have the wave equation for **B**

$$\nabla^2 \mathbf{B} - \frac{1}{v^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0. \leftarrow \text{ Wave equation for } \mathbf{B} \tag{2.11}$$

We have seen that the field **E** as **B** satisfy the same wave equation. In Optics it is working with the field **E** because it is several orders of magnitude greater than the magnetic field and when electromagnetic waves propagate materials, the field **E** is more efficient exerting forces on the electrons that in atoms inducing optical phenomena.

2.2.3 Helmholtz equation

The Eqs. (2.8) and (2.11) have the same structure, herefore we can denote with the symbol ψ any of the fields (**E** y **B**), and then write a generic way wave equation for both

$$\nabla^2 \psi - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0. \tag{2.12}$$

This equation is known as the vector wave equation of D'Alembert. There are several methods to solve such equations, we start separating the spatial variables of the temporary

$$\psi(\mathbf{r}, t) = U(\mathbf{r})\mathrm{T}(t), \tag{2.13}$$

Substituting this in Eq. (2.12)

$$\mathrm{T}(t)\nabla^2 U(\mathbf{r}) - U(\mathbf{r})\frac{1}{v^2}\frac{\partial^2 \mathrm{T}(t)}{\partial t^2} = 0, \tag{2.14}$$

implies

$$\frac{\nabla^2 U(\mathbf{r})}{U(\mathbf{r})} = \frac{1}{v^2 \mathrm{T}(t)} \frac{\partial^2 \mathrm{T}(t)}{\partial t^2}, \quad \forall \mathbf{r} \in \Re^3, t \in \Re. \tag{2.15}$$

we can see that in this equation the left-hand side depends only on the variable $\mathbf{r}$ and the right-hand side only depends of t, and the speed is constant, so each side of the equation must be equal to a constant, without loss of generality we let us denote the constant as $-k^2$, this way two independent equations are obtained

$$\mathrm{T}''(t) + (vk)^2 \mathrm{T}(t) = 0, \tag{2.16}$$

$$\nabla^2 U(\mathbf{r}) + k^2 U(\mathbf{r}) = 0, \tag{2.17}$$

the Eq. (2.17) is known as vector Helmholtz equation. To simplify Eq.(2.16) we do $\omega = vk$, so

$$\mathrm{T}''(t) + \omega^2 \mathrm{T}(t) = 0. \tag{2.18}$$

To this point, we only imposed the condition that the solution $\mathbf{U}$ is separable in space and time and thus we come to two equations, one that is a partial differential equation known as the vector Helmholtz equation and the other is a harmonic differential equation in the time.

2.2.4 Monochromatic waves

The spatial and temporal equations [Eq. (2.17) and Eq. (2.18) respectively] depend parametrically on the separation constant (k^2 or ω^2). The condition k^2 (ω^2) no negative and we have the solution from Eq. (2.18)

$$\mathrm{T}(t) = \mathrm{T}_0\, \mathrm{e}^{-i\omega t}. \tag{2.19}$$

is oscillating in time, ω is precisely the amount determining the rate of these oscillations; T_0 is a constant to be determined by the conditions of the problem, the minus sign in the exponential is by convention, equally it could be positive. The solutions

$$\psi(\mathbf{r}, t) = U(\mathbf{r}) \mathrm{e}^{-i\omega t}. \tag{2.20}$$

are called monochromatic harmonic solutions, constant T_0 has been absorbed in the function $\psi(\mathbf{r})$.

Of course there are also solutions with $k^2 < 0$, but opposed to these previous [Eq. (2.20)], are evanescent both space and time. In this thesis we report are only important oscillatory solutions, hence the emphasis we have placed on them is clear. In the thesis that we report here only oscillatory solutions are important, hence the emphasis we have placed on them is clear.

Maxwell's equations to these monochromatic oscillatory solutions in a medium free of charges and currents take the following simplified form

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}) &= 0, \\
\nabla \cdot \mathbf{B}(\mathbf{r}) &= 0, \\
\nabla \times \mathbf{B}(\mathbf{r}) &= \frac{i\omega}{v^2}\mathbf{E}(\mathbf{r}), \\
\nabla \times \mathbf{E}(\mathbf{r}) &= -i\omega\mathbf{B}(\mathbf{r}).
\end{aligned} \tag{2.21}$$

Waves, in general, propagate energy. If light is a wave, light must carry energy. A fundamental property of lightś electric and magnetic fields is that they store energy (energy creates them). Using electromagnetic equations, we can find the energy stored in the electric and magnetic fields of electromagnetic radiation. Maxwell showed the following relationship between the maximum values of the field:

$$c = \frac{|\mathbf{E}_{max}|}{|\mathbf{B}_{max}|}. \tag{2.22}$$

Because light waves travel in a particular direction, the Poynting vector, an equation based on electricity and magnetism theory, provides the rate at which light waves transport energy to a unit area of surface. The Poynting vector shows the direction that the wave travels in based on the orientation of the electric and magnetic fields. It also tells you the rate at which energy is delivered per unit of surface area. Because the fields follow a sinusoidal pattern, talking about the intensity or irradiance, the average power per area delivered by the wave is often useful, especially if the fields wiggle very quickly (like a million-billion times in a second). You calculate the intensity from the following equation:

$$\mathrm{I} = \frac{1}{2}\frac{|\mathbf{E}_{max}|}{\mu c}, \tag{2.23}$$

where

I is the intensity of the electromagnetic wave. It has units of watts/square meter. **E** is the amplitude of the oscillating electric field. μc is the impedance of vacuum (a constant) and has a value of 377 ohms.

The intensity of the electromagnetic wave is a useful parameter for numerous applications, such as optical data transmission and laser machining.

3 Coordinate Systems

In the study of different kind of beams, it's important understand how changes the waist of beams like the Gaussian beams, it's for that we study the geometry of elliptic cylindrical coordinates, because this helps to represent the waist in the Gaussian beams as shown later. Also this coordinates will help us for connect the Mathie's Differential Equation with Weber's Differential Equation.

3.3.1 Polar coordinates

In $3-D$ we can have differents kinds of coordinate system[4]. The most important are orthogonal coordinate system, we study the cylindrical coordinates because they hep to describe the propagation of beams, in all they $z = z$, and the plane $x - y$ we have differents kind of coordinate system one of them is the Polar coordinates, where

Definition 3.1 Polar Coordinates

$$x = r\cos\theta, \tag{3.1}$$

$$y = r\sin\theta \tag{3.2}$$

where r is real positive, $\theta \in [0, 2\pi)$.

the coordinate system results in

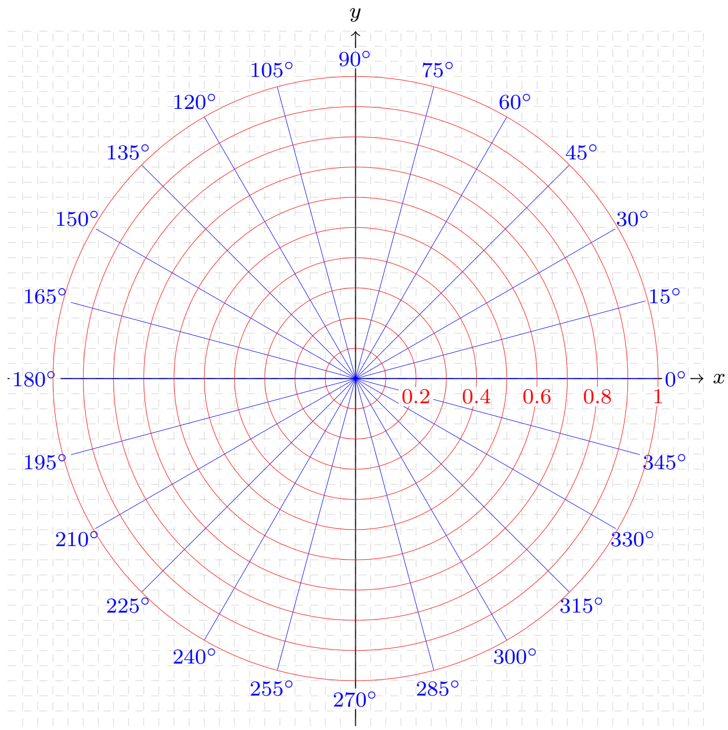


Figure 3.1: Polar coordinates.

we note that the curves constants for $r = \text{cte}_r$ and $\theta = \text{cte}_\theta$ this is when we solve for $\cos\theta$ and $\sin\theta$ in Eq. (3.1) and Eq. (3.2) respectively

$$\cos^2\theta + \sin^2\theta = \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1, \tag{3.3}$$

so we have

$$\boxed{x^2 + y^2 = r^2}. \tag{3.4}$$

this equation means that we have circles with different radius r, and this doesn't depend of θ.

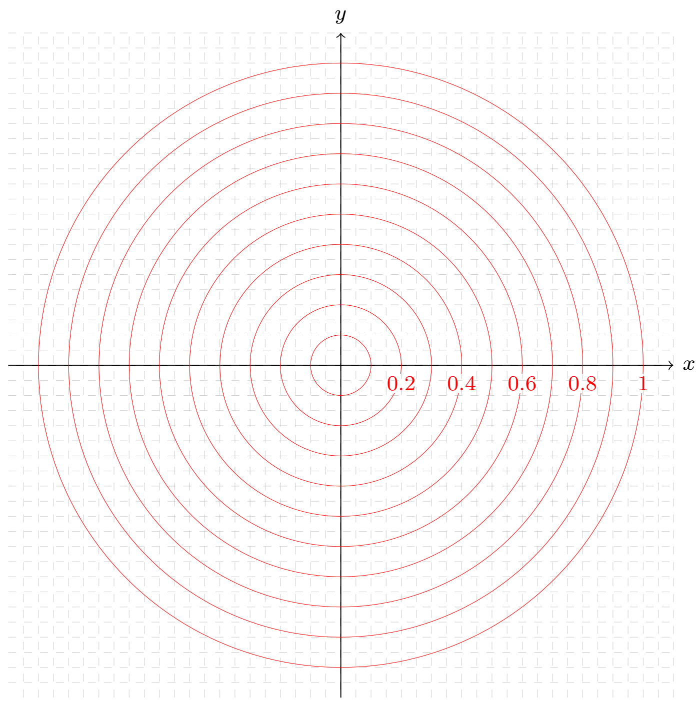


Figure 3.2: circles with radius r.

Now if we solve for r in Eq. (3.1) and Eq. (3.2) we have

$$r = \frac{x}{\cos\theta}, \tag{3.5}$$

and

$$r = \frac{y}{\sin\theta}, \tag{3.6}$$

we divide Eq. (3.5) from Eq. (3.6)

$$1 = \frac{\frac{x}{\cos\theta}}{\frac{y}{\sin\theta}} = \frac{x\sin\theta}{y\cos\theta}, \tag{3.7}$$

finally we have

$$\boxed{\tan\theta = \frac{y}{x},} \tag{3.8}$$

this equation means straight lines with different inclination $\tan\theta$ for each $\theta \in [-\pi/2, \pi/2)$, and this doesn't depend of r. Also the straight lines are from 0 to some r, **not negatives**, this is important because it helps us for understand the geometry of elliptic coordinates

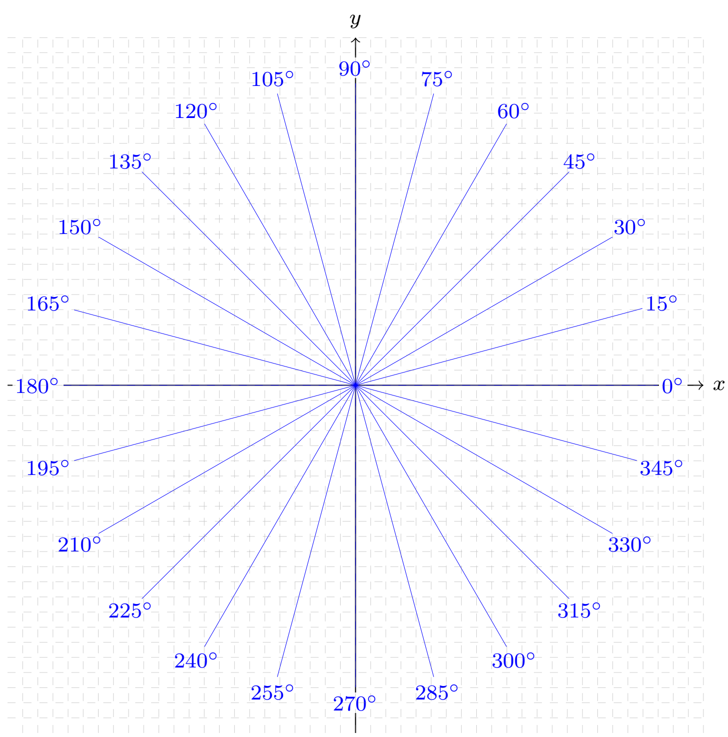


Figure 3.3: straight lines with slope $\tan\theta$.

Example 3.1

The point $(-\sqrt{3}, 1)$ in cartesian coordinates we want represent in polar coordinates, so using the Eq. (3.4) we have

$$r = \sqrt{\sqrt{3}^2 + 1^2} = 2, \tag{3.9}$$

we only take the plus sign in the square, because the distances are positives, and using Eq. (3.8) we have

$$\theta = \arctan\frac{1}{-\sqrt{3}} = -\arctan\frac{1}{\sqrt{3}} = -\frac{\pi}{6} = -30^\circ \tag{3.10}$$

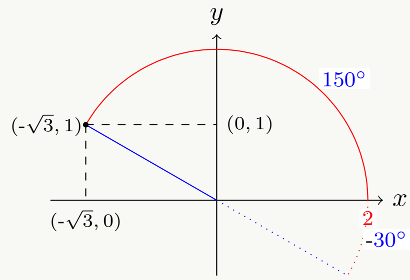


Figure 3.4: Example of change of Cartesian coordinates to Polar coordinates.

We note that the angle we need is 150° not −30° is because the function arctan its domain is $[-\pi/2, \pi/2]$, this for that we need the next function for θ

$$\theta = \operatorname{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases} \tag{3.11}$$

with this function we have $\theta = 150^\circ = 5\pi/6$, this is the angle that we need, so in polar coordinates our point is $(2, 5\pi/6)$.

It is important to know the inverse transformation, because we need know the new coordinates in terms of old coordinates.

3.3.2 Elliptic Coordinates

Now we study the elliptic coordinates [4] where the transformation is

Definition 3.2 Elliptic Coordinates

$$x = s\cosh\xi\cos\eta, \tag{3.12}$$

$$y = s\sinh\xi\sin\eta, \tag{3.13}$$

with ξ is a nonnegative real number and $\eta \in [0, 2\pi)$.

It is important to understand the Eq. (3.12) and Eq. (3.13) and their geometry, as we will see later.

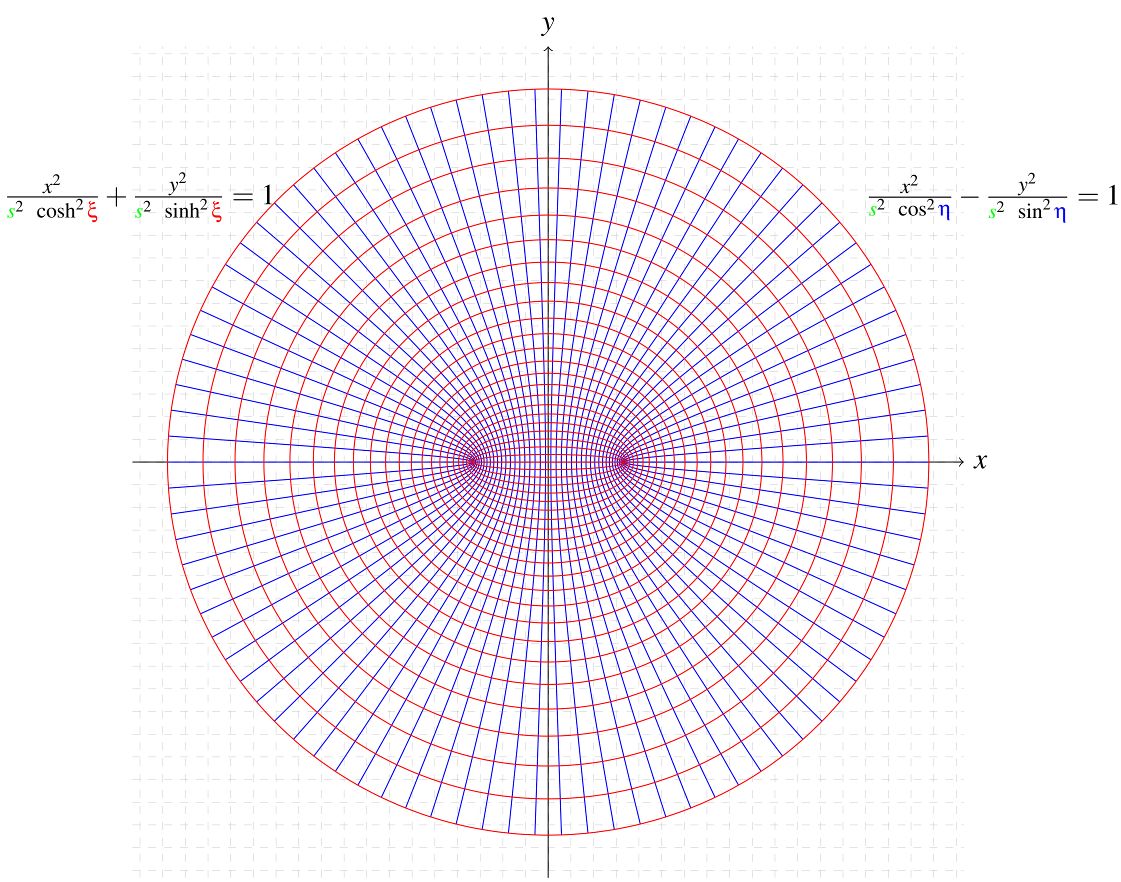


Figure 3.5: Elliptic cylindrical coordinates.

The most important in the elliptic coordinate system is to define a focus s and set ourselves as change hyperbolas and ellipses on this focus.

3.2.3.2.1 Ellipses

First we note that in Eq. (3.12) and Eq. (3.13) if we solve for $\cos\eta$ and $\sin\eta$ respectively, and for the trigonometric identity we have

$$\cos^2\eta + \sin^2\eta = \frac{x^2}{s^2\cosh^2\xi} + \frac{y^2}{s^2\sinh^2\xi} = 1, \tag{3.14}$$

we note that in the case of polar coordinates when we solve for the angles, we can factorize the denominators, but in this case we can't , so instead of circles with radius r, thus, the family of curves characterized by the parameters ξ = constant are ellipses having their centers at the origin. In addition, since $\xi \geq 0$ then $\cosh\xi \geq 0$, $\sinh\xi \geq 0$, and

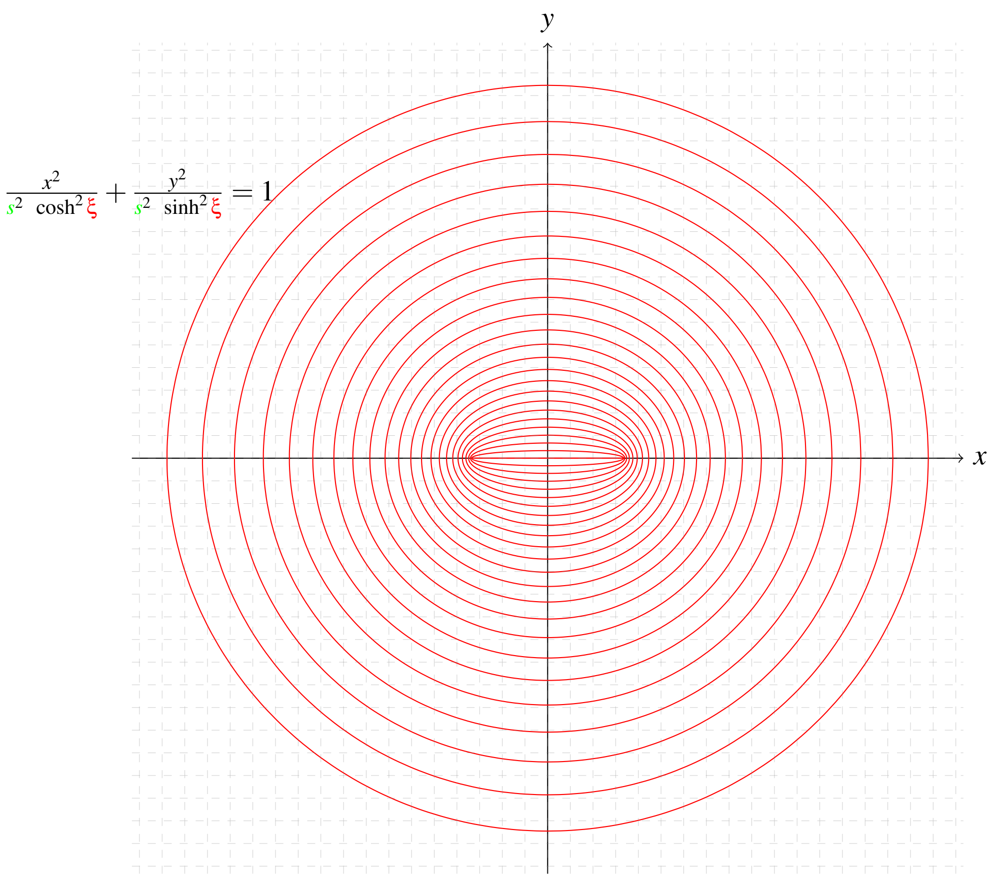


Figure 3.6: Ellipses.

with semimajor axis

$$\boxed{a_e = s\cosh\xi,} \tag{3.15}$$

and semiminor axis

$$\boxed{b_e = s\sinh\xi,} \tag{3.16}$$

both semimajor and semiminor axis dependent of s and ξ, i.e. $a_e = a_e(s,\xi)$ and $b_e = b_e(s,\xi)$. For geometry the distance from each focus to the center is

$$f_e^2 = a_e^2 - b_e^2, \tag{3.17}$$

for this case

$$\boxed{f_e = s,} \tag{3.18}$$

this is for all η, from which it follows that the family of ellipses are confocal; that is, every ellipse of the family has the same foci. The two foci are on the x axis at the points $(x = \pm s, y = 0)$.

we can do next

$$a_e + b_e = s\mathrm{e}^{\xi}, \tag{3.19}$$

and

$$a_e - b_e = s\mathrm{e}^{-\xi}, \tag{3.20}$$

then

$$\frac{a_e + b_e}{a_e - b_e} = \frac{\mathrm{e}^{\xi}}{\mathrm{e}^{-\xi}}, \tag{3.21}$$

therefore

$$\boxed{\xi = \frac{1}{2}\ln\left(\frac{a_e + b_e}{a_e - b_e}\right),} \tag{3.22}$$

this equation expresses the parameter ξ in terms of the lengths of the semiaxes, i.e. $\xi = \xi(a_e, b_e)$. We have different semimajor and semiminor axis for each ellipse, for that it has different eccentricity e_e which is defined as

$$e_e = \sqrt{1 - \frac{b_e^2}{a_e^2}}, \tag{3.23}$$

this is in terms of a_e, b_e, i.e. $e_e = e_e(a_e, b_e)$, if we replace the values of semiaxes

$$e_e = \sqrt{1 - \frac{\sinh^2 \xi}{\cosh^2 \xi}} = \sqrt{1 - \tanh^2 \xi} = \operatorname{sech}\xi. \tag{3.24}$$

then

$$\boxed{e_e = \operatorname{sech}\xi.} \tag{3.25}$$

So $e_e = e_e(\xi)$, we have many different ellipses, but all they have the same focus s, different semiaxis, and their eccentricity that dependence only of ξ.

e_e has values in the interval $[0, 1]$. Now we will have a only ellipse for each cases, for a focus given, first if $\xi \gg f_e$ we have the next ellipse

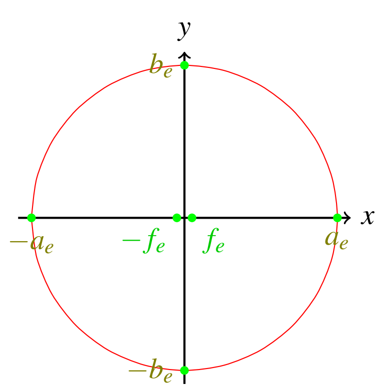


Figure 3.7: Ellipse small focus.

as ξ is large then in Eq. (3.25) we have $e_e \to 0$, this mean that the the ellipse becomes to a circle.

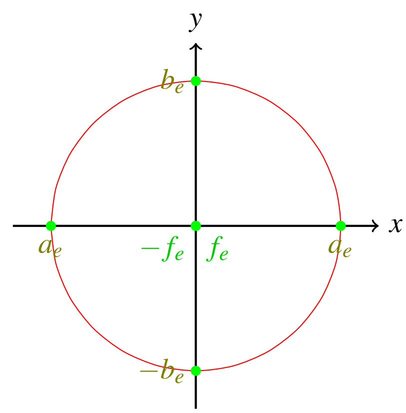


Figure 3.8: Ellipse become to a circle.

in this case $f_e = 0$, $a_e = b_e = r$, and the ellipse equation changes to

$$x^2 + y^2 = r^2, \tag{3.26}$$

circle equation, this case we studied it in the section Polar coordinates.

Now if $\xi \ll f_e$ we have the next ellipse

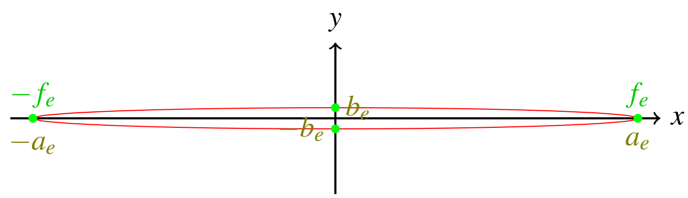


Figure 3.9: Ellipse with ξ small.

as $\xi \to 0$ then in Eq. 3.25 we have $e_e \to 1$, what happen if $\xi = 0$?

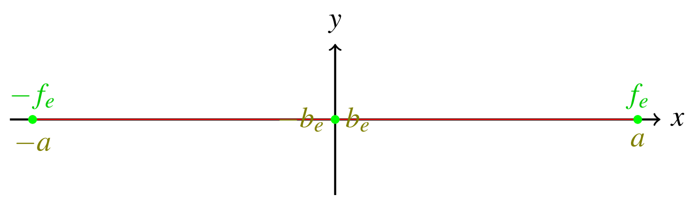


Figure 3.10: Ellipse with ξ equal to zero.

we note that the ellipse collapse in a finite straight line, the semiminor axis b_e go to zero and thus has its eccentricity go to one. The result is a line segment (degenerate because the ellipse is not differentiable at the endpoints) with its foci at the endpoints, and as $\cosh 0 = 1$, then

$$a_e = s, \tag{3.27}$$

this mean the samimajor axis a_e is in the focus f_e.

Another important thing is the inverse transformation for ξ, i.e. $\xi(x, y)$ this we can do with the definition for a ellipse with semimajor axis in x-axis and focus s

$$\sqrt{(x+s)^2+y^2}+\sqrt{(x-s)^2+y^2}=2a_e \tag{3.28}$$

we know a_e, then

$$\sqrt{(x+s)^2+y^2}+\sqrt{(x-s)^2+y^2}=2\cosh\xi \tag{3.29}$$

therefore

$$\boxed{\xi=\cosh^{-1}\left(\frac{\sqrt{(x+s)^2+y^2}+\sqrt{(x-s)^2+y^2}}{2}\right),} \tag{3.30}$$

we have ξ in terms of x, y for a focus given, i.e. $\xi = \xi(x, y)$.

3.2.3.2.2 Hyperbolas

For another hand if in Eq. (3.12) and Eq. (3.13) solve for $\cosh\xi$, and $\sinh\xi$ and with trigonometry identities we have

$$\cosh^2\xi-\sinh^2\xi=\frac{x^2}{s^2\cos^2\eta}-\frac{y^2}{s^2\sin^2\eta}=1, \tag{3.31}$$

in the case of the polar coordinates we can match the radius, but in this case we can't, so instead of straight lines we have parts of hyperbolas, $0<\eta<2\pi$, then $\cos(\eta)$ and $\sin(\eta)$ have values in the interval $[-1, 1]$, therefore $\cos^2(\eta)$ and $\sin^2(\eta)$ have values in the interval $[0, 1]$

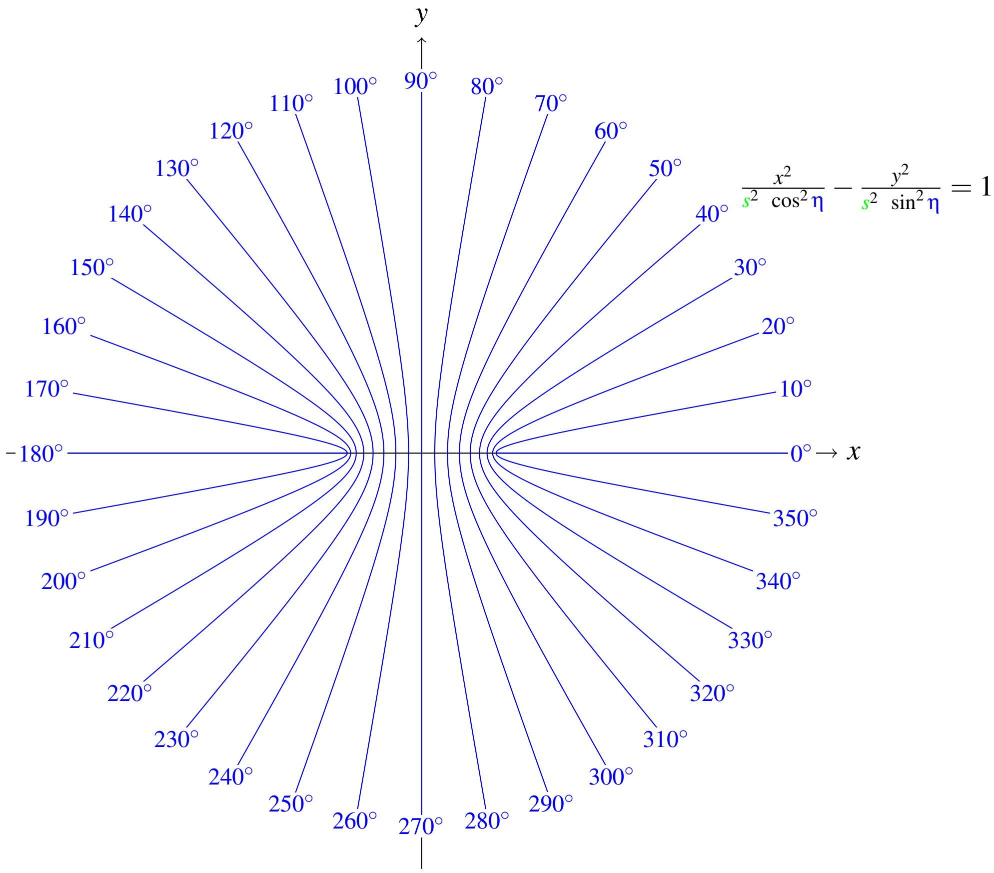


Figure 3.11: Hyperbolas.

Now the semimajor axis are

$$\boxed{a_h = s \left|\cos \eta\right|,} \tag{3.32}$$

and semiminor axis

$$\boxed{b_h = s \left|\sin \eta\right|,} \tag{3.33}$$

we note that for every angle we have a part of hyperbola for each quadrant, for example if $\eta = \pi/4 = 45°$

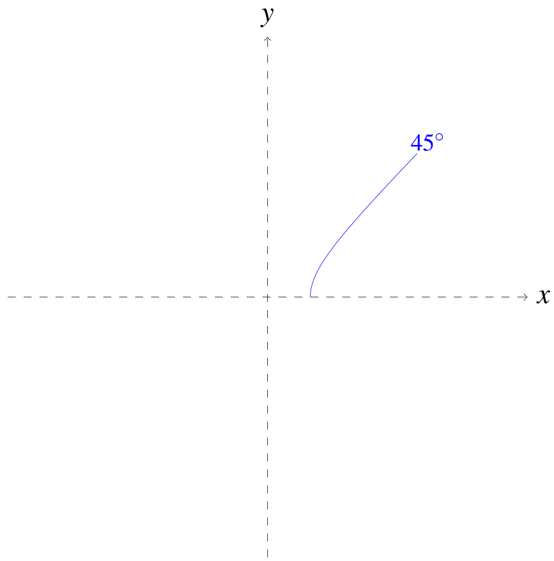


Figure 3.12: Hyperbola 45.

if we want the complete hyperbola we have the angles $\pi - \eta$, $\pi + \eta$, $2\pi - \eta$, thus

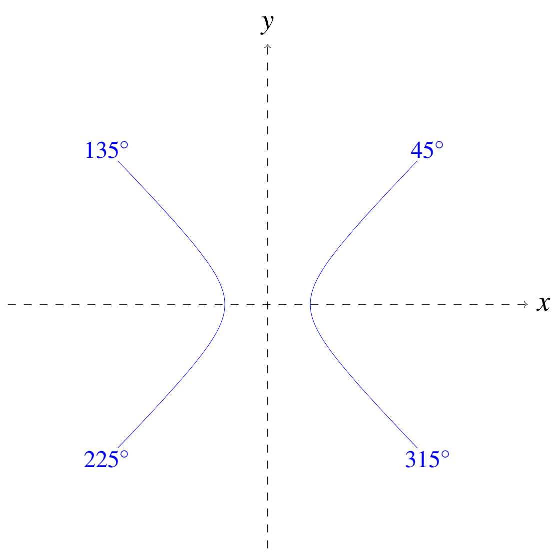


Figure 3.13: Complete hyperbola.

this is because

$$a_h = s \left|\cos\eta\right| = s \left|-\cos\eta\right| = s \left|\cos(\pi - \eta)\right|, \tag{3.34}$$

$$a_h = s \left|\cos\eta\right| = s \left|\cos(-\eta)\right| = s \left|-\cos(-\eta)\right| = s \left|\cos(\pi+\eta)\right|, \tag{3.35}$$

and

$$a_h = s \left|\cos\eta\right| = s \left|\cos(-\eta)\right| = s \left|\cos(2\pi-\eta)\right|. \tag{3.36}$$

All they belong to the same hyperbola, but every angle is different for represent each part of the hyperbola, It is analogous for b_h.

It knows the semi-focal length (the distance from a focus to the hyperbola) is

$$f_h^2 = a_h^2 + b_h^2, \tag{3.37}$$

for this case

$$\boxed{f_h = s,} \tag{3.38}$$

for all ν, in all hyperbolas must satisfies that $a_h \leqslant f_h$, in this case

$$s\left|\cos\eta\right| \leqslant s, \tag{3.39}$$

then

$$\left|\cos\eta\right| \leqslant 1, \tag{3.40}$$

this is always true.

what happen if we take the focus as 1/10 of the previous focus?

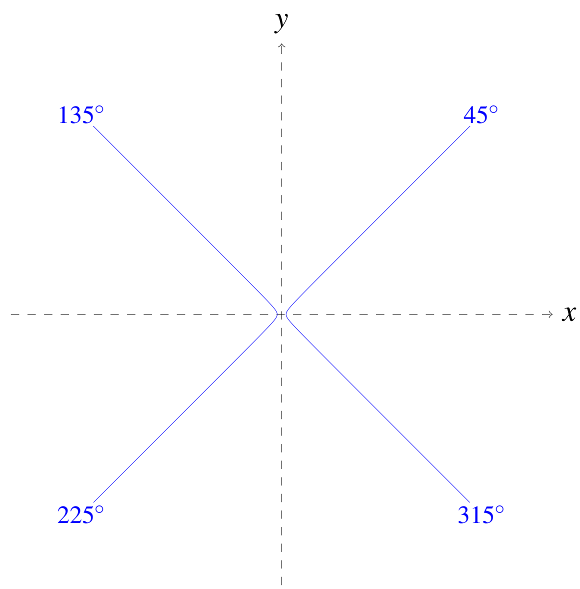


Figure 3.14: Hyperbola with small focus.

this is like we will do zoom out in the graph Fig. 3.13 for a factor of $10x$, we note the branches of the hyperbola tend to straight lines, it is why when we are away from the focus we observe straight lines like the polar coordinates, the case limit is when the focus is zero

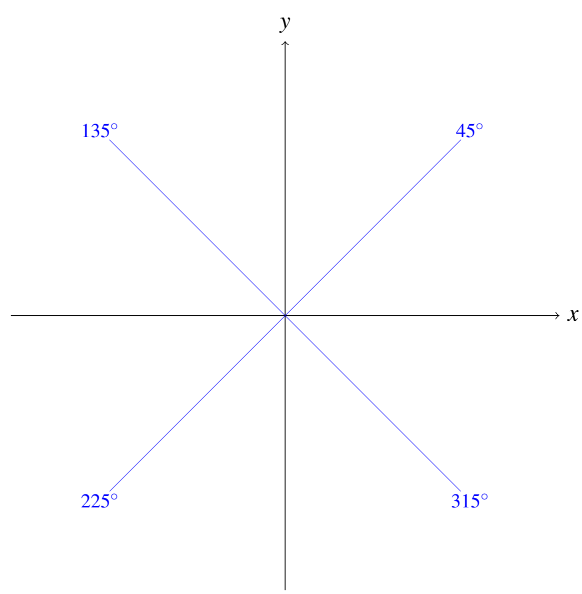


Figure 3.15: Hyperbola becomes to a straight line.

in this case we have straight lines like in the Polar coordinates. we remeber that Eq. (3.8) we have straight lines, but that equation is the same for the point (x, y) and $(-x, -y)$ because in the quotient they have the same sign, the same way for $(x, -y)$ and $(-x, y)$, all they represent different point in the space, but in some cases they have the same angle, like the example that we gave, therefore in the case of elliptic coordinates we have different angles, that represent different points in the space, but that are in one hyperbola.

We have different hyperbolas, because $\eta \in [0, 2\pi)$, then we have different semimajor and semiminor axis, the different is in the eccentricity e_h, then defined as

$$e_h = \frac{f_h}{a_h} \tag{3.41}$$

so in this case

$$e_h = \frac{s}{s\cos\eta} = \frac{1}{\cos\eta}. \tag{3.42}$$

then

$$\boxed{e_h = \frac{1}{\cos\eta}.} \tag{3.43}$$

The eccentricity only dependence of η, i.e. $e_h = e_h(\eta)$.

Now we will study a only hyperbola for each cases like in the case of ellipses, for a focus given, and we will study its geometry, for simplicity, we will have again $\eta = 45°$ in degree or in radians $\eta = \pi/4$ and taking the angles $\pi - \eta$, $\pi + \eta$ and $2\pi - \eta$, for a focus given, so we have the next hyperbola

Now we will study the geometry of the hyperbolas, well we have

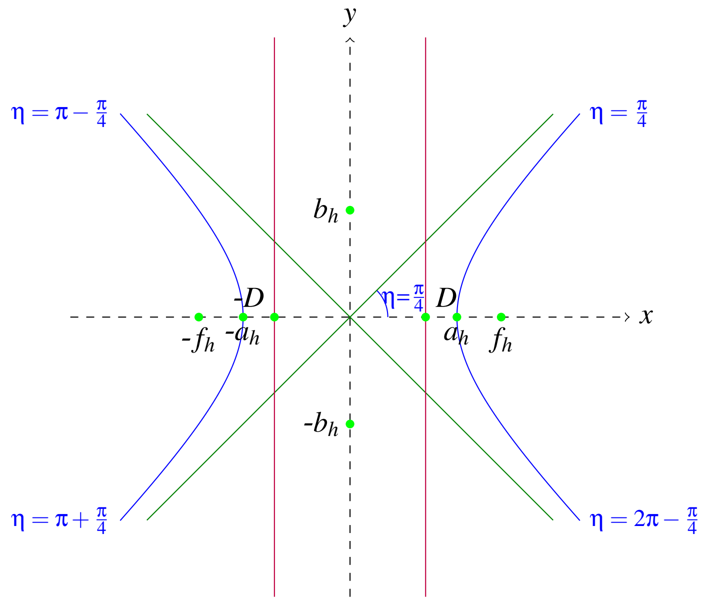


Figure 3.16: Hyperbola Geometry.

where a_h is the semimajor axis, b_h is the semiminor axis, s is the focus, and D is the directrix and

$$D = \frac{a_h}{e_h}, \tag{3.44}$$

then in this case

$$D = \frac{s\cos(\eta)}{1/\cos(\eta)}, \tag{3.45}$$

therefore

$$\boxed{D = s\cos^2(\eta),} \tag{3.46}$$

D depends of the parameters s and η, i.e. $D = D(s,\eta)$.

asymptotes are

$$y = \pm\frac{a_h}{b_h}x = \pm\tan\eta x \tag{3.47}$$

then

$$\boxed{y = \pm \tan \eta x} \tag{3.48}$$

We note that the asymptote gives the angle η.

With this we can have the parameter η in terms of semiaxis a_h, b_h

$$\boxed{\eta = \tan^{-1} \frac{a_h}{b_h},} \tag{3.49}$$

Now if we have small angle as $\eta = 10°$ in degree or in radians $\eta = \pi/18$ we have the next hyperbola

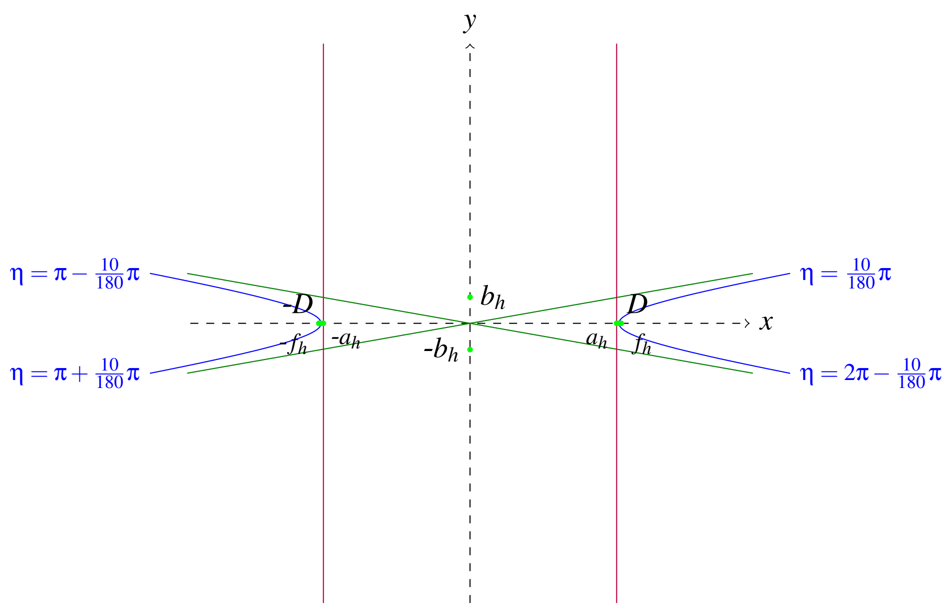


Figure 3.17: Hyperbola with small angle η.

We note that the hyperbolas are more closer to x-axis and the eccentricity is larger because

$$e_h = \frac{1}{\cos \eta} \to 1 \tag{3.50}$$

is large, semimajor axis

$$a_h = s \left|\cos \eta\right| \to s, \tag{3.51}$$

semiminor axis

$$b_h = s\,\left|\sin\eta\right| \to 0. \qquad (3.52)$$

and

$$D \to s. \qquad (3.53)$$

so if we have $\eta = 0$, we have

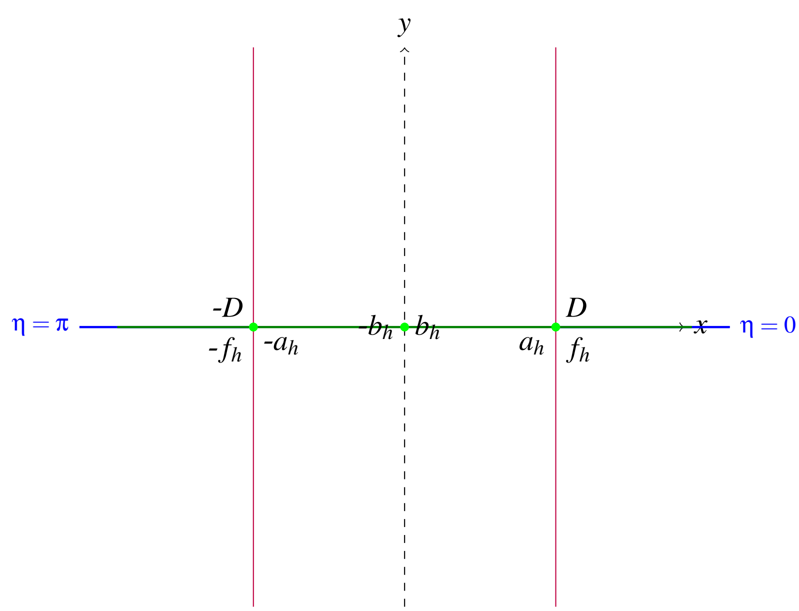


Figure 3.18: Hyperbola with angle $\eta = 0, \pi$.

in this case the hyperbola becomes two straight lines, first have values from minus infinity to -a_h, and the second have values from a_h to infinity, also the each asymptote is in x-axis.

Now we will have to $\eta \to \pi/2$, for example $\eta = 80^\circ$ in degrees or $\eta = \frac{4}{9}\pi$

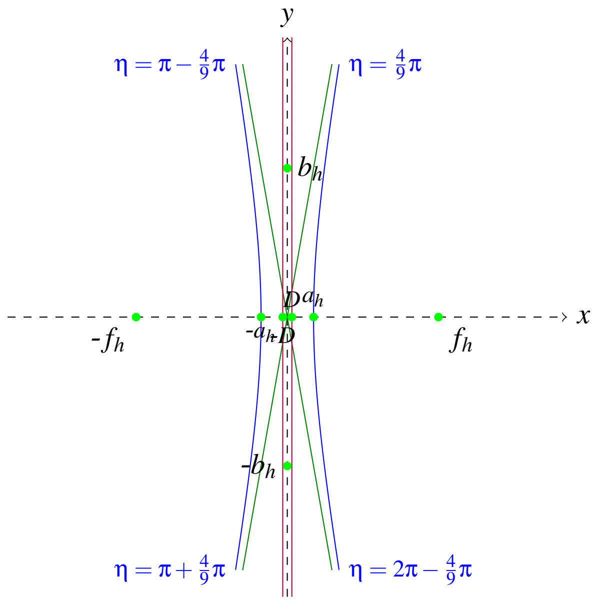


Figure 3.19: Hyperbola with angle $\eta \to \pi/2$.

and directrix is closer to origin of axis, if we do zoom in to the origin we can observe better

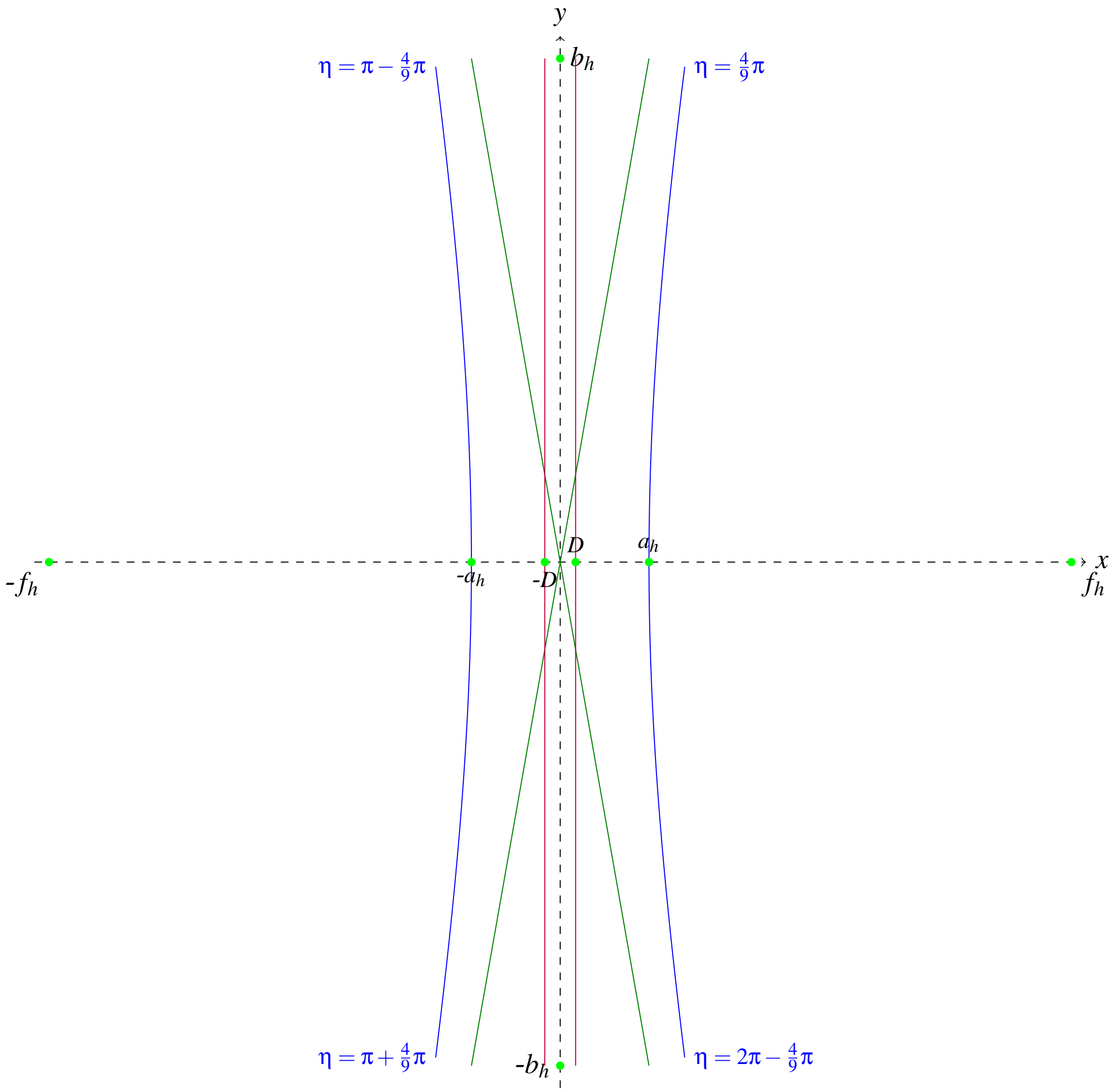


Figure 3.20: Hyperbola with angle $\eta \to \pi/2$ (Zoom in).

we note in the origin the hyperbolas tend to straight lines, and the asymptotes are closer to y-axis, the semimajor axis

$$a_h \to s\cos(v) \to 0, \tag{3.54}$$

semiminor axis

$$b_h \to s\sin(v) \to s, \tag{3.55}$$

and the eccentricity

$$e = \frac{1}{\cos(v)} \to \infty. \tag{3.56}$$

this mean that hyperbola tends to straight lines y in $x = 0$, because $a \to 0$, so What happen if we do $\eta = \pi/2$

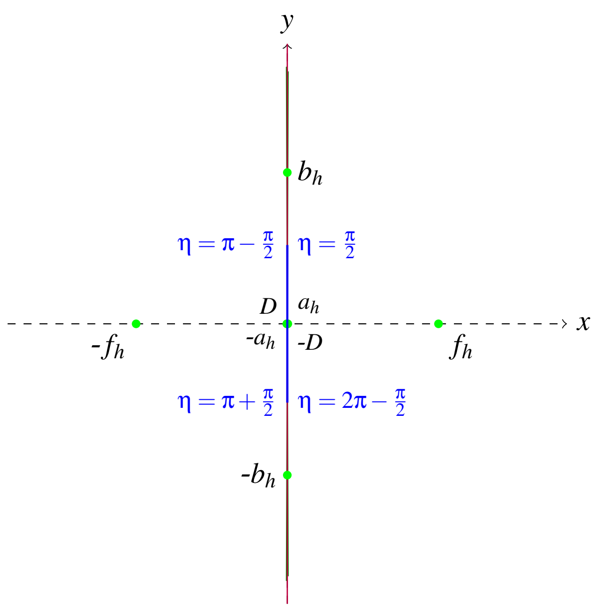


Figure 3.21: Hyperbola with angle $\eta = \pi/2$.

the hyperbola collapse to a straight line y in $x = 0$.

We need η in terms of x, y, i.e. $\eta(x, y)$, well by construction geometry of hyperbola we have

$$\sqrt{(x+s)^2 + y^2} - \sqrt{(x-s)^2 + y^2} = 2a_h \tag{3.57}$$

but we know a_h, then

$$\sqrt{(x+s)^2 + y^2} - \sqrt{(x-s)^2 + y^2} = 2\cos\eta, \tag{3.58}$$

therefore

$$\eta = \cos^{-1}\left(\frac{\sqrt{(x+s)^2+y^2}-\sqrt{(x-s)^2+y^2}}{2}\right), \tag{3.59}$$

so, we have $\eta(x, y)$ for a focus given.

Inverse or Elliptic Coordinates

in the section before we found the $\xi(x, y)$, and $\eta(x, y)$, they are the inverse of elliptic coordinates

$$\xi(x,y) = \cosh^{-1}\left(\frac{\sqrt{(x+s)^2+y^2}+\sqrt{(x-s)^2+y^2}}{2}\right), \tag{3.60}$$

$$\eta(x,y) = \cos^{-1}\left(\frac{\sqrt{(x+s)^2+y^2}-\sqrt{(x-s)^2+y^2}}{2}\right), \tag{3.61}$$

3.2.3.2.3 Ellipses with Hyperbolas

Now we can analyze the ellipses and hyperbolas together, for example when we have a small focus s, and we look at the circles that $\xi \gg s$

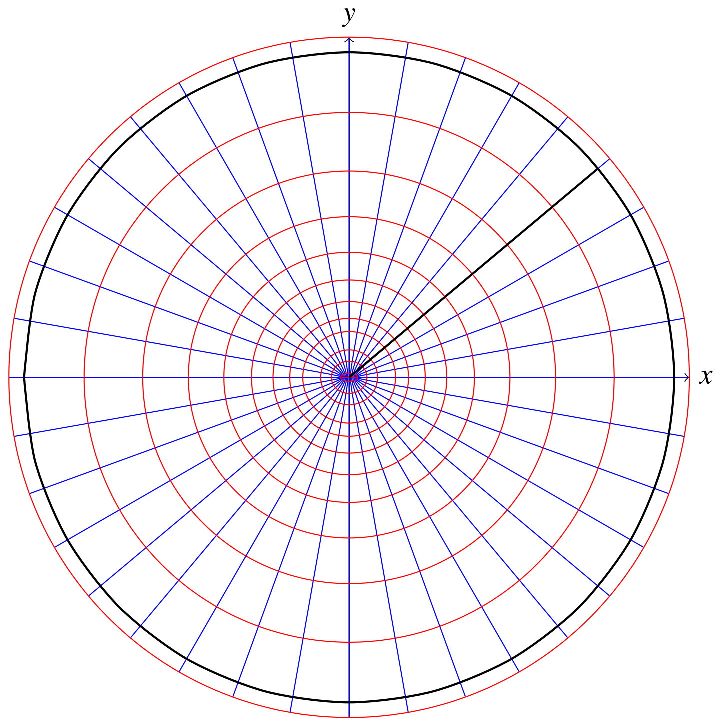


Figure 3.22: Elliptic Coordinates (small focus).

We note that in this case for the curves ξ = cte are like the circles and the curves η = cte are like the straight lines, for that in this region the elliptic coordinates approach to polar coordinates.

Now when we have a small focus s, and $\xi \ll s$

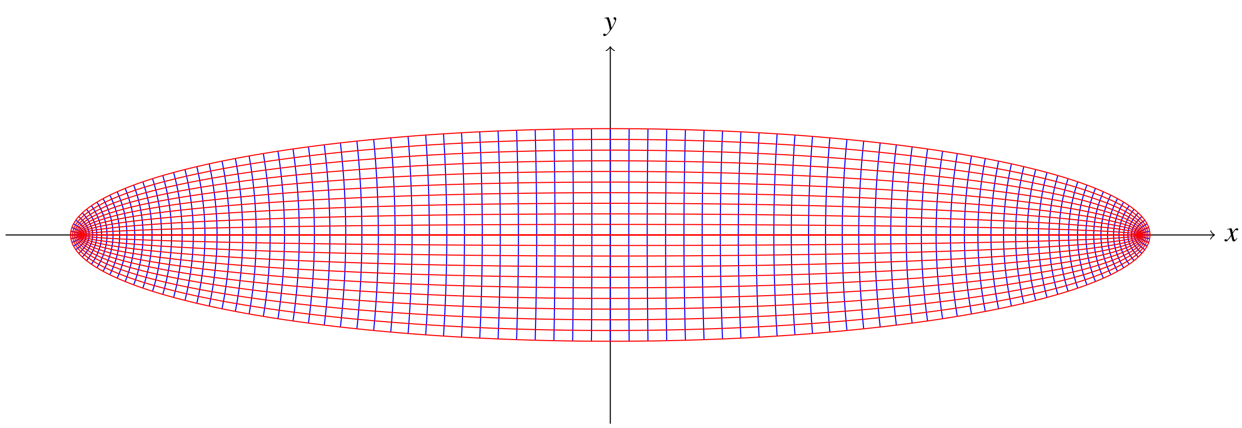


Figure 3.23: Elliptic Coordinates (big focus).

we observe that in this case we have ellipses, but near in the origin we have

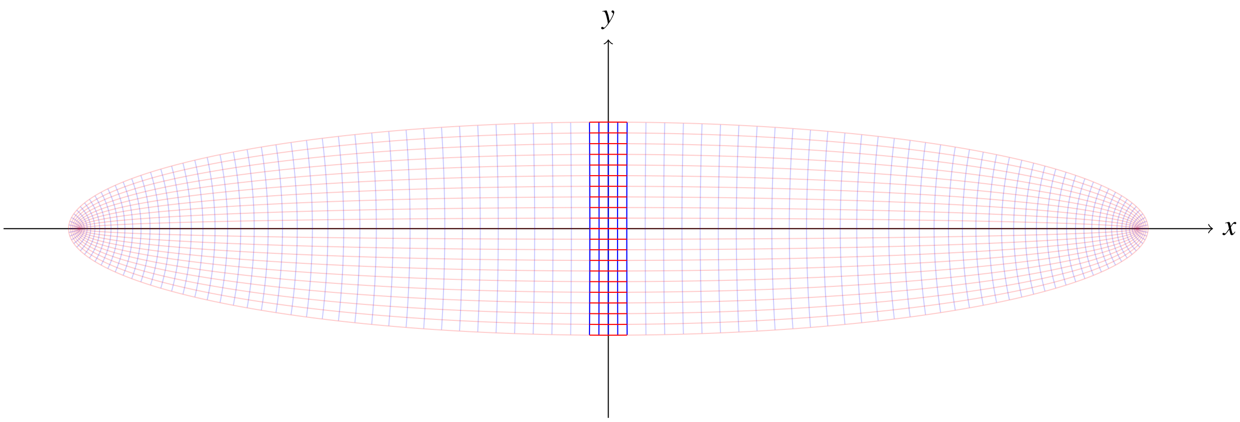


Figure 3.24: Elliptic Coordinates (big focus) near in origin.

Now we analyze when we are near of one of the focus

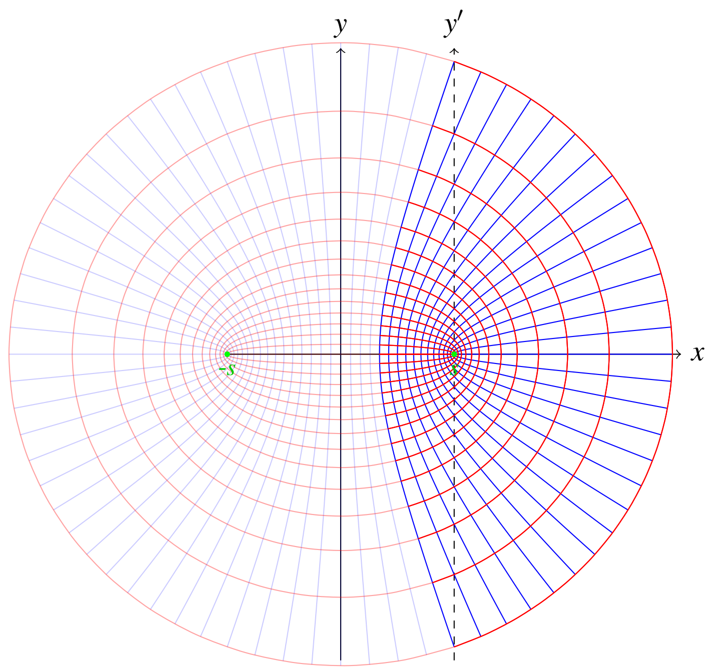


Figure 3.25: Elliptic Coordinates (around in the focus).

this looks like the parabolic coordinates, but only when we are very near in the focus, because if we remember when ξ is large, the eccentricity $e_e \to 0$ this mean that when we are far from the focus the ellipses look like a circles, and when we are near from the focus ξ is small, and $e_e \to 1$ in this case we see parabolas, because the eccentricity of the parabolas are 1. Now for the case η, when this is approximately $\pi/2$, the eccentricity $e_h \to \infty$, the hyperbola looks like a straight line, but when η is small $\eta \to 0$, the eccentricity $e_e \to 1$, i.e. the hyperbola looks likes the parabolas.

we can see better when we have a focus far from the origin (zoom out in the previous figure) and we have small angles for η and ξ is small, then

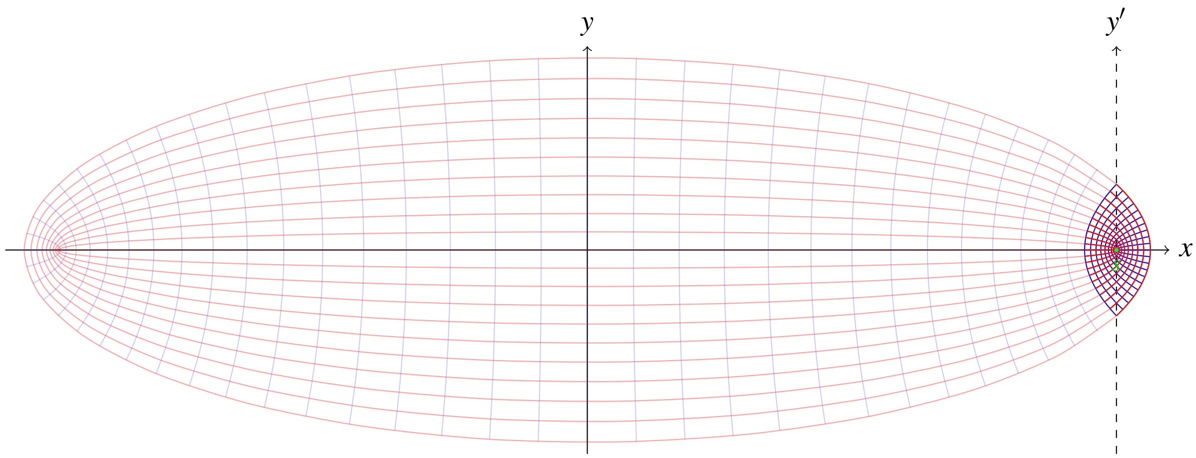


Figure 3.26: Focus far from origin axis x, y.

and we do zoom in around the focus

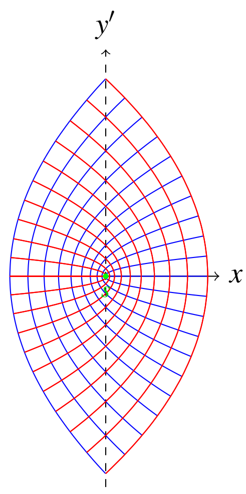


Figure 3.27: Zoom in around the focus.

In this figure we can observe that we have the parabolic coordinates[4]. The parabolic coordinates is

Definition 3.3 Parabolic Coordinates

$$x = c\left(u^2 - v^2\right), \tag{3.62}$$

$$y = 2cuv, \tag{3.63}$$

with $u \in (-\infty, \infty)$ and $u \in [0, \infty)$

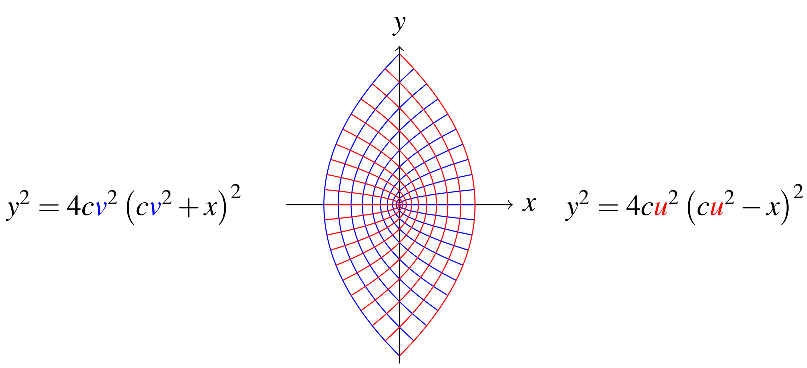


Figure 3.28: Parabolic coordinates.

Summary of Elliptic Coordinates

-	Ellipses	Hyperbolas
semiminor axis	$a_e = s\ \cosh\mu$	$a_h = s\ \lvert\cos v\rvert$
semimajor axis	$b_e = s\ \sinh\mu$	$b_h = s\ \lvert\sin v\rvert$
focus	$f_e = s = \sqrt{a_e^2 - b_e^2}$	$f_h = s = \sqrt{a_h^2 + b_h^2}$
parameters in terms of semiaxis	$\mu = \frac{1}{2}\ln\left(\frac{a_e+b_e}{a_e-b_e}\right)$	$v = \tan^{-1}\left(\frac{a_h}{b_h}\right)$
eccentricity	$e_e = \frac{s}{a_e} = \operatorname{sech}\mu$	$e_h = \frac{s}{a_h} = \frac{1}{\cos v}$
parameters in terms of x, y (inverse function)	$\mu = \cosh^{-1}\left(\frac{\sqrt{(x+s)^2+y^2}+\sqrt{(x-s)^2+y^2}}{2}\right)$	$v = \cos^{-1}\left(\frac{\sqrt{(x+s)^2+y^2}-\sqrt{(x-s)^2+y^2}}{2}\right)$
asympotates	not apply	$y = \pm\frac{b_h}{a_h}x = \pm\tan v x$
directrixes	$D_e = \frac{a_e^2}{s} = s\cosh^2\mu$	$D_h = \frac{a_h^2}{s} = s\cos^2 v$

Summary of Ellipses		
-	$\mu \ll s$	$\mu \gg s$
semiminor axis	$a_e \to f_e \to s$	$a_e \to b_e \to r$
semimajor axis	$b_e \to 0$	$b_e \to a_e \to r$
focus	$f_e \to a_e \to s$	$f_e \to 0$
eccentricity	$e_e \to 1$	$e_e \to 0$
directrixes	D_e	$D_h \to \infty$

Summary of Hyperbolas		
-	$\nu \to 0$	$\nu \to \pi/2$
semiminor axis	$a_h \to f_h \to s$	$a_h \to 0$
semimajor axis	$b_h \to 0$	$b_h \to f_h \to s$
focus	$f_h \to a_e \to s$	$f_h \to \infty$
eccentricity	$e_h \to 1$	$e_h \to 0$
asympotates	tends to x-axis	tends to y-axis
directrixes	$D_h \to f_h s$	$D_h \to 0$

Example 3.2

Now we represent a point $\left(-\frac{5}{6}\sqrt{2}, \frac{2}{3}\sqrt{2}\right)$ in cartesian coordinates to elliptic coordinates.

If we remember we need a specific focus, in this example we take focus $s = 1$

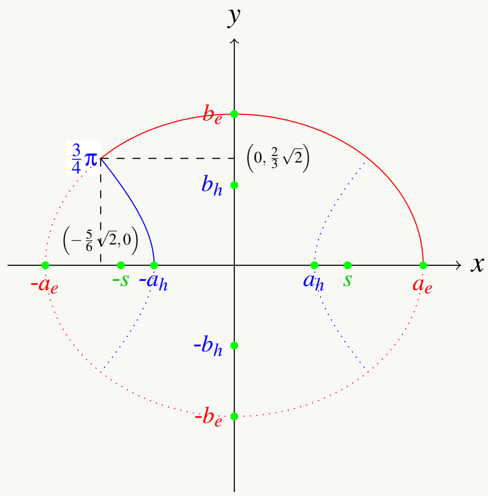


Figure 3.29: Cartesian Coordinates to Elliptic Coordinates.

We use the Eq. (3.60) and Eq. (3.59), for the first equation we have

$$\begin{aligned}
\xi &= \cosh^{-1}\left(\frac{\sqrt{\left(-\frac{5}{6}\sqrt{2}+1\right)^2+\left(\frac{2}{3}\sqrt{2}\right)^2}+\sqrt{\left(-\frac{5}{6}\sqrt{2}-1\right)^2+\left(\frac{2}{3}\sqrt{2}\right)^2}}{2}\right) && (3.64)\\
&= \cosh^{-1}\left(\frac{\sqrt{\frac{50}{36}+1-\frac{10\sqrt{2}}{6}+\frac{8}{9}}+\sqrt{\frac{50}{36}+1+\frac{10\sqrt{2}}{6}+\frac{8}{9}}}{2}\right) && (3.65)\\
&= \cosh^{-1}\left[\frac{1}{2}\left(\sqrt{\frac{59}{18}-\frac{5\sqrt{2}}{3}}+\sqrt{\frac{59}{18}+\frac{5\sqrt{2}}{3}}\right)\right] && (3.66)\\
&= \cosh^{-1}\left[\frac{1}{2}\left(\frac{10}{3}\right)\right] = \cosh^{-1}\left(\frac{10}{6}\right) && (3.67)
\end{aligned}$$

we can use the identity

$$\cosh^{-1}(z) = \ln\left(z+\sqrt{z^2-1}\right) \tag{3.68}$$

therefore

$$\xi = \cosh^{-1}\left(\frac{10}{6}\right) = \ln(3). \tag{3.69}$$

In the Eq. (3.59) we have

$$\eta = \cos^{-1}\left(\frac{\sqrt{(x+s)^2+y^2}-\sqrt{(x-s)^2+y^2}}{2}\right) \tag{3.70}$$

$$= \cos^{-1}\left(\frac{\sqrt{\frac{50}{36}+1-\frac{10\sqrt{2}}{6}+\frac{8}{9}}-\sqrt{\frac{50}{36}+1+\frac{10\sqrt{2}}{6}+\frac{8}{9}}}{2}\right) \tag{3.71}$$

$$= \cos^{-1}\left[\frac{1}{2}\left(\sqrt{\frac{59}{18}-\frac{5\sqrt{2}}{3}}-\sqrt{\frac{59}{18}+\frac{5\sqrt{2}}{3}}\right)\right] \tag{3.72}$$

$$= \cos^{-1}\left[\frac{1}{2}\left(-\sqrt{2}\right)\right] = \cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) \tag{3.73}$$

therefore

$$\eta = \frac{3}{4}\pi \tag{3.74}$$

so the point $\left(-\frac{5}{6}\sqrt{2}, \frac{2}{3}\sqrt{2}\right)$ in elliptic coordinates with focus $s = 1$ is $\left(\ln(3), \frac{3}{4}\pi\right)$

4.4.1 Gaussian Beams

Turning to physical observation for inspiration, the output of many lasers is a highly directional beam with Gaussian intensity profile, i.e.

$$I(x, y, z_0) \backsim I_0 e^{-\frac{x^2+y^2}{w^2}}, \tag{4.1}$$

where the beam is propagating in the z-direction and the effective width of the beam is denoted by w.

In almost of the literature the solution of E field such that the distance in the field decays as e^{-1} [5, 6, 7], but the intensity will be $I(x, y, z_0) \backsim I_0 \exp\left(-2(x^2 + y^2)/w^2\right)$, this is bad because the waist for this guassian function is $w/\sqrt{2}$, and we would want it to be w, because we measure w.

Furthermore, it has been observed that the shape of such Gaussian beams does not change as the beam propagates; only the width of the Gaussian and its brightness change. Additional Gaussian-like "shape-invariant" beams may be derived; in this section we show that a complete set of such beams can be represented in Cartesian coordinates using the Hermite–Gauss functions. We begin by deriving the propagation characteristics of Gaussian beams.

We are interested in finding solutions to the Helmholtz equation (Eq. (2.17)) which have a Gaussian intensity of the form of Eq. (4.1). Because a laser is known to produce a highly directional output, we restrict ourselves to solutions which look more like beams,

$$U(\mathbf{r}) = u(\mathbf{r}) \exp(ikz), \tag{4.2}$$

where the function $u(\mathbf{r})$ is assumed to be such that its variations in the z-direction are negligible over the distance of a wavelength, i.e.

$$\lambda \left|\frac{\partial u}{\partial z}\right| \ll |u|, \tag{4.3}$$

and

$$\lambda \left| \frac{\partial^2 u}{\partial z^2} \right| \ll \left| \frac{\partial u}{\partial z} \right|. \tag{4.4}$$

These assumptions basically enforce the requirement that the beam does not change its size and shape appreciably as it propagates in the z-direction, i.e. that it is directional. If we substitute from Eq. (4.2) into the Helmholtz equation (Eq. (2.17)), we may expand the z-derivative,

$$\frac{\partial^2}{\partial z^2} u e^{ikz} = \left(\frac{\partial^2}{\partial z^2} u + 2ik \frac{\partial}{\partial z} u - k^2 u \right) e^{ikz} \sim \left(2ik \frac{\partial}{\partial z} u - k^2 u \right) e^{ikz}, \tag{4.5}$$

where in the last step we have used our assumption of directionality. On substitution into the Helmholtz equation, that equation takes on the form

$$\boxed{\nabla_T^2 u + 2ik \frac{\partial}{\partial z} u = 0,} \tag{4.6}$$

where

$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{4.7}$$

is referred to as the transverse Laplacian. Equation (4.6) is known as the paraxial wave equation. We now try to construct a solution of the paraxial wave equation of Gaussian form whose shape is invariant on propagation, i.e.

$$u(r) = A_0 e^{ik(x^2+y^2)/2q(z)} e^{ip(z)}, \tag{4.8}$$

where $q(z)$ and $p(z)$ are z-dependent, possibly complex, functions to be determined. On substitution of this form into the paraxial equation, we find that

$$A_0 \left[\frac{k^2}{q^2} (x^2 + y^2) \left(\frac{dq}{dz} - 1 \right) - 2k \left(\frac{dp}{dx} - \frac{i}{q} \right) \right] = 0. \tag{4.9}$$

Since $p(z)$ and $q(z)$ only depend on z, this equation will only be satisfied if

$$\frac{dq}{dz} = 1, \tag{4.10}$$

and

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{i}{q}, \tag{4.11}$$

We can solve these equations quite readily, first integrating the q equation and then substituting this result into the $p(z)$ equation. The results are

$$\boxed{q(z) = z + q_0,} \tag{4.12}$$

$$\boxed{p(z) = i \ln\left(\frac{z+q_0}{q_0}\right),} \tag{4.13}$$

where $q_0 = q(0)$ and we have assumed that $p(0) = 0$. The quantity $q(z)$ is in general a complex number, and it is convenient to write it in the form

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{i\lambda}{\pi w^2(z)}, \tag{4.14}$$

where $R(z)$ and $w(z)$ are real. The latter term was chosen to match the intensity to the "observed" intensity profile, given by Eq. (4.1). With this choice of $q(z)$, we may write

$$\mathrm{e}^{ip(z)} = \exp\left(-\ln\left(\frac{z+q_0}{q_0}\right)\right) = \frac{1}{1 + z/R_0 + i\lambda z/\pi w_0^2}, \tag{4.15}$$

where R_0 and w_0 are the values of $R(z)$ and $w(z)$ at $z = 0$. If we match the real parts of Eq.(4.13) and Eq.(4.12), we can readily find that

$$\frac{1}{R(z)} = \frac{z + \mathrm{Re}(q_0)}{|q_0|^2 + 2z\mathrm{Re}(q_0) + z^2}. \tag{4.16}$$

From this we note that there exists some value of z for which $1/R \to 0$, or $R \to \infty$. Because q_0 is unspecified, let us choose $R_0 \to \infty$, so that

$$\boxed{\frac{1}{q_0} = \frac{i\lambda}{\pi w_0^2},} \tag{4.17}$$

and

$$\boxed{z_0 = \frac{\pi w_0^2}{\lambda},} \tag{4.18}$$

we may then write

$$\boxed{R(z) = z + \frac{z_0^2}{z},} \tag{4.19}$$

and

$$\boxed{w(z) = w_0\sqrt{1 + \frac{z^2}{z_0^2}}.} \tag{4.20}$$

With its definition, and the specification of $R_0 \to \infty$, we further find that

$$e^{ip(z)} = \frac{1}{1 + iz/z_0} = \frac{1}{1 + z^2/z_0^2} e^{-i\Phi(z)}, \tag{4.21}$$

where

$$\boxed{\Phi(z) = \tan^{-1}(z/z_0).} \tag{4.22}$$

Our solution for a Gaussian beam may be written completely in the form

$$\boxed{U(\mathbf{r}) = A_0 e^{-i\Phi(z)} \left[\frac{1}{\sqrt{1 + z^2/z_0^2}} e^{ikz} e^{ik(x^2+y^2)/2R(z)} \right] e^{-(x^2+y^2)/(2w(z)^2)}.} \tag{4.23}$$

Each of these terms has a clear physical meaning. The term A_0 is the amplitude of the beam. The last term, dependent on $w(z)$, represents the amplitude profile of the beam as a function of z; it is a Gaussian profile which decreases in widthas z increases towards the plane $z = 0$, where it is minimum, and increases again afterwards. The plane $z = 0$ is known as the beam waist, and represents the focal plane of a Gaussian beam. We do $z_o = z_R$. The intensity profile of a Gaussian beam is illustrated in Fig 4.1.

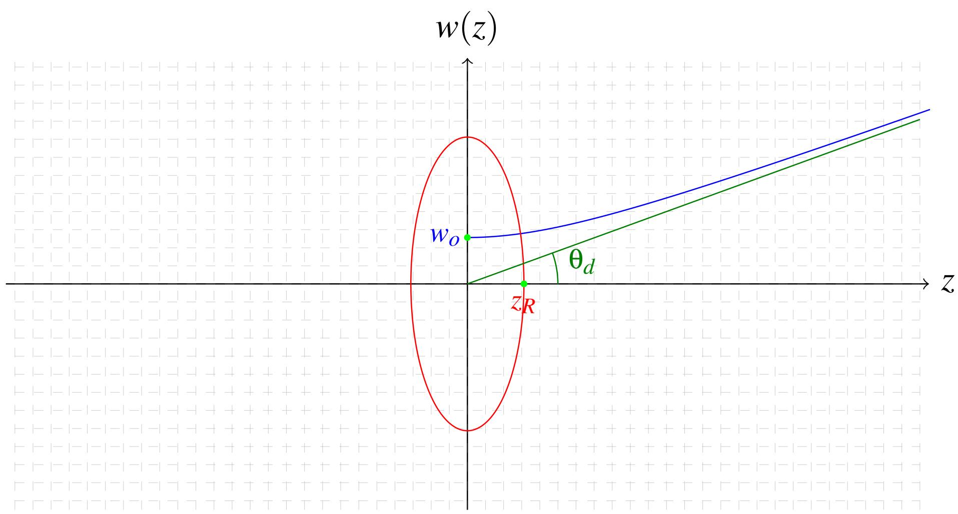


Figure 4.1: Waist of Gaussian Beam in terms of z.

we observe the blue curve is a hyperbola, this is because the Eq. (4.20) can write as

$$\frac{w^2(z)}{w_o^2} - \frac{z^2}{z_R^2} = 1, \tag{4.24}$$

in this form is more easy to see the hyperbola equation, and we can construct the ellipse that will be orthogonal to the hyperbola, in this case with semiminor axis in z_R, so we have

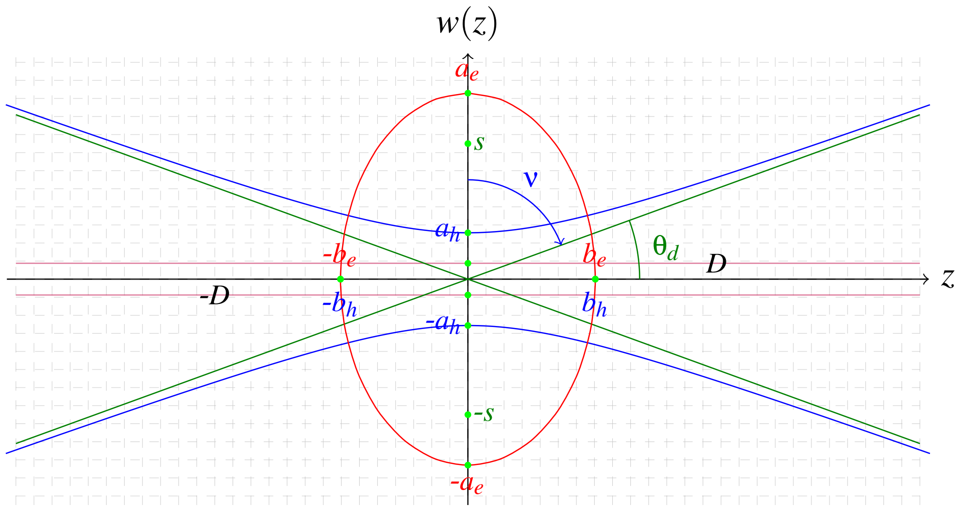


Figure 4.2: Waist of Gaussian Beam in terms of z.

this is almost the elliptic coordinates but rotated $-\pi/2$, and flipped in $w(z)$ so we can do the next parametrization

$$z = s \sinh \mu \sin \nu, \tag{4.25}$$

$$w(z) = s \cosh \mu \cos \nu, \tag{4.26}$$

with μ is a nonnegative real number and $\nu \in [0, 2\pi)$, but we will want that the angle of parametrization is measured in the z-axis not in the $w(z)$-axis, so we can take the next parametrization

Definition 4.1 Parametrization of Gaussian Beams

$$z = s \sinh \mu \cos \nu, \tag{4.27}$$

$$w(z) = s \cosh \mu \sin \nu, \tag{4.28}$$

with $\mu \in \mathbf{R}$ and $\nu \in [-\pi/2, \pi/2)$

this parametrization is better because we see in the gaussian beams how to change the waist with the distance z, this equation is like the elliptic coordinates, and the angle ν is the same as θ_d, thus we have the next results

4.1.4.1.1 Hyperbolas and parameters of Gaussian Beams

from Eq. (4.25) and Eq. (4.26) we have

$$\frac{w^2(z)}{s^2\ \sin^2 \nu} - \frac{z^2}{s^2\ \cos^2 \nu} = 1. \tag{4.29}$$

and the Eq. (4.24) we have

$$a_h = w_o = s\ |\sin \nu|\,, \tag{4.30}$$

and

$$b_h = z_R = s\ |\cos \nu|\,, \tag{4.31}$$

therefore

$$\boxed{a_h = w_o,} \tag{4.32}$$

$$\boxed{b_h = z_R,} \tag{4.33}$$

from

$$s^2 = a_h^2 + b_h^2, \tag{4.34}$$

we have

$$\boxed{s^2 = z_R^2 + w_o^2,} \tag{4.35}$$

this equation gives us the focus from the parameters of beam z_R and w_o, so with a_h, b_h and s we can determinate the another parameters of hyperbola like the eccentricity

$$e_h = \frac{s}{a_h} = \frac{\sqrt{z_R^2 + w_o^2}}{w_o}, \tag{4.36}$$

then

$$\boxed{e_h = \sqrt{\frac{z_R^2}{w_o^2} + 1}}, \tag{4.37}$$

the directrix

$$D = \frac{a_h}{e_h} = \frac{w_o}{\sqrt{\frac{z_R^2}{w_o^2} + 1}}, \tag{4.38}$$

so

$$\boxed{D = \frac{w_o^2}{\sqrt{z_R^2 + w_o^2}}}, \tag{4.39}$$

the asymptotes are

$$w(z) = \pm \frac{a_h}{b_h} z = \pm \frac{w_o}{z_R} z \tag{4.40}$$

$$\boxed{w(z) = \pm \frac{w_o}{z_R} z,} \tag{4.41}$$

the slope of the straight line is the angle divergence

$$\theta_d = \frac{w_o}{z_R} \tag{4.42}$$

then we have

$$\boxed{\cot \nu = \frac{w_o}{z_R},} \tag{4.43}$$

4.1.4.1.2 Ellipses and parameters of Gaussian Beams

from Eq. (4.25) and Eq. (4.26) we have

$$\frac{z^2}{s^2\sinh^2\mu}+\frac{w^2(z)}{s^2\cosh^2\mu}=1, \tag{4.44}$$

The ellipses are verticals so the semi axis are

$$a_e = s\cosh\mu, \tag{4.45}$$

$$b_e = s\sinh\mu, \tag{4.46}$$

we know that

$$s^2 = a_e^2 - b_e^2, \tag{4.47}$$

and if we take

$$\boxed{b_e = z_R}, \tag{4.48}$$

then

$$s^2 = a_e^2 - z_R^2, \tag{4.49}$$

from Eq. (4.35) we have

$$z_R^2 + w_o^2 = a_e^2 - z_R^2, \tag{4.50}$$

then

$$\boxed{a_e = \sqrt{2z_R^2 + w_o^2}}, \tag{4.51}$$

$$e_e = \sqrt{1 - \frac{b_e^2}{a_e^2}} = \sqrt{1 - \frac{z_R^2}{2z_R^2 + w_o^2}}, \tag{4.52}$$

so

$$\boxed{e_e = \sqrt{\frac{w_o^2 + z_R^2}{w_o^2 + 2z_R^2}}}, \tag{4.53}$$

we know that

$$b_e = z_R = s \sinh \mu, \tag{4.54}$$

and

$$s = \sqrt{z_R^2 + w_o^2}, \tag{4.55}$$

then

$$z_R = \sqrt{z_R^2 + w_o^2} \sinh \mu, \tag{4.56}$$

so

$$\mu = \sinh^{-1} \frac{z_R}{\sqrt{z_R^2 + w_o^2}} = \sinh^{-1} \frac{1}{\sqrt{1 + \frac{w_o^2}{z_R^2}}}, \tag{4.57}$$

therefor

$$\boxed{\mu = \sinh^{-1}\left(\frac{1}{\sqrt{1 + w_o^2/z_R^2}}\right)}, \tag{4.58}$$

We will put all results about the waist of Gaussian Beams with hyperbolas and ellipses in the next table

Summary of Connection of Elliptic Coordinates and parameters of Gaussian Beam

-	Ellipses	Hyperbolas
semiminor axis	$a_e = s\ \cosh\mu = \sqrt{2z_R^2 + w_o^2}$	$a_h = s\ \lvert\sin\nu\rvert = w_o$
semimajor axis	$b_e = s\ \sinh\mu = z_R$	$b_h = s\ \lvert\cos\nu\rvert = z_R$
focus	$f_e = s = \sqrt{a_e^2 - b_e^2} = \sqrt{z_R^2 + w_o^2}$	$f_h = s = \sqrt{a_h^2 + b_h^2} = \sqrt{z_R^2 + w_o^2}$
parameters in terms of semiaxis	$\mu = \frac{1}{2}\ln\left(\frac{a_e+b_e}{a_e-b_e}\right) = \frac{1}{2}\ln\left(\frac{\sqrt{2z_R^2+w_o^2}+z_R}{\sqrt{2z_R^2+w_o^2}-z_R}\right)$	$\nu = \cot^{-1}\left(\frac{a_h}{b_h}\right) = \cot^{-1}\left(\frac{w_o}{z_R}\right)$
eccentricity	$e_e = \frac{s}{a_e} = \operatorname{sech}\mu = \frac{z_R^2+w_o^2}{\sqrt{2z_R^2+w_o^2}}$	$e_h = \frac{s}{a_h} = \frac{1}{\cos\nu} = \sqrt{\frac{z_R^2}{w_o^2}+1}$
asympotates	not apply	$w(z) = \pm\frac{b_h}{a_h}z = \pm\cot\nu z = \pm\frac{w_o}{z_R}z$
directrixes	$D_e = \frac{a_e^2}{s} = s\cosh^2\mu = \frac{2z_R^2+w_o^2}{\sqrt{z_R^2+w_o^2}}$	$D_h = \frac{a_h^2}{s} = s\sin^2\nu = \frac{w_o^2}{\sqrt{z_R^2+w_o^2}}$

4.4.2 Hermite Gaussian Beams

With the Gaussian solution of Eq. (4.23) developed, we now look for other shape invariant beams, one of them are the Hermite polynomials.

The most important functions are the next Hermite (Weber) functions

$$y(\eta) = N\mathrm{H}_\nu(\zeta)\mathrm{e}^{-\frac{\zeta^2}{2}}, \tag{4.59}$$

where $\mathrm{H}_\nu(\zeta)$ are the Physical Hermite polynomials, with N a constant, in this case the waist of gaussian is $\sqrt{2}$ and in the square of function the waist is 1, but in the experiments we measure a different waist w, so we should connect this functions with quantities measured in the lab, we can the change

$$\zeta = \frac{x}{w(z)}, \tag{4.60}$$

so in the equation Eq. (4.59) we have

$$y(z) = M\mathrm{H}_\nu\left(\frac{x}{w(z)}\right)\mathrm{e}^{-\frac{'zeta^2}{2w(z)^2}}, \tag{4.61}$$

we will use functions such that the weight factor Gaussian is

$$e^{-\frac{\zeta^2}{2w(z)^2}} \tag{4.62}$$

and therefore the argument of the function must be of the form ξ/w. Then we propose the next ansatz as solution of Helmholtz Paraxial Equation

$$v(r) = f\left[\frac{x}{w(z)}\right] g\left[\frac{y}{w(z)}\right] u(r) e^{i\Phi(z)}, \tag{4.63}$$

where $f[x/w(z)]$ and $g[x/w(z)]$ are the propagation-dependent transverse profiles to be determined. The function $\Phi(z)$ is a propagation dependent phase shift. The transverse intensity of this solution scales in size by a factor $w(z)/w_0$ on propagation.

On substituting from Eq. (4.63) into the paraxial wave equation Eq. (4.6), we have

$$gu\partial_x^2 f + 2g\partial_x u\partial_x f + fu\partial_y^2 g + 2f\partial_y u\partial_y g + 2ik\left[gu\partial_z f + fu\partial_z g\right] + fg\left(\partial_x^2 u + \partial_y^2 u + 2ik\partial_z u\right) - 2kfgu\partial_z\Phi = 0, \tag{4.64}$$

The term in the curly braces satisfies Eq. (4.6), and is therefore equal to zero. We change to coordinates

$$\xi = x/w(z), \tag{4.65}$$

and

$$\eta = y/w(z), \tag{4.66}$$

and note by the chain rule that

$$\frac{\partial}{\partial x} = \frac{1}{w}\frac{\partial}{\partial \xi}, \tag{4.67}$$

$$\frac{\partial}{\partial y} = \frac{1}{w}\frac{\partial}{\partial \eta}, \tag{4.68}$$

$$\frac{\partial f}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial f}{\partial \xi} = -\frac{\xi w'}{w}\frac{\partial f}{\partial \xi}, \tag{4.69}$$

$$\frac{\partial g}{\partial z} = \frac{\partial \eta}{\partial z}\frac{\partial g}{\partial \eta} = -\frac{\xi w'}{w}\frac{\partial g}{\partial \eta}, \tag{4.70}$$

and $w = \partial w/\partial z$. Using these transformations in Eq. (4.64), and then dividing by $2fgu/w^2$ we have

$$\frac{\partial_\xi^2 f}{f} + \left(\frac{ikw^2}{R} - 2\right)\xi\frac{\partial_\xi f}{f} + \frac{\partial_\eta^2 g}{g} + \left(\frac{ikw^2}{R} - 2\right)\eta\frac{\partial_\eta g}{g} - ikww'\left(\frac{\xi\partial_\xi f}{f} + \frac{\eta\partial_\eta g}{g}\right) - kw^2\partial_z\Phi = 0. \quad (4.71)$$

We note that

$$ww' = w_0^2\frac{z}{z_0} = \frac{w^2}{R}. \quad (4.72)$$

The imaginary terms in the above equation cancel, and we are left with

$$\frac{\partial_\xi^2 f}{f} + -2\xi\frac{\partial_\xi f}{f} + \frac{\partial_\eta^2 g}{g} + -2\eta\frac{\partial_\eta g}{g} - kw^2\partial_z\Phi = 0. \quad (4.73)$$

This equation may be grouped into terms which depend only upon a single variable. As in separation of variables, each grouping must therefore be equal to a constant: -2m for the first, -2n for the second, and C for the third. The constants satisfy the equation

$$2n + 2m = C \quad (4.74)$$

We get the following separated set of equations

$$\partial_\xi^2 f - 2\xi\partial_\xi f + 2mf = 0, \quad (4.75)$$

$$\partial_\eta^2 g - 2\eta\partial_\eta g + 2ng = 0, \quad (4.76)$$

$$\partial_z\Phi = \frac{C}{kw_0^2}\frac{1}{1 + z^2/z_0^2}. \quad (4.77)$$

The equation for Φ can be directly integrated, to find

$$\Phi(z) = \frac{C}{2}\arctan(z/z_0). \quad (4.78)$$

4.2.4.2.1 Two Solution for the Hermite Differential Equation

The Eqs. (4.75) and (4.76) have the same structure, so we solve this equation (Hermite Differential Equation) for some methods for look the two solutions of this equation

Frobenius Method

We have the next Hermite differential equation that we will solve

$$u''(\zeta) - 2\zeta u'(\zeta) + 2\alpha u(\zeta) = 0. \tag{4.79}$$

In general this equation is solved for the Frobenius method, and its general solution is written like

$$u(\zeta) = \sum_{k=0}^{\infty} a_n \zeta^n. \tag{4.80}$$

replacing this in Eq. (4.79) we obtain

$$2\alpha a_0 + 2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + 2(\alpha - k)a_k]\zeta^k = 0 \tag{4.81}$$

where the following relationships are satisfied

$$2\alpha a_0 + 2a_2 = 0 \tag{4.82}$$

and

$$a_{k+2} = -2\frac{\alpha - k}{(k+1)(k+2)} a_k, \quad \text{for} \quad k = 1, 2, 3, \cdots \tag{4.83}$$

and therefore having the expression for each a_k the two series are hermite even and hermite odd series

$$\boxed{u_1(\zeta) = a_0\left(1 - \alpha\zeta^2 + 2\frac{2\alpha(\alpha-2)}{4!}\zeta^4 - 2^3\frac{\alpha(\alpha-2)(\alpha-4)}{6!}\zeta^6 + \cdots\right),} \tag{4.84}$$

and

$$u_2(\zeta) = a_1\left(\zeta - 2\frac{(\alpha-1)}{3!}\zeta^3 + 2^2\frac{(\alpha-1)(\alpha-3)}{5!}\zeta^5 - 2^3\frac{(\alpha-1)(\alpha-3)(\alpha-5)}{7!}\zeta^7 + \cdots\right). \tag{4.85}$$

One particular case is when α has any integer value. If $\alpha = 2l$ is even, with l integer, we have $a_0 \neq 0$, and $a_1 = 0$, so the term $a_{k+2} = -2\frac{2l-k}{(k+1)(k+2)}a_k$ will be zero when $k = 2l$, then the serie becomes in a even polynomial. The same applies when $\alpha = 2l+1$ is odd, with l integer, we have $a_0 = 0$, and $a_1 \neq 0$ so the term $a_{k+2} = -2\frac{2l+1-k}{(k+1)(k+2)}a_k$ will be zero when $k = 2l+1$ then the serie becomes in a odd polynomial. So for α integer the solution is $H_\alpha(\zeta)$, with the floor function, and to do $a_0 = 1$, $a_1 = 1$ the polynomials can be written as

$$H_\alpha(\zeta) = \alpha! \sum_{m=0}^{\lfloor\frac{\alpha}{2}\rfloor} \frac{(-1)^m}{m!(\alpha-2m)!}(2\zeta)^{\alpha-2m}. \tag{4.86}$$

With this method we have two solutions but when α is a integer number (odd or even number) we have to do zero a_0 or a_1 depending on the case, and with this we just have one solution.

For the functions converge for large values of ξ and η, m and n are constrained to integer values. The Eqs. (4.75) and (4.76) have the same structure and their solutions are Hermite polynomials ($H_m(\xi)$ and $H_n(\eta)$ respectively).

We find that there exist an infinite number of solutions to the paraxial wave equation of the form

$$v(r) = H_m\left(\frac{x}{w(z)}\right)H_n\left(\frac{y}{w(z)}\right)u(r)\exp\left(i\Phi(z)\right). \tag{4.87}$$

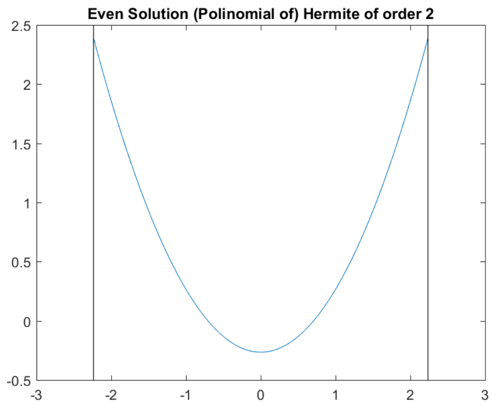
with

$$\Phi(z) = (m+n)\arctan(z/z_0) \tag{4.88}$$

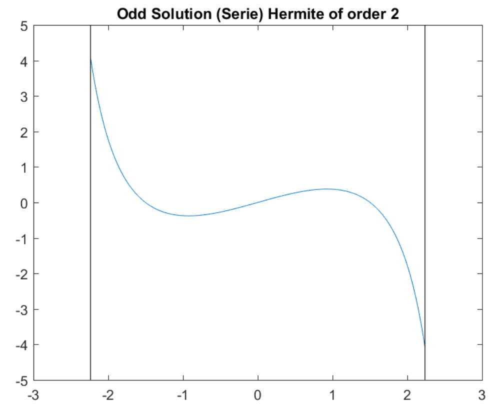
In the plane $z = 0$, these solutions appear as

$$v(x,y,0) = H_m\left(\frac{x}{w_0}\right)\exp\left(-x^2/2w_0^2\right)H_n\left(\frac{y}{w_0}\right)\exp\left(-y^2/2w_0^2\right). \tag{4.89}$$

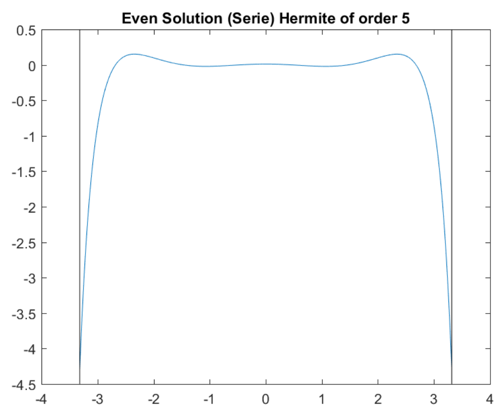
The Eqs. (4.75) and (4.76) are ordinary differential equations, and they must have two solutions, so we need the other solution, we will see next that the two solutions are the series in Eq. (4.84) and Eq. (4.85).



(a) Hermite even serie $\alpha = 2$

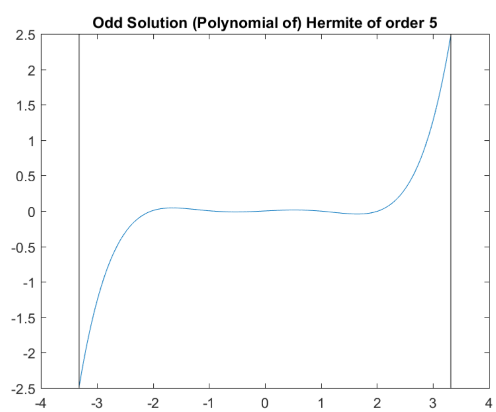


(b) Hermite odd serie $\alpha = 2$



(c) Hermite even serie $\alpha = 5$

Figure 4.3: Solutions for Hermite DIfferential Equation (Series).



(a) Hermite odd serie $\alpha = 5$

Hermite Differential Equation to Weber Differential Equation

We can transform Hermite Differential Equation to Weber Differential Equation if we do $u(\zeta) = y(\zeta)\mathrm{e}^{\frac{\zeta^2}{2}}$ [8], therefore we obtain

$$\boxed{\frac{\mathrm{d}^2 y(\zeta)}{\mathrm{d}\zeta^2} + \left(2\alpha + 1 - \zeta^2\right) y(\zeta) = 0.} \tag{4.90}$$

this equation is the Weber Differential Equation for Hermite Physics Polynomials, and their solutions are

$$y_1(\zeta) = u_1(\zeta)\,\mathrm{e}^{-\frac{\zeta^2}{2}}, \tag{4.91}$$

and

$$y_2(\zeta) = u_2(\zeta)\,\mathrm{e}^{-\frac{\zeta^2}{2}}, \tag{4.92}$$

if we make change $\zeta = \sqrt{2}\xi$ we obtain

$$\boxed{\frac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2} + \left(\alpha + \frac{1}{2} - \frac{\xi^2}{4}\right) y(\xi) = 0.} \tag{4.93}$$

this is the Weber Differential Equation for Hermite Probabilistic Polynomials, and their solutions are the parabolic cylinder functions $y_1(\xi) = D_\alpha(\xi)$, $y_2(z) = D_{-\alpha-1}(i\xi)$, but we don't interested in this kind of solutions where the argument of functions is complex.

if we do $\beta = -(\alpha + 1/2)$ in the Eq. (4.93) then we have

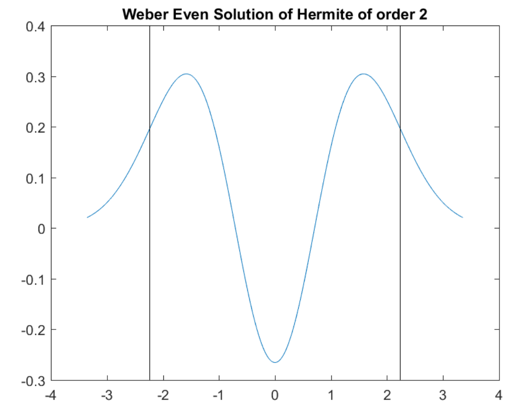
$$\frac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2} - \left(\beta + \frac{\xi^2}{4}\right) y(\xi) = 0. \tag{4.94}$$

and their solutions are

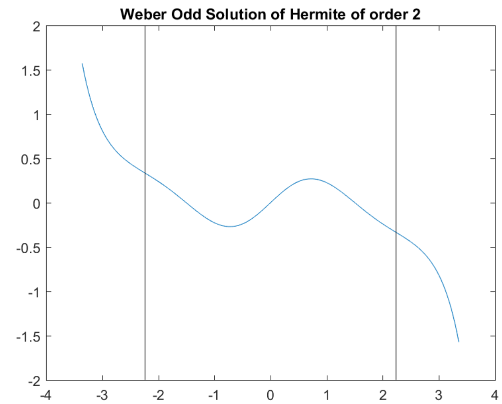
$$y_1(\xi) = u_1\left(\sqrt{2}\xi\right)\mathrm{e}^{-\xi^2}, \tag{4.95}$$

and

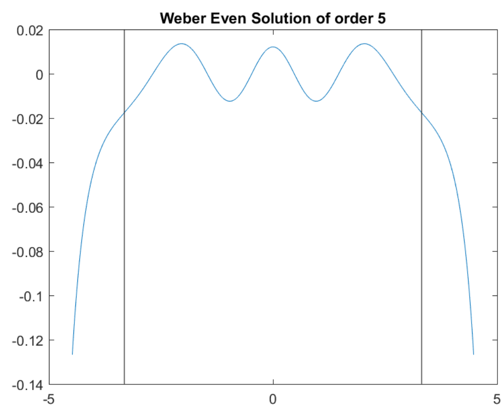
$$y_2(\xi) = u_2\left(\sqrt{2}\xi\right) e^{-\xi^2}, \tag{4.96}$$



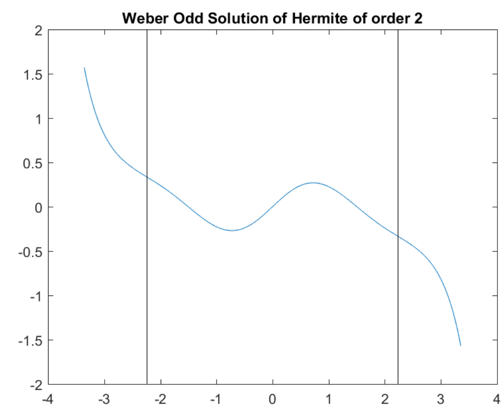
(a) Weber even serie $\alpha = 2$



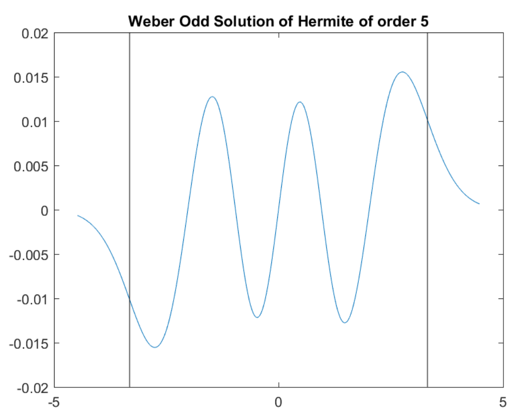
(b) Weber odd serie $\alpha = 2$



(c) Weber even serie $\alpha = 5$



(d) Weber odd serie $\alpha = 2$



(e) Weber odd serie $\alpha = 5$

Figure 4.5: Solutions for Weber Differential Equation.

Weber Differential Equation to Confluent Hypergeometric Differential Equation

We can transform Weber Differential Equation (Eq. (4.93)) to Confluent Hypergeometric Differential Equation (Eq: (4.104)) [8] if we do the next change

$$y(\xi) = \mathrm{e}^{\frac{-\xi^2}{4}} w\left(\frac{\xi^2}{2}\right), \tag{4.97}$$

and

$$\epsilon = \frac{\xi^2}{2}. \tag{4.98}$$

we need to apply change rule, then we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{d}\xi} = \xi\frac{\mathrm{d}}{\mathrm{d}\epsilon} = \sqrt{2\epsilon}\frac{\mathrm{d}}{\mathrm{d}\epsilon} \tag{4.99}$$

so

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} = \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} = \sqrt{2\epsilon}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\sqrt{2\epsilon}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right) = \sqrt{2\epsilon}\left(\frac{1}{\sqrt{2\epsilon}}\frac{\mathrm{d}}{\mathrm{d}\epsilon} + \sqrt{2\epsilon}\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\right) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} + 2\epsilon\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} \tag{4.100}$$

therefore in the Eq. (4.93) we have

$$\left(2\epsilon\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} + \frac{\mathrm{d}}{\mathrm{d}\epsilon}\right)\mathrm{e}^{-\frac{\epsilon}{2}} w(\epsilon) + \left(\alpha + \frac{1}{2} - \frac{\epsilon}{2}\right)\mathrm{e}^{-\frac{\epsilon}{2}} w(\epsilon) = 0, \tag{4.101}$$

then

$$2\epsilon\,\mathrm{e}^{-\frac{\epsilon}{2}}\left[w''(\epsilon) - w(\epsilon) + \frac{w(\epsilon)}{4}\right] + \mathrm{e}^{-\frac{\epsilon}{2}}\left[w'(\epsilon) - \frac{w(\epsilon)}{2}\right] + \left(\alpha + \frac{1}{2} - \frac{\epsilon}{2}\right)\mathrm{e}^{-\frac{\epsilon}{2}} w(\epsilon) = 0, \tag{4.102}$$

finally simplifying

$$\epsilon w''(\epsilon) + \left(\frac{1}{2} - \epsilon\right) w'(\epsilon) + \frac{\alpha}{2} w(\epsilon) = 0. \tag{4.103}$$

if we compare this equation with Confluent Hypergeometric Differential Equation [8]

$$xg''(x) + (c - x)g' - ag = 0. \tag{4.104}$$

whose solutions are

$$g(x) = C_1 M(a, c, x) + C_2 x^{1-c} M(a + 1 - c, 2 - c, x), \tag{4.105}$$

provided that c is not an integer. For our case $c = 1/2$ and $a = -\alpha/2$, therefore this is true, and where M is the confluent hypergeometric functions of the first type.

therefor the solution for the Eq. (4.103) are

$$w(\epsilon) = C_1 M\left(-\frac{\alpha}{2}, \frac{1}{2}, \epsilon\right) + C_2 \epsilon^{1/2} M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, \epsilon\right), \tag{4.106}$$

we denote

$$w_1(\epsilon) = M\left(-\frac{\alpha}{2}, \frac{1}{2}, \epsilon\right), \tag{4.107}$$

as one solution and

$$w_2(\epsilon) = \epsilon^{1/2} M\left(-\frac{\alpha - 1}{2}, \frac{3}{2}, \epsilon\right). \tag{4.108}$$

as another solution. Now if α takes integer values, let's see each case when α is odd or even.

If $\nu = 2k$, with k in naturals, then

$$w(\epsilon) = C_1 M\left(-k, \frac{1}{2}, \epsilon\right) + C_2 \epsilon^{1/2} M\left(-k + \frac{1}{2}, \frac{3}{2}, \epsilon\right) = C_1 w_1(\epsilon) + C_2 w_2(\epsilon). \tag{4.109}$$

$w_1(\epsilon)$ has the first parameter negative integer, so the number is reduced to a polynomial and this is directly related to the Hermite polynomials of order pair as follows

$$w_1(\epsilon) = M\left(-k, \frac{1}{2}, \epsilon\right) = (-1)^k \frac{k!}{(2k)!} H_{2k}(\sqrt{\epsilon}), \tag{4.110}$$

While $w_2(\epsilon)$ is still a series, so the complete solution for when $\alpha = 2k$ can be written as

$$w(\epsilon) = C_1' H_{2k}(\sqrt{\epsilon}) + C_2 \epsilon^{1/2} M\left(-k + \frac{1}{2}, \frac{3}{2}, \epsilon\right). \tag{4.111}$$

with

$$C_1' = C_1 (-1)^k \frac{k!}{(2k)!}. \tag{4.112}$$

Now Let's see when $\alpha = 2k+1$ with k in naturals, then

$$w(\epsilon) = C_1 M\left(-k - \frac{1}{2}; \frac{1}{2}; \epsilon\right) + C_2 \epsilon^{1/2} M\left(-k, \frac{3}{2}, \epsilon\right) = C_1 w_1(\epsilon) + C_2 w_2(\epsilon), \tag{4.113}$$

in this case the second solution is directly related to the odd Hermite polynomials, i.e.

$$M\left(-k, \frac{3}{2}, \epsilon\right) = (-1)^k \frac{k!}{(2k+1)! 2\sqrt{\epsilon}} H_{2k+1}(\sqrt{\epsilon}), \tag{4.114}$$

while the first solution is still a series, which completes the solution when $\alpha = 2k+1$ can be written as

$$w(\epsilon) = C_1 w_1(\epsilon) + C_2' H_{2k+1}(\sqrt{\epsilon}), \tag{4.115}$$

where

$$C_2 = C_2' (-1)^k \frac{k!}{(2k+1)!2}, \tag{4.116}$$

This confirms that hypergeometric solutions Eq. (4.103) are related to the even and odd series of Hermite and when α is even or odd, the respective series becomes a polynomial odd or even degree, besides that $w_1(\epsilon)$ it is an even function and $w_2(\epsilon)$ is odd function.

The solutions Wronskian

$$W\left[M\left(-\frac{\alpha}{2}, \frac{1}{2}, \epsilon\right), M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, \epsilon\right)\right] = \frac{1}{\pi} \sin\left(\frac{\pi}{2}\right) \epsilon^{-\frac{1}{2}} \mathrm{e}^{\epsilon} = \frac{1}{\pi} \epsilon^{-\frac{1}{2}} \mathrm{e}^{\epsilon}. \tag{4.117}$$

So the two solutions are linearly independent and satisfy the differential equation.

Returning variable changes to Eq(4.93), We find that their solutions are given by

$$\boxed{u_1(z) = e^{\frac{-z^2}{4}} M\left(-\frac{\alpha}{2}; \frac{1}{2}; \frac{z^2}{2}\right).} \tag{4.118}$$

as we saw this is an even function and the other solution

$$\boxed{u_2(z) = \frac{z}{2} e^{\frac{-z^2}{4}} M\left(-\frac{\alpha}{2} + \frac{1}{2}; \frac{3}{2}, \frac{z^2}{2}\right),} \tag{4.119}$$

It is an odd solution, so that the complete solution is written as

$$u(z) = C_1 u_1(z) + C_2 u_2(z). \tag{4.120}$$

and these become polynomials depending on the value of α.

We can conclude that two solutions that we have of the Hermite differential equation are the two series obtained by the method of Frobenius and its various representations (Weber, Hypergeometric) to make changes variable on the Hermite differential equation.

We will rewrite both solutions of Eq. (4.79) as $u_{\alpha_{odd}}(\zeta)$ for the odd solution and $u_{\alpha_{even}}(\zeta)$, for the same eigenvalor α.

4.4.3 Waist for Hermite-Weber functions

It's Very Important to know the waist of Weber-Hermite functions [9], because this is different to gaussian function, so we determinate this waist.

First we have the next Hermite-Weber functions

$$y(z) = N H_n(z) e^{-\frac{z^2}{2}}, \tag{4.121}$$

where N is a constant of normalization such that

$$\int_{-\infty}^{\infty} |y(z)|^2 \, dz = 1, \tag{4.122}$$

in this case

$$N = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}}. \tag{4.123}$$

the Eq. (4.121) satisfies the differential equation

$$y''(z) + (2n+1-z^2)y(z) = 0, \tag{4.124}$$

with this now we determinate the waist of Hermite-Weber function, using the expected value

$$\sigma_n^2 = \frac{2\int_{-\infty}^{\infty} z^2 N^2 H_n^2(z) \mathrm{e}^{-z^2} \mathrm{d}z}{\int_{-\infty}^{\infty} N^2 H_n^2(z) \mathrm{e}^{-z^2} \mathrm{d}z}, \tag{4.125}$$

the integral from the denominator is equal to 1 because the function is normalized, and the integral from the numerator is

$$\int_{-\infty}^{\infty} z^2 N^2 H_n^2(z) \mathrm{e}^{-z^2} \mathrm{d}z = \frac{2n+1}{2}, \tag{4.126}$$

replacing this integral in the Eq. (4.125) we have

$$\sigma_n^2 = 2n+1, \tag{4.127}$$

so we have the waist Hermite-Weber functions are

$$\sigma_n = \sqrt{2n+1}, \tag{4.128}$$

this coincides with the classical limit with the turning points oscillator in quantum mechanics, let's remember that we have the relationship $\varepsilon = 2n+1$ then

$$\sigma_n^2 = \varepsilon, \tag{4.129}$$

In optics we have a different waist for Gaussian function, so we make the change $z = \frac{x}{w}$, therefore in the Eq. (4.121) we have

$$y(x) = MH_n\left(\frac{x}{w}\right)\mathrm{e}^{-\frac{x^2}{2w^2}}, \tag{4.130}$$

the relation $z = \frac{x}{w}$ is very important because it allows us to connect the Weber-Hermite functions with Weber-Hermite functions for beams, we need nother normalization factor M, where

$$M = \frac{1}{\sqrt{\int_{-\infty}^{\infty} H_n^2\left(\frac{x}{w}\right) \mathrm{e}^{-\frac{x^2}{w^2}} \mathrm{d}x}}, \tag{4.131}$$

and we have

$$M = \frac{N}{\sqrt{w}}, \tag{4.132}$$

.

the normalization is only renormalizated with a factor of $1/\sqrt{w}$.

What happen with the associated differential equation?, we need change the differential using the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}z} = w\frac{\mathrm{d}}{\mathrm{d}x}, \tag{4.133}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} = w^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \tag{4.134}$$

so in Eq. (4.124) we have

$$w^2 y''(x) + \left[\varepsilon - \frac{x^2}{w^2}\right] y(x) = 0, \tag{4.135}$$

then

$$y''(x) + \left[\frac{\varepsilon}{w^2} - \frac{x^2}{w^4}\right] y(x) = 0, \tag{4.136}$$

We observe that index of Weber-Hermite functions is still being associated with $\varepsilon = 2n+1$, and the waist for this Weber-Hermite function is

$$\sigma_n^2(x) = \frac{2\int_{-\infty}^{\infty} z^2 M^2 H_n^2\left(\frac{x}{w}\right) \mathrm{e}^{-\frac{x^2}{w^2}} \mathrm{d}x}{\int_{-\infty}^{\infty} M^2 H_n^2\left(\frac{x}{w}\right) \mathrm{e}^{-\frac{x^2}{w^2}} \mathrm{d}x}, \tag{4.137}$$

the integral from denominator is equal to 1, and the integral from numerator is equal to

$$\int_{-\infty}^{\infty} x^2 M^2 H_n^2\left(\frac{x}{w}\right) \mathrm{e}^{-\frac{x^2}{w^2}} \mathrm{d}x = \frac{w^2}{2}[2n+1], \tag{4.138}$$

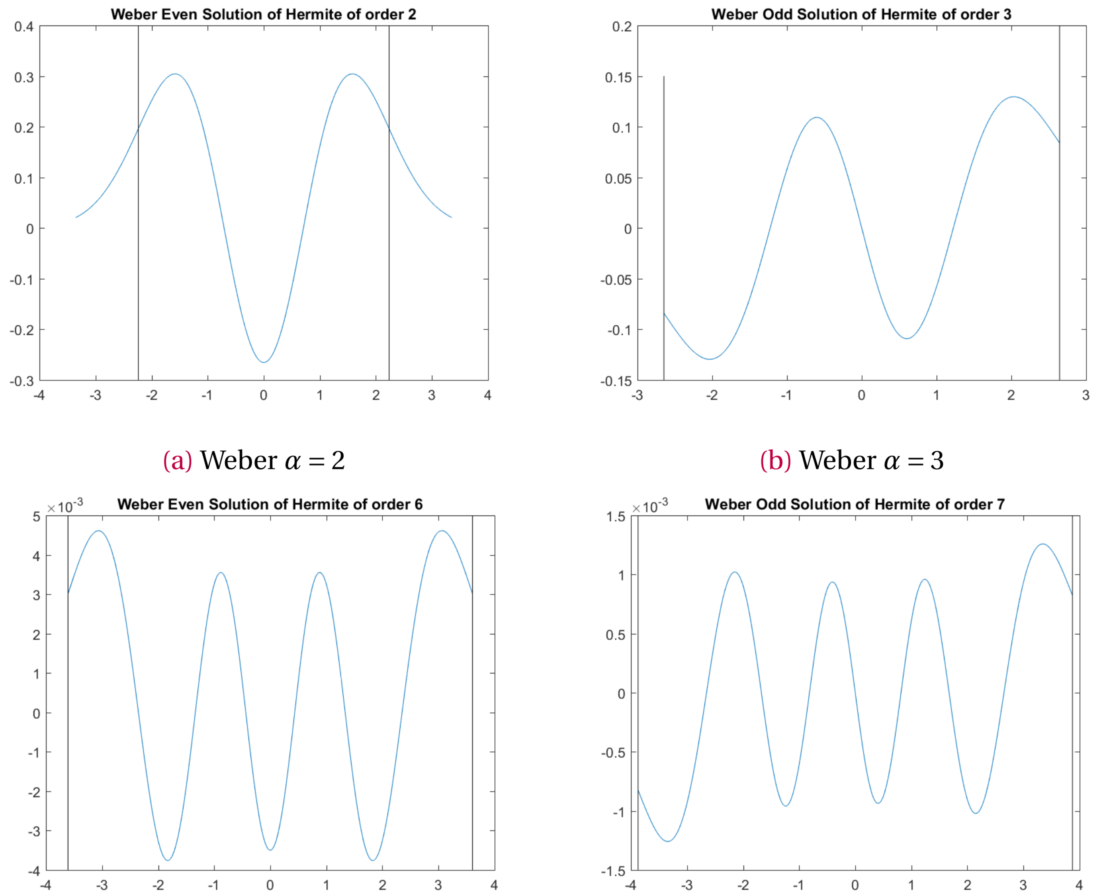
replacing this integral in the Eq. (4.137) we obtain

$$\sigma_n^2 = [2n+1]\, w^2, \tag{4.139}$$

therefor

$$\boxed{\sigma_n = \sqrt{2n+1}\, w.} \tag{4.140}$$

a particular case is $n = 0$, and we back the waist Gaussian function.



(a) Weber $\alpha = 2$

(b) Weber $\alpha = 3$

(c) Weber odd serie $\alpha = 6$

(d) Weber even serie $\alpha = 7$

Figure 4.6: black vertical straigth line is the waist of Weber functions.

4.4.4 Third Kind function of Weber Differential Equation

We can build the third kind function of Weber Differential Equation as the analogous case in Bessel functions, or the form polar $y(x) = e^{ix} = \cos(x) + i\sin(x)$ which is summing the even and odd functions, then for Weber-Physics Eq. (4.124) case with eignevalor n we have

$$HW_n(z) = y_{n_{odd}}(x) + i\, y_{n_{even}}(x), \tag{4.141}$$

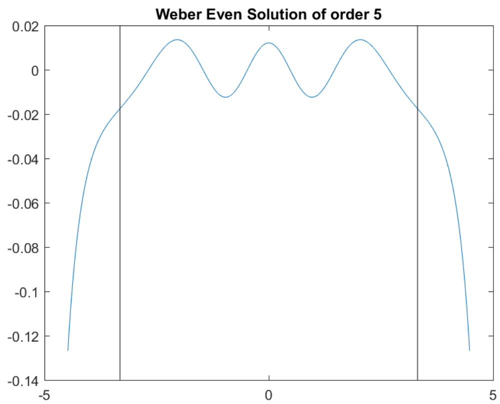
with

$$y_{n_{odd}}(x) = NH_{n_{odd}}(x)\mathrm{e}^{-\frac{x^2}{2}}, \tag{4.142}$$

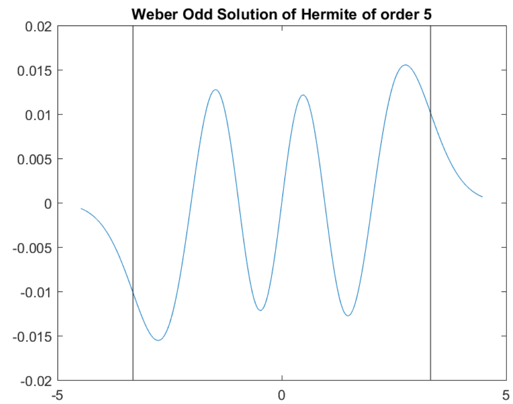
and

$$y_{n_{even}}(x) = NH_{n_{even}}(x)\mathrm{e}^{-\frac{x^2}{2}}. \tag{4.143}$$

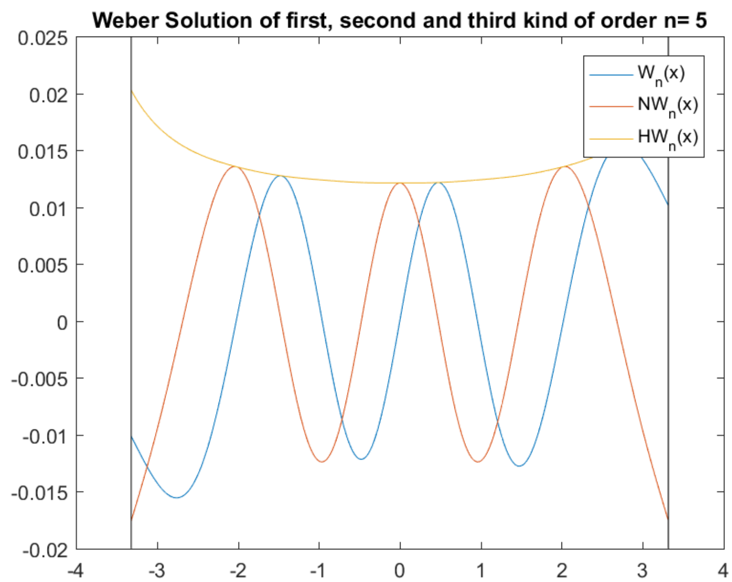
for $n = 5$ we have



(a) $y_{n_{even}}(x)$ with $n = 5$



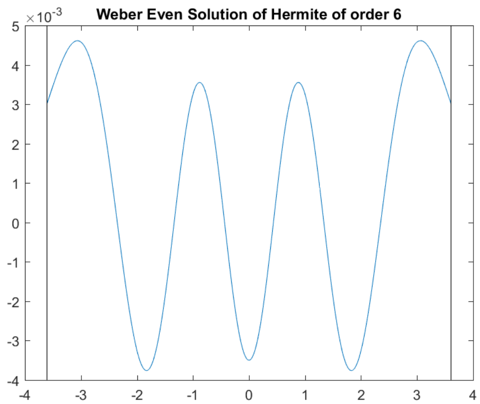
(b) $y_{n_{odd}}(x)$ with $n = 5$



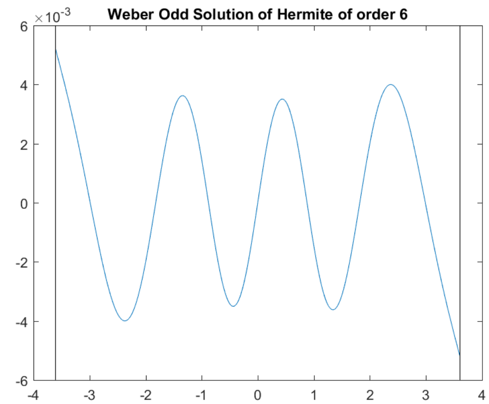
(c) $HW_n(z) = y_{n_{odd}}(x) + i\,y_{n_{even}}(x)$ with $n = 5$

Figure 4.7: Third kind function for Weber functions wih $n = 5$.

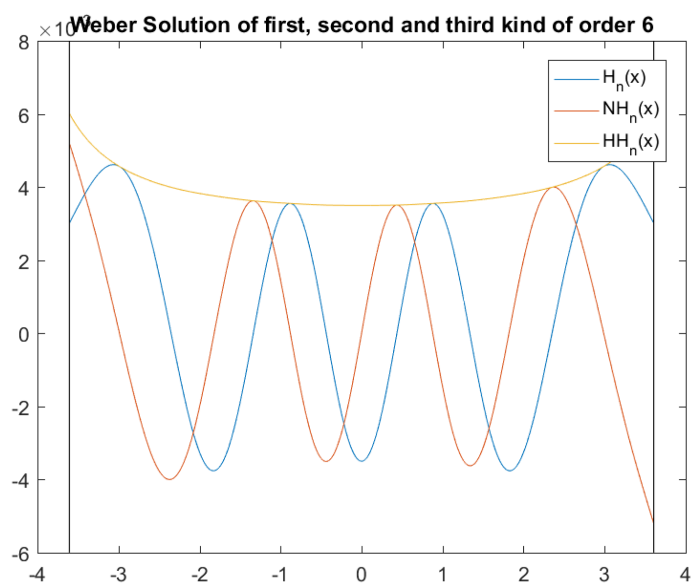
and $n = 6$



(a) $y_{\alpha_{even}}(x)$ with $n = 6$



(b) $y_{n_{odd}}(x)$ with $n = 6$



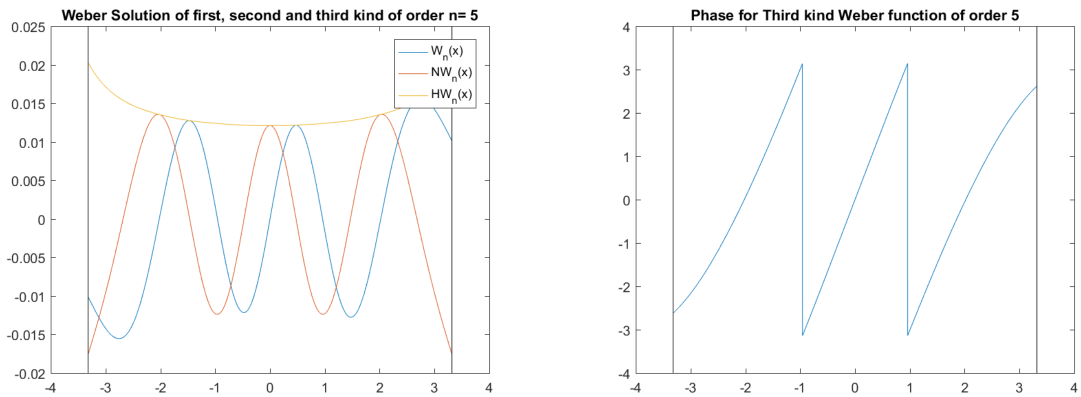
(c) $HW_n(z) = y_{n_{odd}}(x) + i\, y_{n_{even}}(x)$ with $\alpha = 6$

Figure 4.8: Third kind function for Weber functions wih $n = 6$.

4.4.5 Phase for Third Kind Weber function

We can determinate the phase of the Third Kind Weber function only dividing $y_{n_{even}}(x)$ above $n_{odd}(x)$

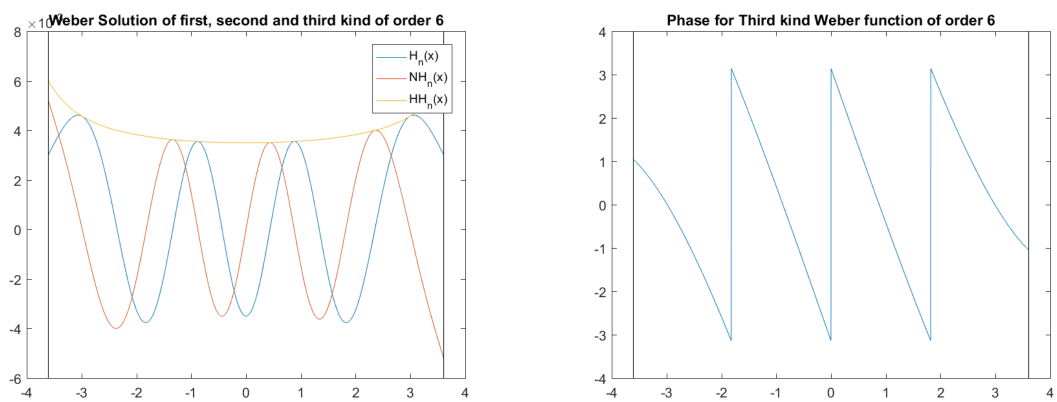
for $n = 5$ we have



(a) $HW_n(z) = y_{\alpha_{odd}}(x) + i\, y_{\alpha_{even}}(x)$ with $n = 5$ (b) angle $HW_n(z)$ with $n = 5$

Figure 4.9: Phase for Third kind Weber functions wih $n = 5$.

and for $n = 6$ we have



(a) $HW_n(z) = y_{\alpha_{odd}}(x) + i\, y_{\alpha_{even}}(x)$ with $n = 6$ (b) angle $HW_n(z)$ with $n = 6$

Figure 4.10: Phase for Third kind Weber functions wih $n = 6$.

5 Helmholtz Equation in Elliptic Cylinder Coordinates

In this chapter we will obtain the connection between the paraxial equation in Cartesian coordinates (4.6) with the Mathieu Differential Equation, for this we will solve the Helmholtz equation from the Elliptic Cylinder and we determinate how to get to obtain the Weber Differential Equation from the Mathieu Differential Equation.

We solve for the coordinates x and z, and it will be the same for the analogue y and z.

Instead of using instead the conventional coordinates [4], we use the parametrization for η angle measured from z-axis as follows

Definition 5.1 Elliptic Cylinder Coordinates with parametrization for η angle measured from z-axis

$$z = s \sinh \xi_{zx} \cos \eta_{zx}, \tag{5.1}$$

$$x = s \cosh \xi_{zx} \sin \eta_{zx}, \tag{5.2}$$

with $\xi_{zx} \in \mathbf{R}$ and $\eta_{zx} \in [-\pi/2, \pi/2)$

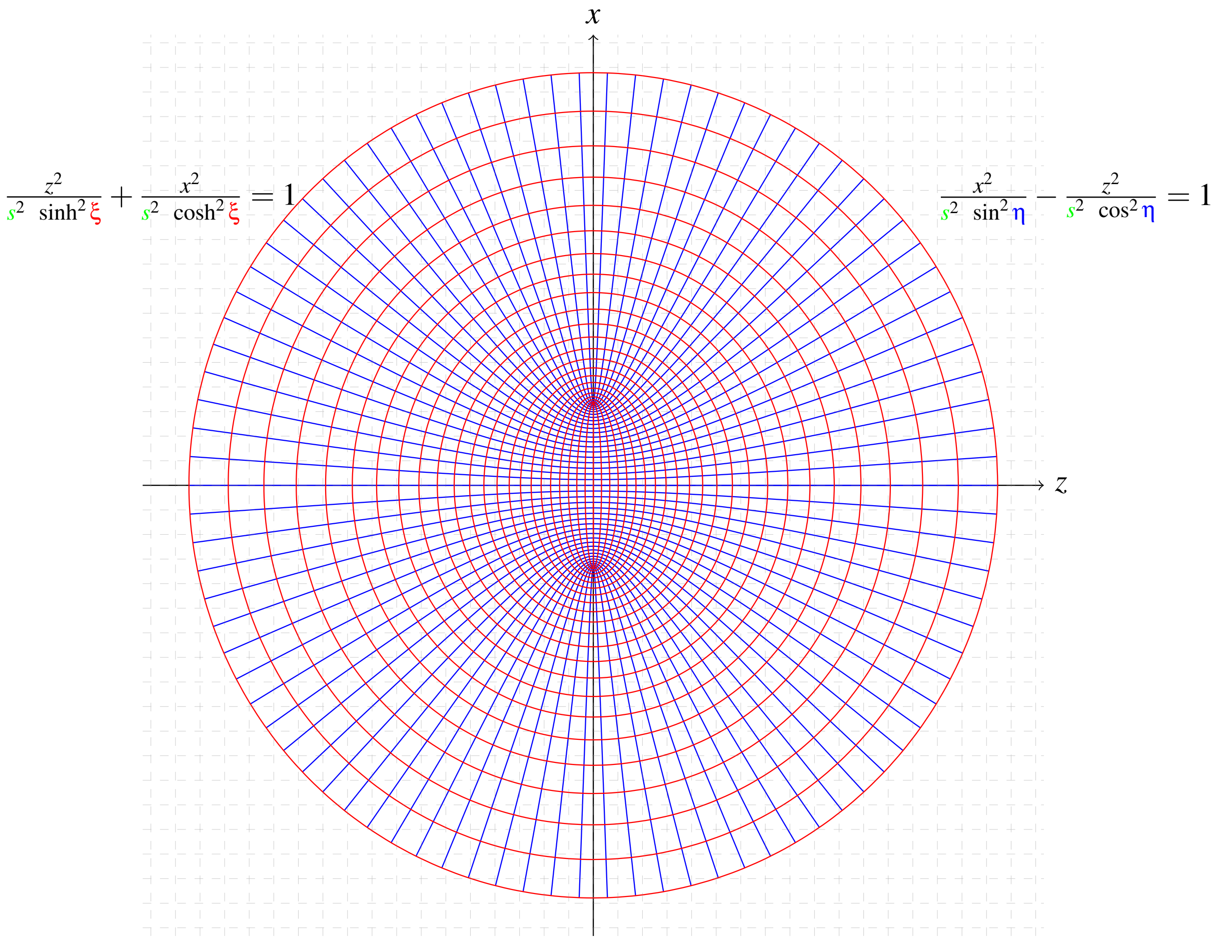


Figure 5.1: Elliptic cylindrical coordinates with η_{zx} angle measured from z-axis.

the ellipses are

$$\frac{z^2}{s^2\ \sinh^2 \xi} + \frac{x^2}{s^2\ \cosh^2 \xi} = 1$$

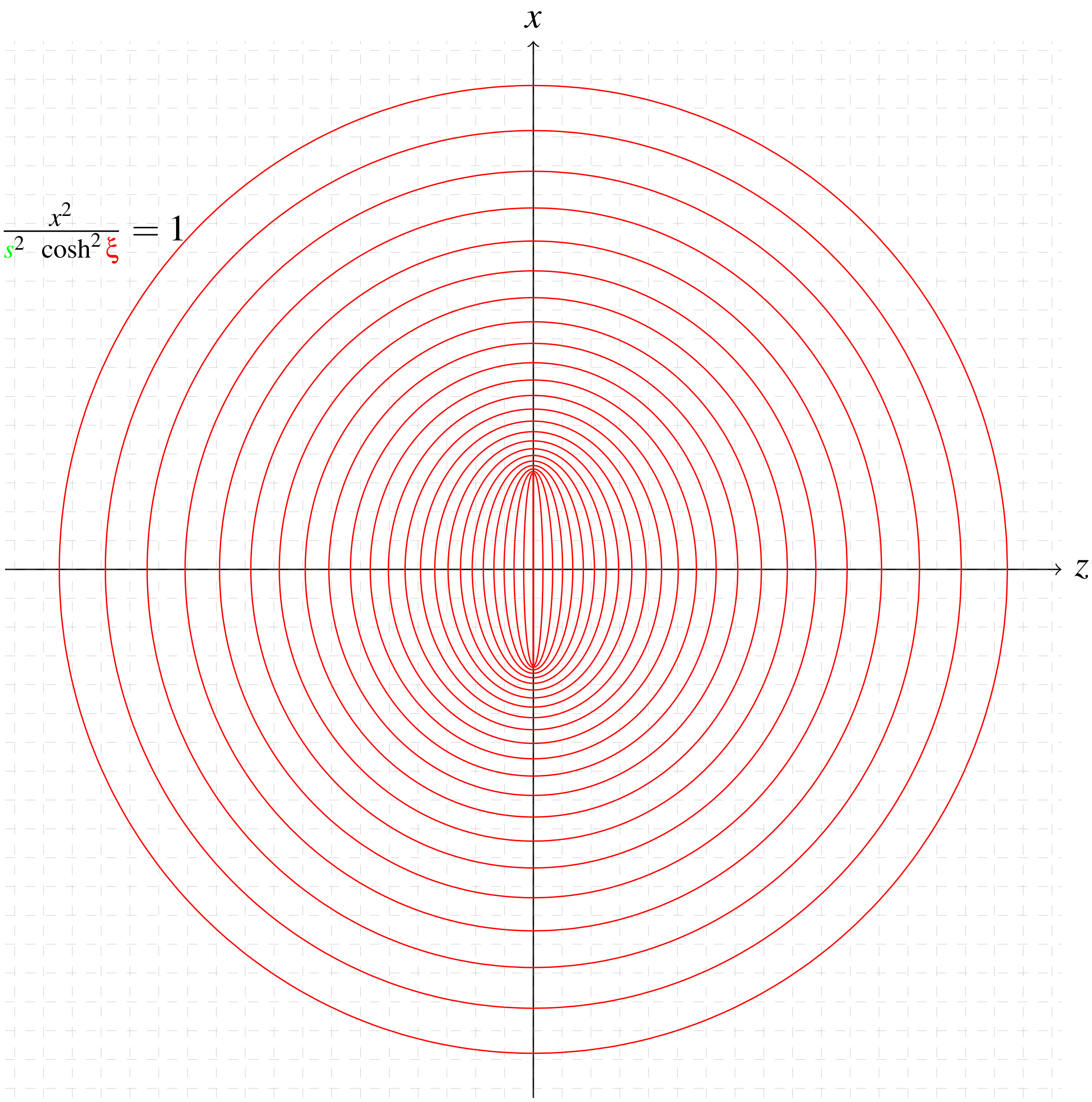


Figure 5.2: Ellipses.

and hyperbolas are

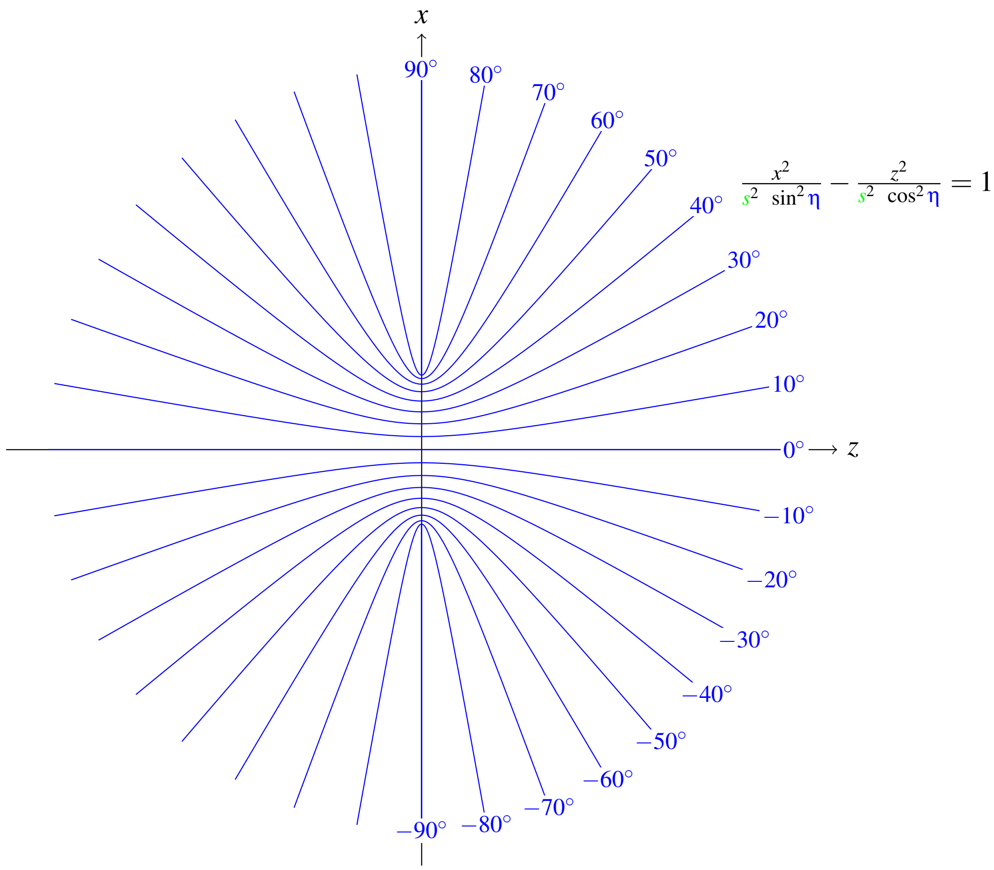


Figure 5.3: Hyperbolas.

the new scale factors are

$$h_1 = h_2 = s\sqrt{\sinh^2 \xi_{zx} + \cos^2 \eta_{zx}}, \tag{5.3}$$

therefor the Laplacian is

$$\nabla^2 = \frac{1}{s^2(\sinh^2 \xi_{zx} + \cos^2 \eta_{zx})}\left[\frac{\partial}{\partial \xi_{zx}}\left(\frac{s\sqrt{\sinh^2 \xi_{zx} + \cos^2 \eta_{zx}}}{s\sqrt{\sinh^2 \xi_{zx} + \cos^2 \eta_{zx}}}\right)\frac{\partial}{\partial \xi_{zx}} + \frac{\partial}{\partial \eta_{zx}}\left(\frac{s\sqrt{\sinh^2 \xi_{zx} + \cos^2 \eta_{zx}}}{s\sqrt{\sinh^2 \xi_{zx} + \cos^2 \eta_{zx}}}\right)\frac{\partial}{\partial \eta_{zx}}\right], \tag{5.4}$$

simplifying

$$\nabla^2 = \frac{1}{s^2(\sinh^2 \xi_{zx} + \cos^2 \eta_{zx})} \left[\frac{\partial^2}{\partial \xi_{zx}{}^2} + \frac{\partial^2}{\partial \eta_{zx}{}^2} \right], \tag{5.5}$$

then this in the Helmholtz equation we have

$$\frac{1}{s^2(\sinh^2 \xi_{zx} + \cos^2 \eta_{zx})} \left[\frac{\partial^2}{\partial \xi_{zx}{}^2} + \frac{\partial^2}{\partial \eta_{zx}{}^2} \right] \phi(\xi_{zx}, \eta_{zx}) = 0, \tag{5.6}$$

we propose $\phi(\xi_{zx}, \eta_{zx}, z) = U(\xi_{zx}) V(\eta_{zx})$ separable then

$$\frac{1}{s^2(\sinh^2 \xi_{zx} + \cos^2 \eta_{zx})} \left[V(\eta_{zx}) \frac{\mathrm{d}^2 U(\xi_{zx})}{\mathrm{d}\xi_{zx}{}^2} + U(\xi_{zx}) \frac{\mathrm{d}^2 V(\eta_{zx})}{\mathrm{d}\eta_{zx}{}^2} \right] + k^2 U(\xi_{zx}) V(\eta_{zx}) = 0, \tag{5.7}$$

dividing for $\phi(\xi_{zx}, \eta_{zx})$ we have

$$\frac{1}{s^2(\sinh^2 \xi_{zx} + \cos^2 \eta_{zx})} \left[\frac{U''(\xi_{zx})}{U(\xi_{zx})} + \frac{V''(\eta_{zx})}{V(\eta_{zx})} \right] + k^2 = 0, \tag{5.8}$$

or

$$\frac{1}{s^2(\sinh^2 \xi_{zx} + \cos^2 \eta_{zx})} \left[\frac{U''(\xi_{zx})}{U(\xi_{zx})} + \frac{V''(\eta_{zx})}{V(\eta_{zx})} \right] = -k^2 \tag{5.9}$$

this equation is separable in ξ_{zx}, η_{zx}, then regrouping

$$\left[\frac{U''(\xi_{zx})}{U(\xi_{zx})} + k^2 s^2 \sinh^2 \xi_{zx} \right] + \left[\frac{V''(\eta_{zx})}{V(\eta_{zx})} + k^2 s^2 \cos^2 \eta_{zx} \right] = 0, \tag{5.10}$$

here we have a separable equation, then

$$\left[\frac{U''(\xi_{zx})}{U(\xi_{zx})} + k^2 s^2 \sinh^2 \xi_{zx} \right] = c, \tag{5.11}$$

$$\left[\frac{V''(\eta_{zx})}{V(\eta_{zx})} + k^2 s^2 \cos^2 \eta_{zx} \right] = -c, \tag{5.12}$$

then we have the next equations

$$U''(\xi_{zx}) - \left(c - k^2 s^2 \sinh^2 \xi_{zx}\right) U(\xi_{zx}) = 0, \tag{5.13}$$

and

$$V''(\eta_{zx}) + \left(c + k^2 s^2 \cos^2 \eta_{zx}\right) V(\eta_{zx}) = 0, \tag{5.14}$$

it can use trigonometry identities $\cos^2\theta = (1+\cos(2\theta))/2$ and $\sinh^2\theta = (\cosh(2\theta)-1)/2$, and we rewrite this equations as

$$U''(\xi_{zx}) - \left[\left(c - \frac{k^2 s^2}{2}\right) + \frac{m^2 s^2}{2}\cosh(2\xi_{zx})\right] U(\xi_{zx}) = 0, \tag{5.15}$$

and

$$V''(\eta_{zx}) + \left[\left(c - \frac{k^2 s^2}{2}\right) + \frac{k^2 s^2}{2}\cos(2\eta_{zx})\right] V(\eta_{zx}) = 0. \tag{5.16}$$

if $\boxed{a = c - k^2 s^2/2}$, and $\boxed{q = -k^2 s^2/4}$, so

$$\boxed{U''(\xi_{zx}) - \left(a - 2q\cosh(2\xi_{zx})\right) U(\xi_{zx}) = 0,} \tag{5.17}$$

$$\boxed{V''(\eta_{zx}) + \left(a - 2q\cos(2\eta_{zx})\right) V(\eta_{zx}) = 0.} \tag{5.18}$$

The Eq. (5.18) is the Mathieu differential equation, and the Eq. (5.17) is modifiqued Mathieu differential equation[10, 11].

We note that the parameter a depends the constants of separations m, c, also the coordinate s, and q depends of s and m, i.e $a = a(s, c, m)$ and $q = q(s, m)$.

The limit when q is zero, we have in Eq. (5.18)

$$V''(\eta_{zx}) + aV(\eta_{zx}) = 0, \tag{5.19}$$

this equation is the same like the simple harmonic oscillator equation, and its solutions are linear combination of sines and cosines, but what happen when q is not equal to zero?, the solutions of Eq. (5.18) are Ce(a, q, η_{zx}) and Se(a, q, η_{zx}) functions this function are elliptic sines and elliptic cosines.

5.5.1 The Hermite Gaussian beams in the paraxial limit of the Mathieu Cartessian beams

We show how to transform the Mathieu Differential Equation to Weber Differential Equation, doing a approximation for small angles η_{zx}.

We have Mathieu Differential Equation

$$V''(\eta_{zx}) + \left[a - 2q\cos(2\eta_{zx})\right] V(\eta_{zx}) = 0, \tag{5.20}$$

for small angles we know that $\cos(\theta) \approx 1 - \theta^2/2$, then

$$V''(\eta_{zx}) + \left[a - 2q\left(1 - \frac{(2\eta_{zx})^2}{2}\right)\right] V(\eta_{zx}) = 0, \tag{5.21}$$

simplifying

$$V''(\eta_{zx}) + \left[a - 2q + 4q{\eta_{zx}}^2\right] V(\eta_{zx}) = 0, \tag{5.22}$$

this equation is for small angles, in the Coordinates Systems Chapter, we studied the limit in the coordinates, when η_{zx} is near to $\pi/2$ we have straight lines, in this case occurs when η_{zx} is near to 0, and we have straight lines parallel to z-axis.

We want the differential equation Mathieu for small angles, it will extend to the period π, and to match the differential equation of Weber-Hermite make the change of variable $\eta_{zx} = \frac{\nu}{\pi}$, so we get

$$\frac{\mathrm{d}}{\mathrm{d}\eta_{zx}} = \frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\eta_{zx}} = \pi\frac{\mathrm{d}}{\mathrm{d}\nu}, \tag{5.23}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}{\eta_{zx}}^2} = \pi^2\frac{\mathrm{d}^2}{\mathrm{d}\nu^2}, \tag{5.24}$$

substituting this in (5.22) we obtain

$$\pi^2 V''(v) + \left[a + 2q - 4q\frac{x^2}{\pi^2}\right] V(v) = 0, \tag{5.25}$$

then

$$V''(v) + \left[\frac{a+2q}{\pi^2} - \frac{4q}{\pi^4}v^2\right] V(v) = 0, \tag{5.26}$$

this equation is in the domain $v \in [-\pi/2, \pi/2]$, with this we resize the Differential Equation and its solutions.

Analogously for y,z we have

$$\boxed{U''(\xi_{zy}) - \left(a - 2q\cosh(2\xi_{zy})\right) U(\xi_{zy}) = 0,} \tag{5.27}$$

and

$$\boxed{V''(\eta_{zy}) + \left(a - 2q\cos(2\eta_{zy})\right) V(\eta_{zy}) = 0.} \tag{5.28}$$

5.1.5.1.1 Mathieu Differential Equation to Weber Differential Equation

The Mathieu Differential Equation Eq. (5.26) and Weber Differential Equation (4.90) have same structure but different domain, we should resize the Weber-Hermite function, so we will "normalize" with respect to the Hermite waist, thus every Weber-Hermite functions will be in the domain $[-\pi/2, \pi/2]$ because their waist will be equal to 1, so the Weber-Hermite functions are

$$y(\tau) = MH_\alpha(\tau)\mathrm{e}^{-\tau^2/2} \tag{5.29}$$

where τ will be normalized with respect to the waist of Weber-Hermite functions, doing $\tau = \sigma_\alpha\zeta$, then the Differential Equation associated is

$$y''(\tau) + (\varepsilon - \tau^2)y(\tau) = 0, \tag{5.30}$$

as we have $\tau = \sigma_\alpha\zeta$, the Differential Equation associated is

$$y''(\zeta) + [\sigma_\alpha^2 \varepsilon - \sigma_\alpha^4 \zeta^2]\, y(\zeta) = 0, \tag{5.31}$$

by comparing this equation with (5.26), we have

$$\sigma_\alpha^4 = \frac{4q}{\pi^2}, \tag{5.32}$$

or

$$q = \left(\frac{\pi}{2}\sigma_\alpha^2\right)^2, \tag{5.33}$$

substituing the value of σ_α we have

$$\boxed{q = \frac{\pi^2}{4}(2\alpha + 1)\,,} \tag{5.34}$$

and

$$\sigma_\alpha^2 \varepsilon = \frac{a + 2q}{\pi^2}, \tag{5.35}$$

for Eq. (5.32) we have

$$\frac{2\sqrt{q}}{\pi^2}\varepsilon = \frac{a + 2q}{\pi^2}, \tag{5.36}$$

then

$$2\sqrt{\pi}\varepsilon = a + 2q, \tag{5.37}$$

so

$$a = 2\left(\sqrt{q}\varepsilon - q\right), \tag{5.38}$$

substituting this in Eq. (5.34)

$$a = 2\left[\frac{\pi}{2}\sigma_\alpha^2 \varepsilon - \left(\frac{\pi}{2}\sigma_\alpha\right)^2\right], \tag{5.39}$$

replacing the value of σ_α and ϵ, we obtain

$$a = 2\left[\frac{\pi}{2}(2\alpha+1)(2\nu+1) - \frac{\pi^2}{4}(2\alpha+1)\right], \tag{5.40}$$

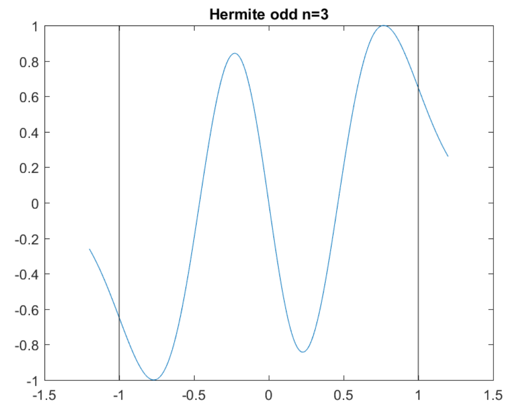
finally we obtain

$$\boxed{a = (2\alpha+1)\pi\,(4-\pi)\,/4,} \tag{5.41}$$

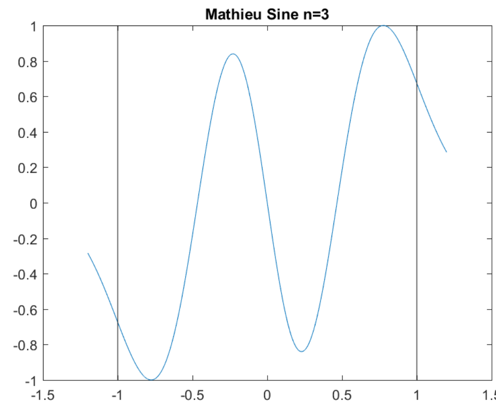
TheEq. (5.34) and Eq. (5.34) are the specific quantities that connect the parameters the Mathieu Differential Equation and Weber Differential Equation

We show the solutions for Weber Differential Equation and Mathieu Differential Equation for different values of α integers.

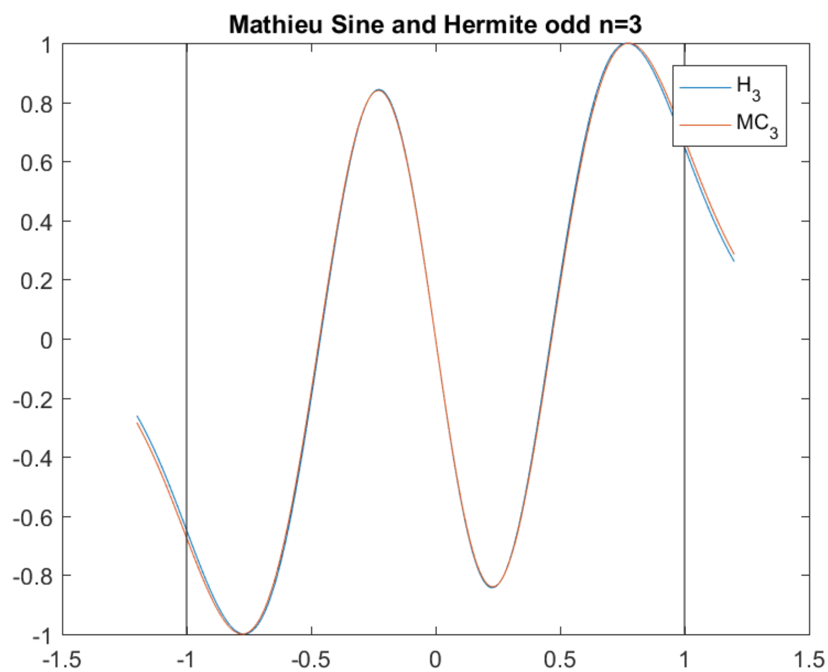
For $\alpha = n = 3$ the solutions odd that we have is



(a) Hermite odd $\alpha = 3$



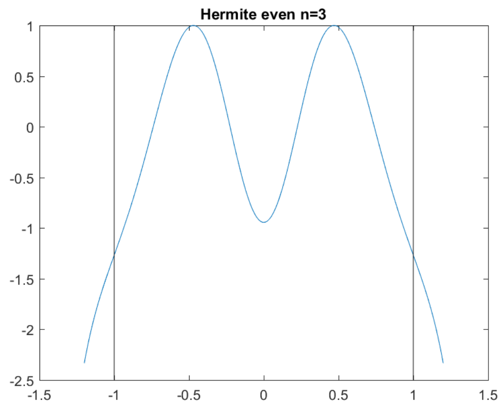
(b) Mathieu Sine $\alpha = 3$



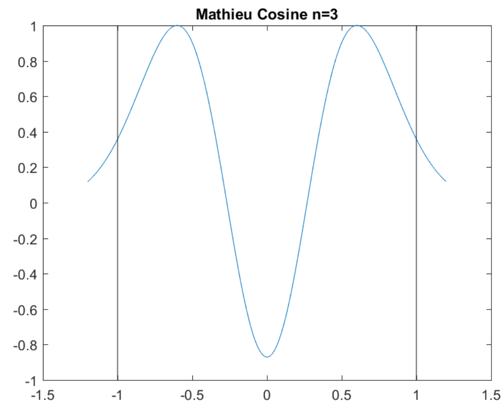
(c) Hermite - Mathieu $\alpha = 3$

Figure 5.4: Solutions odd for Mathieu Differential Equation and Hermite DIfferential, and their comparations for order $\alpha = 3$.

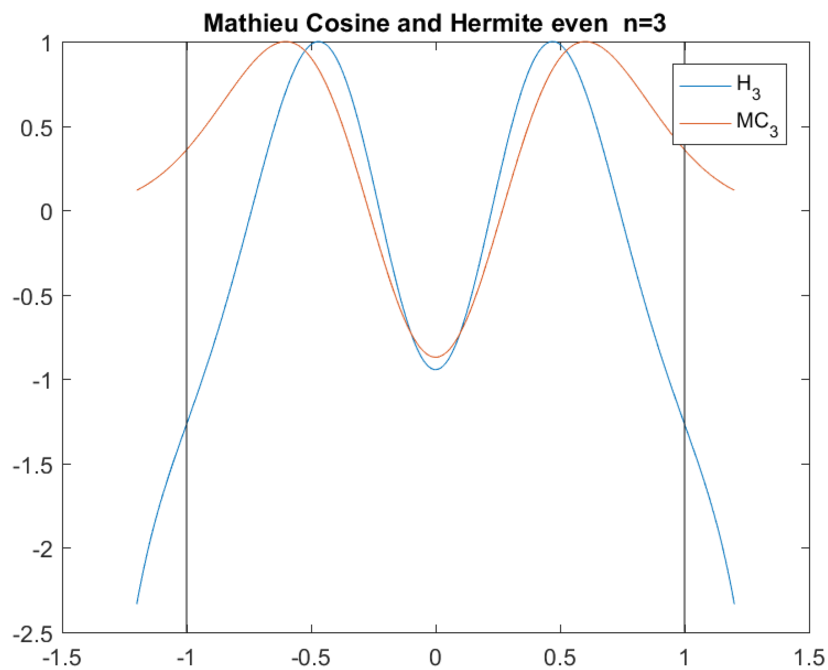
we can observe that both solutions are the same, and even solution for $\alpha = n = 3$



(a) Hermite even $\alpha = 3$



(b) Mathieu Cosine $\alpha = 3$

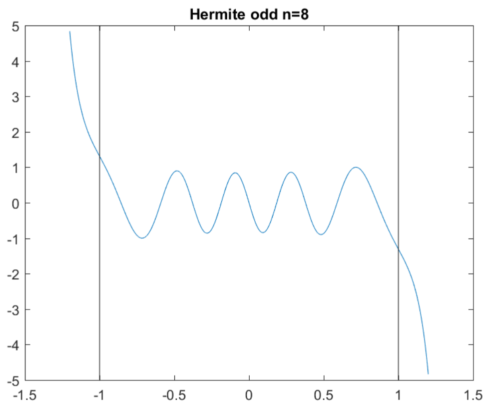


(c) Hermite - Mathieu $\alpha = 3$

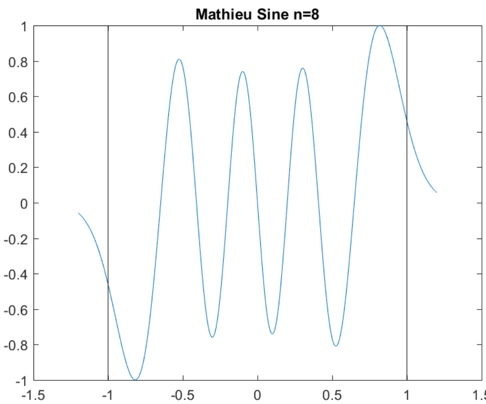
Figure 5.5: Solutions even for Mathieu Differential Equation and Hermite DIfferential, and their comparations for order $\alpha = 3$.

we observer that in even solution we have differences between the Weber Differential Equation and Mathieu Differential Equation, while in the first the solution diverges the second not.

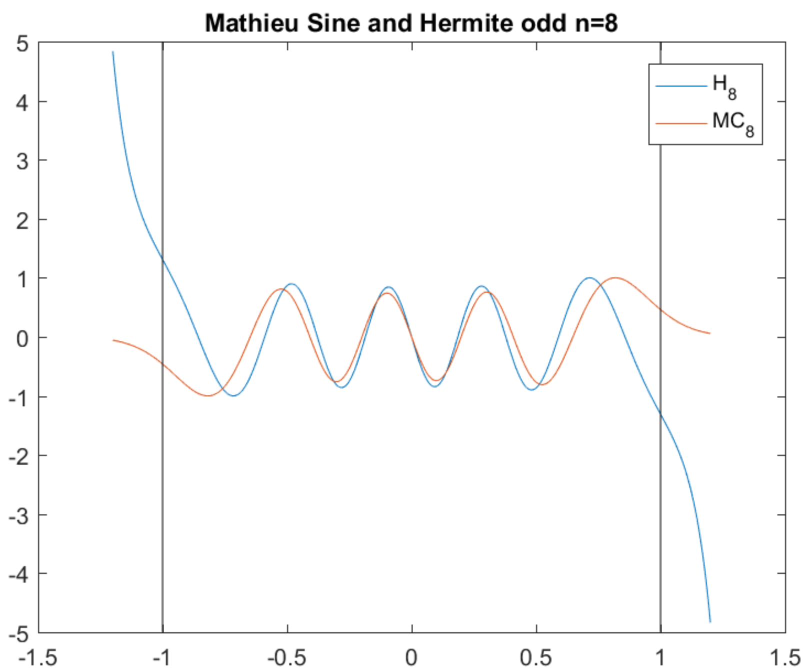
Now for $\alpha = n = 8$ we have the odd solutions is



(a) Hermite odd $\alpha = 8$



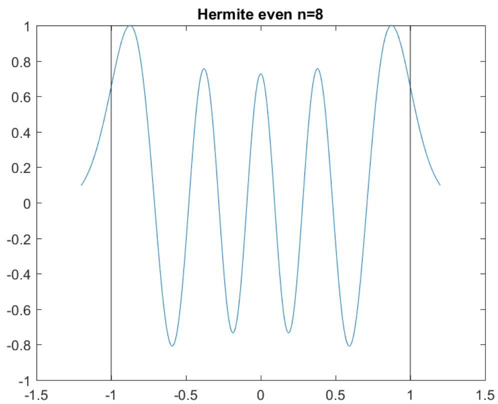
(b) Mathieu Sine $\alpha = 8$



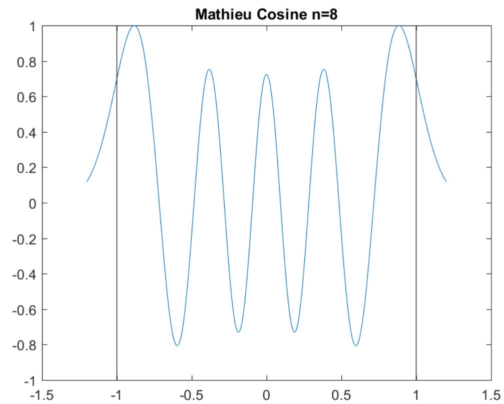
(c) Hermite - Mathieu $\alpha = 8$

Figure 5.6: Solutions odd for Mathieu Differential Equation and Hermite DIfferential, and their comparations for order $\alpha = 8$.

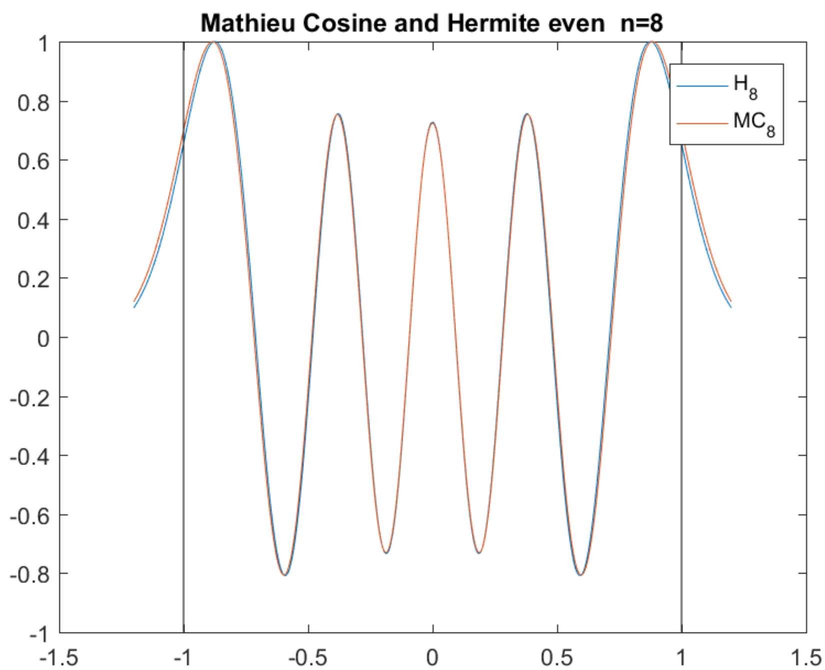
in this case for odd Weber solution diverge while the Mathieu Sine not, and for even solution we have



(a) Hermite even $\alpha = 8$



(b) Mathieu Cosine $\alpha = 8$



(c) Hermite - Mathieu $\alpha = 8$

Figure 5.7: Solutions even for Mathieu Differential Equation and Hermite DIfferential, and their comparations for order $\alpha = 8$.

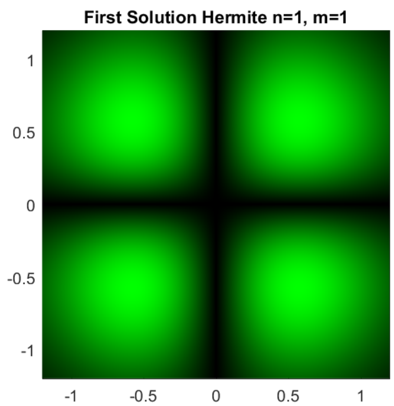
both are the same.

5.5.2 Hermite Gaussian Beams and Mahtieu Cartesian Beams in $2D$

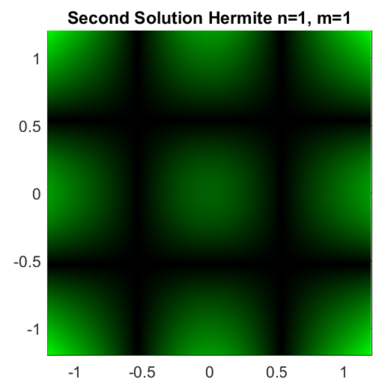
We remember that we can construct the functions in $2D$, because is the analogous for y and z, so we will use n-order for x and z and m-the order for x and z.

We will name F for the first solution and G for the second solution.

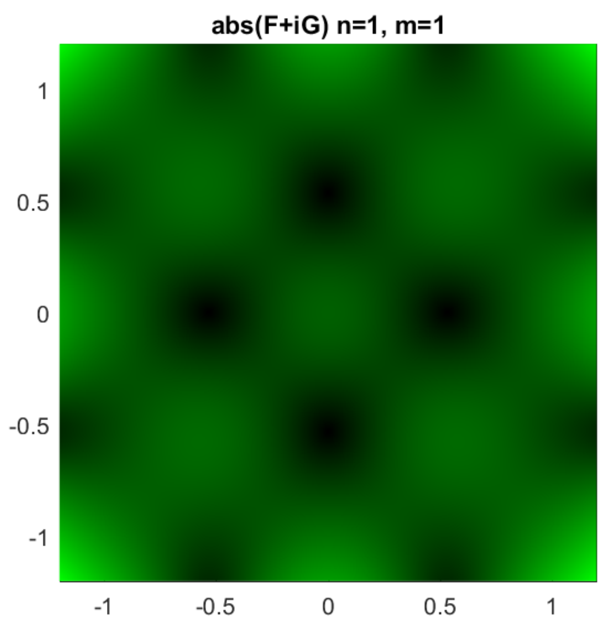
for the case of Hermite with $n = 1$ and $m = 1$ we have



(a) Hermite first Solution $m = 1$, $n = 1$

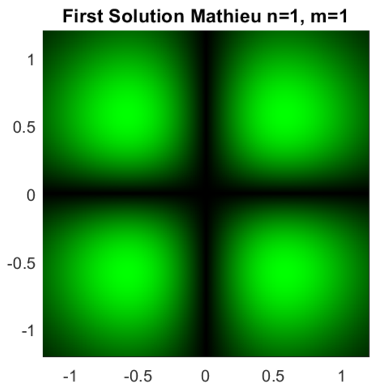


(b) Hermite Second Solution $m = 1$, $n = 1$

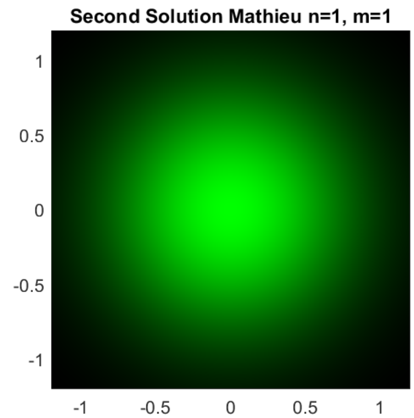


(c) Module of $abs(F + iG)$ with $m = 1$, $n = 1$

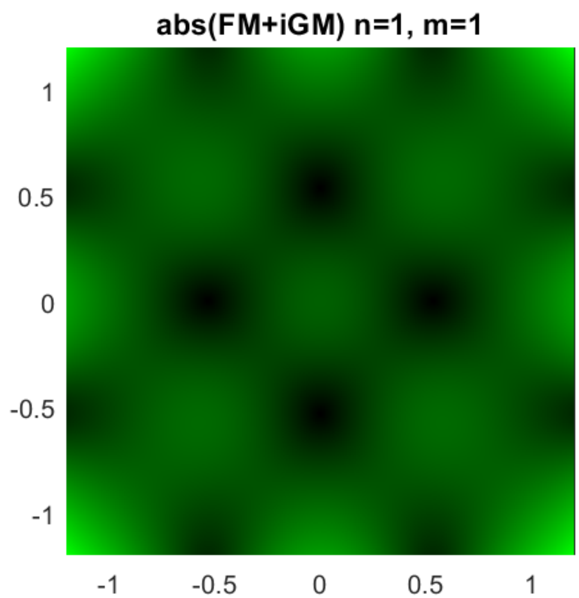
Figure 5.8: Two Solutions for Hermite Differential with $m = 1$, $n = 1$



(a) Mathieu first Solution $m = 1$, $n = 1$

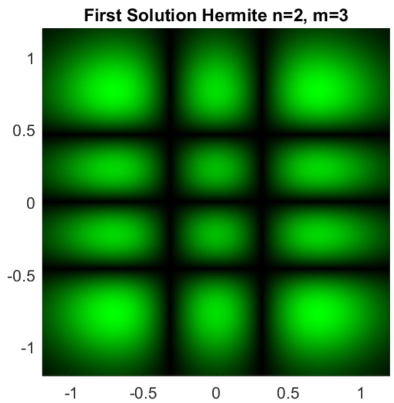


(b) Mathieu Second Solution $m = 1$, $n = 1$

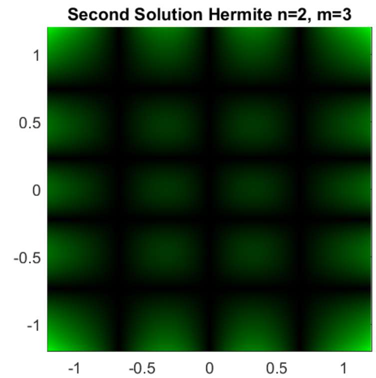


(c) Module of $FM + iGM$ with $m = 1$, $n = 1$

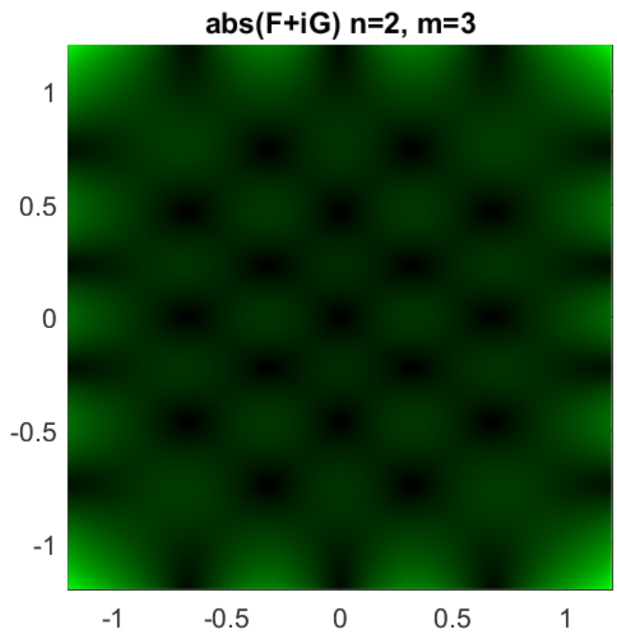
Figure 5.9: Two Solutions for Mathieu Differential Equation (limit case to Hermite) with $m = 1$, $n = 1$



(a) Hermite first Solution $m = 2$, $n = 3$

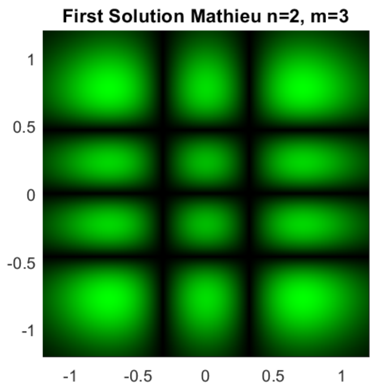


(b) Hermite Second Solution $m = 2$, $n = 3$

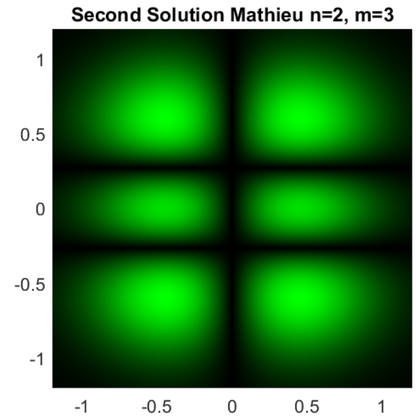


(c) Module of $abs(F + iG)$ with $m = 2$, $n = 3$

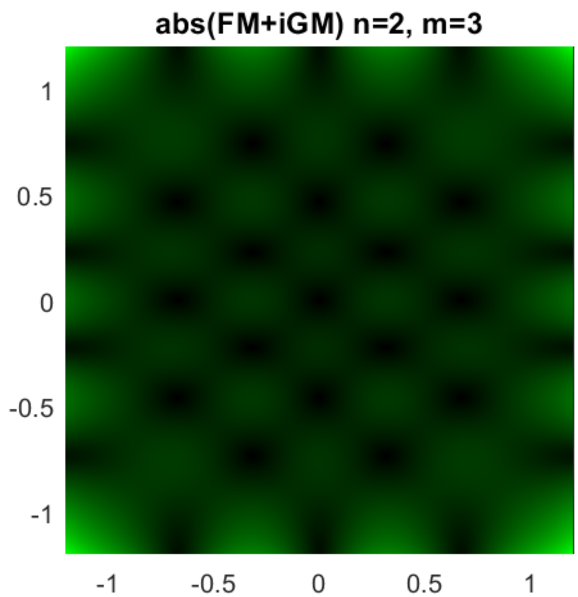
Figure 5.10: Two Solutions for Hermite Differential with $m = 2$, $n = 3$



(a) Mathieu first Solution $m = 2$, $n = 3$

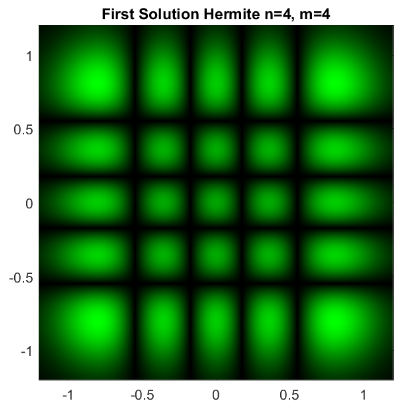


(b) Mathieu Second Solution $m = 2$, $n = 3$

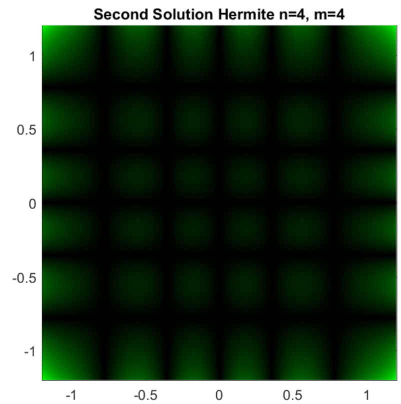


(c) Module of $abs(FM + iGM)$ with $m = 2$, $n = 3$

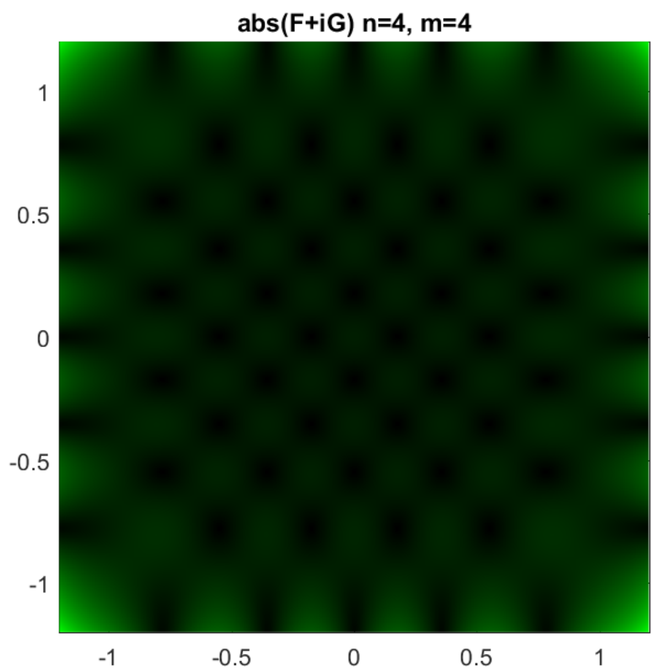
Figure 5.11: Two Solutions for Mathieu Differential Equation (limit case to Hermite) with $m = 2$, $n = 3$



(a) Hermite first Solution $m = 4$, $n = 4$

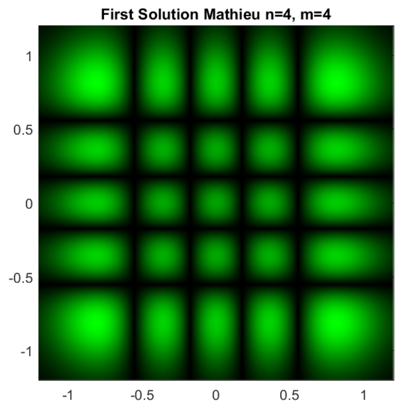


(b) Hermite Second Solution $m = 4$, $n = 4$

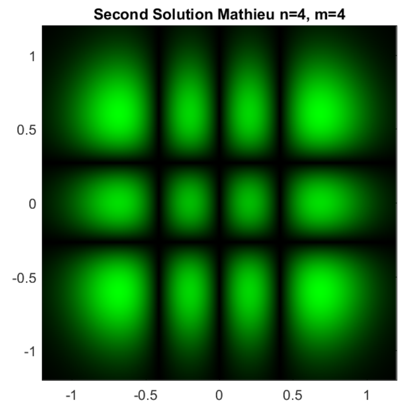


(c) Module of $abs(F + iG)$ with $m = 4$, $n = 4$

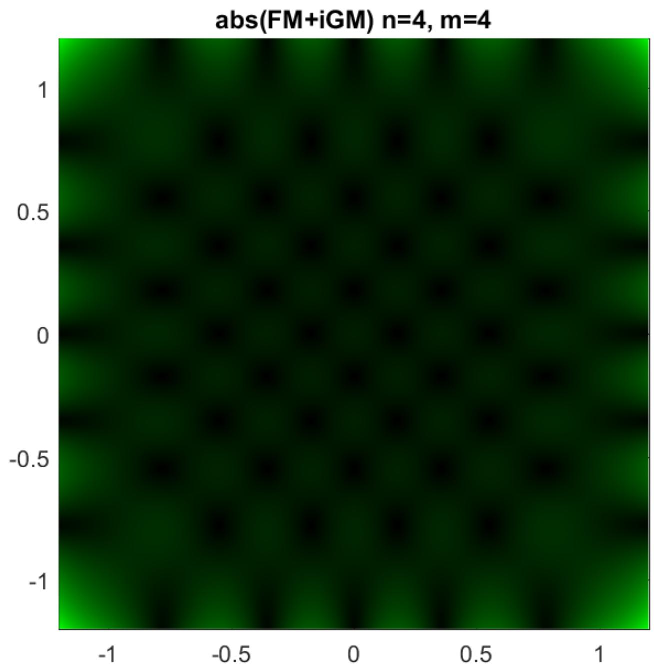
Figure 5.12: Two Solutions for Hermite Differential with $m = 4$, $n = 4$



(a) Mathieu first Solution $m=4$, $n=4$



(b) Mathieu Second Solution $m=4$, $n=4$



(c) Module of $abs(FM+iGM)$ with $m=2$, $n=3$

Figure 5.13: Two Solutions for Mathieu Differential Equation (limit case to Hermite) with $m=4$, $n=4$

6 Propagation and self-healing (obstruction in beams)

One important technique to simulate the propagation of beams is Angular Spectrum Method [5], for that we study briefly

6.6.1 Angular Spectrum Method

The theory of diffraction was originally developed using the Huygens-Fresnel principle, i.e. the idea that a wavefield can be mathematically decomposed into a collection of secondary spherical waves. When light is propagating in a homogeneous medium, it is also possible to decompose the field into a collection of plane waves, in what is known as an angular spectrum representation of wavefields.

We consider again a monochromatic scalar wavefield $V(\mathbf{r},t) = U(\mathbf{r})e^{-i\omega t}$ within the halfspace $z > 0$, where $\mathbf{r} = (x, y, z)$. There exist no sources within the half-space, and the medium is assumed to be vacuum. The space-dependent part of the field satisfies the Helmholtz equation,

$$\left(\nabla^2 + k^2\right) U(\mathbf{r}) = 0, \tag{6.1}$$

where $k = \omega/c$.

We make the reasonable assumption that within any plane of constant z, the field may be represented as a two-dimensional Fourier integral, i.e.

$$U(x, y, z) = \iint_{-\infty}^{\infty} \widetilde{U}(u, v, z)e^{i(ux+vy)}\mathrm{d}u\mathrm{d}v. \tag{6.2}$$

The corresponding inverse representation is

$$\widetilde{U}(u, v, z) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} U(x, y, z)e^{-i(ux+vy)}\mathrm{d}x\mathrm{d}y, \tag{6.3}$$

What does this assumption tell us about the form of the field? If we substitute the above formula into the Helmholtz equation, we find that

$$\iint_{-\infty}^{\infty} (\nabla^2 + k^2)\widetilde{U}(u, v, z)e^{-i(ux+vy)}\mathrm{d}u\mathrm{d}v = 0, \tag{6.4}$$

where we have interchanged the order of integration and differentiation. The derivatives with respect to x and y may now be taken directly, and we are left with the equation

$$\iint_{-\infty}^{\infty} \left[(-u^2 - v^2 - k^2)\widetilde{U}(u,v,z) + \frac{\partial^2 \widetilde{U}(u,v,z)}{\partial z^2} \right] \mathrm{e}^{-i(ux+vy)} \mathrm{d}u\mathrm{d}v = 0, \tag{6.5}$$

The Helmholtz equation must hold for all values of x and y, and Eq. (6.5) must therefore hold for each (u, v), pair. This implies that the function $\widetilde{U}(u,v,z)$ must satisfies the differential equation

$$\frac{\partial^2 \widetilde{U}(u,v,z)}{\partial z^2} + \omega^2 \widetilde{U}(u,v,z) = 0, \tag{6.6}$$

where

$$\omega^2 = k^2 - u^2 - v^2. \tag{6.7}$$

It is to be noted that there are values of ω which are imaginary; we therefore pick a particular branch of ω,

$$\omega = \begin{cases} (k^2 - u^2 - v^2)^{1/2} & \text{when } u^2 + v^2 \le k^2 \\ i(u^2 + v^2 - k^2)^{1/2} & \text{when } u^2 + v^2 > k^2 \end{cases} \tag{6.8}$$

Equation Eq. (6.6) is simply the harmonic oscillator equation, and has the solution

$$\widetilde{U}(u,v,z) = A(u,v)\mathrm{e}^{i\omega z} + B(u,v)\mathrm{e}^{-i\omega z}, \tag{6.9}$$

where A and B are functions that characterize the behavior of a given wavefield. The general solution of the Helmholtz equation may be written in the form

$$U(x,y,z) = \iint_{-\infty}^{\infty} A(u,v)\mathrm{e}^{i(ux+vy+\omega z)} \mathrm{d}u\mathrm{d}v + \iint_{-\infty}^{\infty} B(u,v)\mathrm{e}^{i(ux+vy-\omega z)} \mathrm{d}u\mathrm{d}v. \tag{6.10}$$

This formula represents the solution to the Helmholtz equation as a superposition of four types of plane waves. These types are:

1. $\mathrm{e}^{i(ux+vy+\omega z)}$, with $u^2 + v^2 \le k^2$. These are (homogeneous) plane waves that propagate in the positive z-direction.

2. $e^{i(ux+vy+\omega z)}$, with $u^2+v^2>k^2$. Because $\omega = i(k^2-u^2-v^2)^{1/2}$ the z-component of this wave is exponentially decaying in the positive z-direction, i.e. $e^{i\omega z} = e^{-|\omega|z}$ This is referred to an inhomogeneous plane wave or evanescent wave.

3. $e^{i(ux+vy-\omega z)}$, with $u^2+v^2 \leq k^2$. These are (homogeneous) plane waves that propagate in the megative z-direction.

4. $e^{i(ux+vy+\omega z)}$, with $u^2+v^2>k^2$. This is also an evanescent wave, but one whose z-component is exponentially decaying in the negative z-direction, i.e. $e^{i\omega z} = e^{|\omega|z}$

Equation (6.10) represents the decomposition of an arbitrary field into a collection of (homogeneous and inhomogeneous) plane waves, and is referred to as an angular spectrum representation of the wavefield.

Why is it called an "angular spectrum"? The direction of a particular plane wave is completely specified by the values of u and v. These coordinates are equivalent to specifying the angle at which the plane wave is propagating; hence, angular spectrum.

Physically, the first two classes consist of waves propagating in the positive z-direction, while the last two classes are waves propagating in the negative z-direction. If we are interested only in waves which are diffracted from the plane $z=0$ into the positive halfspace, we may set $B(u,v)=0$. We are then left with

$$U(x,y,z) = \iint_{-\infty}^{\infty} A(u,v)e^{i(ux+vy+\omega z)}\mathrm{d}u\mathrm{d}v, \tag{6.11}$$

The wavefield propagating into a half-space can be represented as a sum of homogeneous and inhomogeneous plane waves propagating in the positive z-direction. Because $u^2+v^2=k^2$ is the boundary between homogeneous and inhomogeneous waves, it is useful to write the angular spectrum representation using slightly different variables,

$$U(x,y,z) = \iint_{-\infty}^{\infty} a(p,q)e^{ik(px+qy+mz)}\mathrm{d}p\mathrm{d}q, \tag{6.12}$$

where $u=kp$, $v=kq$, $w=km$, $a(p,q)=kA(u,v)$, and

$$m = \begin{cases} (1-p^2-q^2)^{1/2} & \text{when } p^2+q^2 \leq k^2 \\ i(p^2+q^2-1^2)^{1/2} & \text{when } p^2+q^2 > k^2 \end{cases} \tag{6.13}$$

The field $U(x,y,0)$ in the plane $z=0$ and the spectral amplitude $a(p,q)$ are related simply by a two-dimensional Fourier transform. First we define

$$\widetilde{U}_0(u,v) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} U(x,y,0)\mathrm{e}^{-i(ux+vy)}\mathrm{d}x\mathrm{d}y, \tag{6.14}$$

From the angular spectrum representation Eq. (6.12), we immediately get an equation for the field at $z=0$,

$$U(x,y,0) = \iint_{-\infty}^{\infty} a(p,q)\mathrm{e}^{ik(px+qy)}\mathrm{d}p\mathrm{d}q, \tag{6.15}$$

If we plug this formula into the Fourier transform formula above, and use the relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i(u-u')x}\mathrm{d}x = \delta(u-u'), \tag{6.16}$$

we immediately find that

$$\widetilde{U}_0(u,v) = \frac{1}{k^2}a\left(\frac{u}{k},\frac{v}{k}\right), \tag{6.17}$$

or

$$a(p,q) = k^2\widetilde{U}_0(kp,kq). \tag{6.18}$$

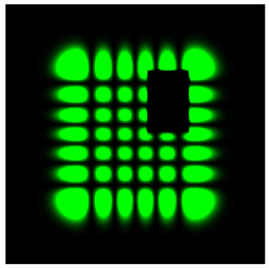
This tells us that the spectral amplitude of each plane wave mode of the angular spectrum representation is specified by a single Fourier component of the boundary value of the field in the plane $z=0$.

It is to be noted that the coordinates u, v are typically referred to as spatial frequencies, as they represent the rate of spatial variation of their Fourier component.

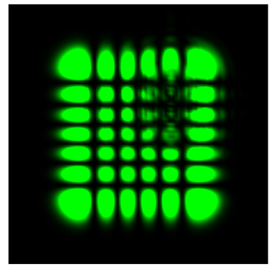
This plane wave representation of an arbitrary field is incredibly useful in optical physics problems because the evolution of a plane wave through a system can often be calculated in a straightforward manner. For instance, exact formulas exist for the reflection and refraction of plane waves through stratified media.

6.6.2 Obstructions in Beams and Self-Healing

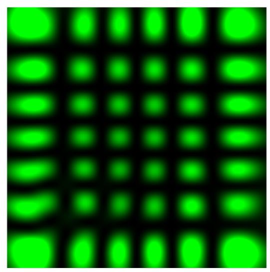
we will propagate in a simulation a Hermite-Gauss beam with a obstruction like the next



(a) Hermite-Gauss Beam $m = 2$, $n = 3$ with obstruction in $z = 0$



(b) Hermite-Gauss Beam $m = 2$, $n = 3$ with obstruction in $z = 5x10^{-10}m$



(c) Hermite-Gauss Beam $m = 2$, $n = 3$ with obstruction in $z = 62x10^{-10}m$

Figure 6.1: Propagating Hermite Beam $m = 2$, $n = 3$ with obstruction

we can observe that the Hermmite-Gauss Beam tries self-hilling itself.

We remember that the Hermite-Gauss Beam that comes of two solutions for build the standing "wave" like in a string, then for propagating we need both solutions for explain the self-hilling of the beam, this because the part of each solution tries to make up for original beam.

7 Conclusions

In this work we showed the importance of having the two solutions of the differential equation of Hermite, because when we study a field in the cavity, is the analogous problem when we study a string held by both sides [1], and in this case we should have two solutions, a travelling wave that moves to the right, and travelling wave that moves to the left, so in more dimensions we need two solutions because we have a second order differential equation.

Beams are generated in cavities, and they we could have more degrees of freedom than one dimension, and different geometries, so it is therefore, that is very important study the curvilinear coordinate system, in this work we introduce in detail the polar coordinates and elliptic coordinates, and each of its parts.

With the elliptic coordinates (with a little difference) we can connect the Mathieu Differential Equation with Weber Differential Equation, and see the differences with their solutions, while in Hermite(Weber) case the second solution diverges the Mathieu case not, so it's is very important to considers if we make the paraxial approximation in the differential equation or solutions, because this is not the same, as we proved.

Having the two solutions of the Hermite Differential Equation, Mathieu Differential Equation, or some second order diferential equation, in one or more dimensions, we can explain why the beams tries self-hilling they self, this is because in the standing wave we have two travelling waves, so these cases is the analogous case to the string but in more dimensions and different geometries.

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