

Instituto Nacional de Astrofísica, Óptica y Electrónica

## ANALYSIS OF SELF-IMAGING AND VORTEX WITH SURFACE PLASMON FIELDS

by

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## Abstract

In this thesis we find a general expression for the surface plasmon modes following a treatment analogous to diffraction free beams, the main result being that the expression for the dispersion relation functions is generalized. This feature allows us the coupling between surface plasmon waves with other kinds of optical fields propagating in homogeneous media such as Bessel beams. We obtain the surface mode solution for the Helmholtz equation using an analog formalism for diffracted free beams. By means of a linear superposition of the surface modes we obtain an expression similar to the angular spectrum model which allows us the study of arbitrary surface diffraction fields. For the understanding of the physical features implicit in this representation, we describe the interaction between two counter-propagating surface modes, generating a standing plasmonic wave, whose nodes then generate a stationary local charge distribution. The study is reinforced by associating extremal features to the surface modes and an eikonal equation for plasmonic fields is obtained, where a plasmonic refractive index appears naturally. This representation allows us to interpret the surface optical field as a geodesic flow which in principle enables us to associate coherence features to plasmon modes and to analyze the stability of the surface fields under small perturbations of the refractive index. We study theoretically the surface plasmon electromagnetic field in the plane of the interface along which it propagates. Arbitrary surface plasmon fields are expressed by means of a linear superposition of elementary surface plasmon modes, thus obtaining an expression for the in-plane components similar to the angular spectrum model, which establishes the formal foundations of a twodimensional surface plasmon optics. From this representation, we obtain the general description for surface plasmon modes as in-plane diffraction free beams. These new modes with their corresponding phase parameters are used to study surface plasmon self-imaging phenomenon and the synthesis of surface plasmon singularity regions (caustics) of surface plasmon fields. Finally, we show the radiometric features for scalar optical fields by means of the spectral density function and the spectral flux function, this model is used to describe the
radiometric features of surface plasmon fields. Hence, we show that the singularities for the phase functions allow us to describe the angular transfer moment in the neighborhood of the focusing regions. We show that the differential equation associated to the singularities remains non-variable under linear transformations which may represent scaling or rotating. These features are the support for the study of dynamical surface plasmon behavior which explains the conditions to generate surface plasmon vortex.

## Resumen

En esta tesis encontramos una expresión general para los modos plasmonicos de superficie siguiendo un tratamiento análogo para haces libres de difracción, el resultado principal es la expresión para la función de relación de dispersión que es generalizada, de modo que con esta característica nos deja el acoplamiento entre ondas plasmonicas de superficie con otros tipos de campos plasmonicos propagándose en un medio homogéneo tales como los haces Bessel. Se obtuvo la solución modal de superficie para la ecuación de Helmholtz utilizando un formalismo análogo para haces libres de difracción. Mediante una superposición lineal de los modos de superficie obtuvimos una solución similar al modelo del espectro angular, el cual nos permite el estudio de campos arbitrarios de difracción de superficie, así como la compresión de las características físicas implícitas en esta representación. Describimos la interacción entre dos modos de superficie contra propagándose que generan una onda plasmonica estacionaria, cuyos nodos producen una distribución local de carga estacionaria: El estudio es reforzado por las características extrémales para los modos de superficie y una ecuación eikonal es obtenida para campos plasmonicos, donde una índice de refracción plasmonico aparece, esta representación nos permite interpretar el campo óptico de superficie como un flujo geodésico que en principio; nos deja asociar características de coherencia, para modos plamonicos y analizar la estabilidad de campos plasmonicos de superficie, bajo pequeñas perturbaciones del índice de refracción. Se estudio teoréticamente el campo electromagnético plasmonico de superficie en el plano a lo largo de la interface en cual se propaga. Los campos arbitrarios plasmonicos de superficie son expresados mediante una superposición lineal de modos plasmonicos de superficie, así obteniendo una expresión similar para las componentes en el plano al modelo del espectro angular, que establece la formación formal de una óptica de plasmones de superficie en dos dimensiones, desde está representación se obtuvo la descripción general para modos plasmonicos de superficie como haces libres de difracción en el plano, estos nuevos modos con sus correspondientes parámetros de fase son
utilizados para estudiar fenómenos de auto imágenes y síntesis de regiones singulares (Causticas) de campos plasmonicos de superficie. Finalmente, se muestra las características radiométricas para campos ópticos escalares mediante la función de densidad espectral y la función de flujo espectral, este modelo es utilizado para describir las características radiométricas de campos plasmonicos de superficie. Por lo tanto, se mostro que las singularidades para la función de fase nos permite describir el momento de transferencia angular en la vecindad de regiones de focalización, se muestra también que la ecuación diferencial asociada a la permanencia de singularidades bajo una transformación lineal no cambia y que puede representar una escalamiento o rotación, estas características son el apoyo para el estudio del comportamiento dinámico de plasmones de superficie, que explica las condiciones para generar vorticidad plasmonica.

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## CHAPTER 1

## 1 INTRODUCTION

The study of contemporary optics such as left-hand materials, atom trapping, etc., implies a two-dimensional analysis of the electromagnetic field propagating on a metal surface. In this context the absorption phenomena plays an important role and the application of the traditional models is not feasible. One alternative approach for avoiding absorption is the description of resonant phenomenon between surface plasmon waves where the structural features characterized by the dispersion relation function is modified. These features offer interesting applications in nano-optics and explain the evolution of light through left hand materials or meta-materials. The fundamental structure to generate resonant effects are the surface plasmons which are waves generated by collective oscillation of surface charges. The elementary surface plasmons waves were predicted by Raether almost twenty five years ago, these waves can be considered analogous to plane waves for homogeneous media. The structural properties of these surface plasmon waves are defined by the phase function and their properties are characterized by the dispersion relation function which essentially describes the coupling between light propagating in the space and the surface
plasmon mode, i.e. this function represents the conservation law of momentum between the optical fields as a function of the frequency. In the present thesis work we find a general expression for surface plasmon modes following a treatment analogous to diffraction free beams, the main result being the expression for a general dispersion relation function. This feature allows us the coupling between surface plasmon waves with modes propagating in homogeneous media such as Bessel beams.

The current theory for surface plasmon waves has been obtained essentially from solid state models; this point of view has the inconvenient that classical optical models are not easy to be implemented in the analysis of the surface plasmon fields. In the present thesis we establish a new point of view. We start the study from the Helmholtz equation following the formalism for diffraction free beams and the dispersion relation function is obtained [1-4]. The refractive index is obtained following the theoretical model proposed by Drude (see Appendix 2). This theory considers the conduction electrons of a metal as a homogenous gas of electrons which are surrounding a positive uniform potential generated by immobile positive charges of red-crystal ions. The most interesting aspect of the Drude model allows us to predict acceptably the electrical and thermal conductivity of the metals; hence it predicts the possibility to generate surface plasmons waves.

As an introduction, we describe the analysis of the conductivity and the possibility to generate surface plasmons as is shown below. To calculate a current on a metal generated by a variable electric field, we use:

$$
\begin{equation*}
E(t)=E(\omega) \exp (-i \omega t) \tag{1}
\end{equation*}
$$

where $E(\omega)$ represents the amplitude field as a function of the frequency $\omega$. The motion equation for a free electron is represented by:

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}=-\frac{1}{\tau} \frac{d x}{d t}-\omega_{0}^{2} x+\frac{e E}{m} \quad * \tag{2}
\end{equation*}
$$

where $e$ is the electron charge and $\tau$ is the average time between successive collisions. We propose a stationary solution of the form:

$$
\begin{equation*}
x(t)=x(\omega) \exp (-i \omega t) \tag{3}
\end{equation*}
$$

Replacing the complex forms $x(t)$ and $E(t)$ in equation (2), which must be satisfied for both real and complex parts, we find:

$$
\begin{equation*}
\omega^{2} \cdot x(\omega)=\frac{i \omega x(\omega)}{\tau}-\omega_{0}^{2} x-\frac{e E(\omega)}{m} \tag{4}
\end{equation*}
$$

Considering a unitary volume of the metal, which contains $n$ free electrons, each of charge $-e$, the total charge will be $-n e$. The total charge crossing a unit area per unit time is $j=-n e v_{d}$, this is the definition of current density and $v_{d}$ is the drift velocity. Therefore, by using the expression for current density we obtain:

$$
\begin{equation*}
j(t)=j(\omega) \exp (-i \omega t) \tag{5a}
\end{equation*}
$$

And

$$
\begin{equation*}
j(\omega)=n e v_{d}=\frac{\left(n e^{2} / m\right) E(\omega)}{1 / \tau-i \omega} \tag{5b}
\end{equation*}
$$

This result is commonly re-written as:

[^0]\[

$$
\begin{equation*}
j(\omega)=\sigma(\omega) E(\omega) \tag{6}
\end{equation*}
$$

\]

where $\sigma(\omega)$ is the alternating conductivity of the material; it depends on the frequency of the electrical field by:

$$
\begin{gather*}
\sigma(\omega)=\frac{\sigma_{0}}{1-i \omega \tau}  \tag{7a}\\
\sigma_{0}=\frac{n e^{2} \tau}{m} \tag{7b}
\end{gather*}
$$

The most important application concerning this result is the study of the propagation of electromagnetic radiation on the metal, however in the Drude model some questions concerning the structure of the electrical field must be considered:

1) For an electromagnetic wave, the electric field $E$ has an associated perpendicular magnetic field $H$ both of them have the same magnitude and they are not included in the motion equation (2).
2) For an electromagnetic wave, the fields vary not only with time but also in the space while the motion equation (2) was concluded by assuming a spatially uniform power (see Appendix 1).

The first obstacle can be not to know why the magnetic field includes a term which is $-e p(\omega) / m c \times H^{*}$ which is $v / c$ times smaller than the electric E term, where $v$ is the average velocity of the electrons. Even for a high density current like $1 \mathrm{~A} / \mathrm{mm}, v=j / n e$ is only $0.1 \mathrm{~cm} / \mathrm{s}$. Hence, the magnetic term is approximately $10^{-10}$ that of the electric term, so it can always be ignored.

[^1]The second difficulty has deep physical implications, the equation of movement for the electrons given by equation (2) implies that at any moment the same power value is on each electron, in general it is not true. This is because the electric field varies with the position. This implies that the current density is function of the position $\mathbf{r}$, the distance must be minor than the average distance between collisions. The collision only takes place after a few free-mean paths from the point $\mathbf{r}$. Hence, if the field is approximately constant over the same distance, then $j(r, t)$ can be correctly calculated by considering that the field in all space is given by the value at the point $\mathbf{r}, E(r, t)$; thus, one obtains:

$$
\begin{equation*}
j(r, \omega)=\sigma(\omega) E(r, \omega) \tag{8}
\end{equation*}
$$

This equation is accepted provided that the wavelength of the field is larger than the electronic free-mean path $l$, so for metals, this is commonly accepted for the visible interval ( $\lambda$ approximately $\left.10^{3} \mathrm{~nm}\right)^{*}$. Next, we assume that the wavelength is larger than the free-mean path. Thus, the Maxwell equations are written by considering a current density $\boldsymbol{j}$ in the form!:

$$
\begin{gather*}
\nabla \cdot D=0  \tag{9}\\
\nabla \cdot B=0  \tag{10}\\
\nabla \times E+\frac{1}{c} \frac{\partial B}{\partial t}=0  \tag{11}\\
\nabla \times H+\frac{1}{c} \frac{\partial D}{\partial t}=\frac{4 \pi}{c} j \tag{12}
\end{gather*}
$$

One seeks a time dependent solution $\exp (-i \omega t)$. According to equation (6) on the

[^2]metal, $\boldsymbol{j}$ can be written by terms of $E$ whereupon by combining with the Maxwell equations, we obtain:
\[

$$
\begin{equation*}
\nabla \times(\nabla \times E)=-\nabla^{2} E=\frac{i \omega}{c} \nabla \times H=\frac{i \omega}{c}\left(\frac{4 \pi}{c} E-\frac{i \omega}{c} E\right) \tag{13}
\end{equation*}
$$

\]

so one finds

$$
\begin{equation*}
-\nabla^{2} E=\frac{\omega^{2}}{c^{2}}\left(1-\frac{4 \pi i \sigma}{\omega}\right) E \tag{14}
\end{equation*}
$$

This equation has the usual form of a wave equation:

$$
\begin{equation*}
-\nabla^{2} E=\frac{\omega^{2}}{c^{2}} \varepsilon(\omega) E \tag{15}
\end{equation*}
$$

The complex dielectric function is expressed as:

$$
\begin{equation*}
\varepsilon(\omega)=1-\frac{4 \pi i \sigma}{c} \tag{16}
\end{equation*}
$$

At sufficiently high frequencies such that $\omega t \gg 1$ the combination of (16) and (7) gives as a result in first approximation:

$$
\begin{equation*}
\varepsilon(\omega)=1-\frac{\omega_{p}^{2}}{\omega^{2}} \tag{17}
\end{equation*}
$$

where $\omega_{p}$ is known as the plasma frequency and is given by:

$$
\begin{equation*}
\omega_{p}^{2}=\frac{4 \pi n e^{2}}{m} \tag{18}
\end{equation*}
$$

One important deduction of equation (17) is that a gas of electrons can support oscillations of charge density, i.e., a perturbation where the electric charge density has a temporal dependence $\exp (-i \omega t)$. From the continuity equation:

$$
\begin{align*}
& \nabla \cdot j=-\frac{\partial \rho}{\partial t}  \tag{19a}\\
& \nabla \cdot j(\omega)=i \omega \rho(\omega) \tag{19b}
\end{align*}
$$

and applying the Gauss law we have

$$
\begin{equation*}
\nabla \cdot E(\omega)=4 \pi \rho(\omega) \tag{20}
\end{equation*}
$$

By considering equation (8), the density of charge is

$$
\begin{equation*}
i \omega \rho(\omega)=4 \pi \sigma(\omega) \rho(\omega) \tag{21}
\end{equation*}
$$

This equation has a solution given by:

$$
\begin{equation*}
1+\frac{4 \pi i \sigma(\omega)}{\omega}=0 \tag{22}
\end{equation*}
$$

Equation (22) is essentially the condition previously found by the propagation of radiation. In this case, it represents the condition which the frequency must satisfy for a charge density wave to be propagated on the surface media.

In fact, the electromagnetic field of these oscillations of charge density is known as plasma or plasmon oscillations where the oscillations can be understood by terms of a simple geometric model, i.e. one can imagine that gas of electrons is moved by a distance $d$ with respect to the stable positive ion. Therefore, the produced surface charge provides an electric field dimension with $4 \pi \rho^{*}$ where $\rho^{*}$ is the charge per area unit in each extreme. (See figure 1.1)


FIGURE 1.1. Model for understanding the oscillations of plasmons or plasmas

Then, gas contains electrons ( $\mathbf{N}$ ) and it satisfies the motion equation by:

$$
\begin{equation*}
N m \ddot{d}=-N e\left|4 \pi \rho^{*}\right|=-N e(4 \pi n e d)=-4 \pi n e^{2} N d \tag{23}
\end{equation*}
$$

This equation shows an oscillation related to the plasma frequency.

### 1.2 SURFACE PLASMON INTERFACE

To improve our understanding of plasma waves, we consider an example. We analyze the characteristics and excitation methods for the surface plasmon waves propagating at an interface using an electromagnetic field generated by the
illumination with a light beam. The interface is placed on the $x y$ plane and it is divided into two semi-infinity spaces of different materials 1 and 2 , of which optical properties are characterized by its complex dielectric constants $\varepsilon_{1}(\omega)$ and $\varepsilon_{2}(\omega)$ respectively, the geometric is shown schematically in figure 2.1.


FIGURE 1.2 System of reference at an interface between two media with dielectric constants $\varepsilon_{1}$ and $\varepsilon_{2}$

In this development the magnetic materials are discarded, i.e. we take $\mu=1$ in all processes. The surface plasmons can only be excited at the interface if the electric field has a normal component at the surface which induces a surface charge $\rho$. Hence, for it to be possible, the electric displacement must be:

$$
\begin{equation*}
z \cdot\left(D_{2}-D_{1}\right)=4 \pi \rho^{*} \tag{24}
\end{equation*}
$$

S-polarized light is propagated along x -axis and has parallel components of electric field at the y-axis, i.e. a TE wave has an electric field $E=\left(0, E_{y}, 0\right)$; therefore, it is unable to generate surface plasmons. Thus, surface plasmons can only be exited by P-polarized light, that is, TM waves with electric field $E=\left(E_{x}, 0, E_{z}\right)$ and $H=\left(0, H_{y}, 0\right)$, so the surface electromagnetic waves are given by:

$$
\begin{equation*}
E_{1}=E_{10} \exp \left(i\left(k_{1} x-\omega t\right)\right)=E_{10} \exp \left(i\left(k_{x 1} x+k_{z 1} z-\omega t\right)\right) \tag{25a}
\end{equation*}
$$

$$
\begin{equation*}
E_{2}=E_{20} \exp \left(i\left(k_{2} x-\omega t\right)\right)=E_{20} \exp \left(i\left(k_{x 2} x+k_{22} z-\omega t\right)\right) \tag{25b}
\end{equation*}
$$

where $E_{1,2}$ represent the electric vector $\boldsymbol{E}$ and the magnetic vector is $\boldsymbol{H}, \boldsymbol{k}_{x 1, x_{2}}$ are the wave vector components along the $x$-axis into media 1 and $2, k_{z 1, z 2}$ are the absolute values of the wave vector components along the z-axis and $\omega$ is the angular frequency. All the components must satisfy the Maxwell equations (9-12). The boundary conditions are written in the form:

$$
\begin{equation*}
E_{1 x}=E_{2 x} \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{1 y}=H_{2 y} \tag{27}
\end{equation*}
$$

Equation (26) is obtained by $k_{x 1}=k_{x 2}=k_{x}$, in addition, if we use Maxwell's equation (12) with $j=0$, because there is no charge flux, the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields given by equation (25), take the form:

$$
\begin{align*}
k_{1 z} H_{y} & =\frac{\omega}{c} \varepsilon_{1} E_{x}  \tag{28}\\
k_{x} H_{y} & =-\frac{\omega}{c} \varepsilon_{1} E_{1 z}  \tag{29}\\
k_{2 z} H_{y} & =-\frac{\omega}{c} \varepsilon_{2} E_{x} \tag{30}
\end{align*}
$$

$$
\begin{equation*}
k_{x} H_{y}=-\frac{\omega}{c} \varepsilon_{2} E_{2 z} \tag{31}
\end{equation*}
$$

Equations (28) and (30) yield the non-trivial solution:

$$
\begin{equation*}
\frac{k_{1 z}}{k_{2 z}}=-\frac{\varepsilon_{1}}{\varepsilon_{2}} \tag{32}
\end{equation*}
$$

By applying Maxwell's equation (14) to $\boldsymbol{E}$ and $\boldsymbol{H}$ fields, one obtains:

$$
\begin{align*}
& -k_{x} E_{1 z}+k_{1 z} E_{x}=\frac{\omega}{c} H_{y}  \tag{33}\\
& -k_{x} E_{2 z}+k_{2 z} E_{x}=\frac{\omega}{c} H_{y} \tag{34}
\end{align*}
$$

Equation (30) shows that the surface plasmons can only be exited on the interface between two media with different dielectric constants of opposite signs. Inside of certain limits, it can be the case of phonons or excitons. The coupling of these conditions up to an electromagnetic field respectively produces the named phonon surface polariton or exciton surface polariton.

We are interested in the coupling of an electromagnetic field with the collective oscillations of plasma for the conduction electrons of a metal at the interface between a metal with dielectric constant $\varepsilon_{m}=\varepsilon_{m}^{\prime}+i \varepsilon_{m}^{\prime}$ and a dielectric with dielectric constant $\varepsilon_{d}=\varepsilon_{d}^{\prime}+i \varepsilon_{d}^{\prime}$. These phenomena were previously named surface plasmon polaritons or surface plasmons.

We will use the subscripts $m$ instead of 1 and $d$ instead of 2 for specifying a dielectric and a metallic medium. By manipulating equations (30), (31) and (34), we have:

$$
\begin{align*}
& k_{x}^{2}+{k_{d z}}^{2}=\left(\frac{\omega}{c}\right)^{2} \varepsilon_{d}  \tag{35}\\
& {k_{x}^{2}}^{2}+{k_{m z}}^{2}=\left(\frac{\omega}{c}\right)^{2} \varepsilon_{m} \tag{36}
\end{align*}
$$

Obtaining

$$
\begin{align*}
& k_{d z}=\sqrt{\varepsilon_{d}\left(\frac{\omega}{c}\right)^{2}-k_{x}^{2}}  \tag{37}\\
& k_{m z}=\sqrt{\varepsilon_{m}\left(\frac{\omega}{c}\right)^{2}-k_{x}^{2}} \tag{38}
\end{align*}
$$

Thus, if we use equation (32), we obtain the dispersion relation of the surface plasmons in a metal/dielectric interface. Then, one obtains:

$$
\begin{equation*}
k_{x}=\frac{\omega}{c} \sqrt{\frac{\varepsilon_{m} \cdot \varepsilon_{d}}{\varepsilon_{m}+\varepsilon_{d}}} \tag{39}
\end{equation*}
$$

At this point is convenient to emphasize some details:
1.- Frequently we only treat the real part of $\omega$, in general, $\varepsilon_{m}$ is complex, and $k_{x}$ is complex too; i.e. $k_{x}=k_{x}^{\prime}+i k_{x}^{\prime \prime}$. Therefore, the surface plasmons, which
propagate at a metal/dielectric interface, show up a finite propagation length $L_{x}$, that is $L_{x}=1 / k_{x}^{\prime \prime}$, this decrease has a great importance since it determines the length of propagation of the plasmon waves.
2.- In the spectral interval of interest, one has:

$$
\begin{equation*}
\frac{\varepsilon_{m} \cdot \varepsilon_{d}}{\varepsilon_{m}+\varepsilon_{d}} \geq \sqrt{\varepsilon_{d}} \tag{40}
\end{equation*}
$$

This has two important consequences. First, equation (38) is deduced, so if we insert equation (40); then, one obtains a z-component for the wave vector in the dielectric that is purely imaginary, from equations (25a, 25b) we can see that it means that the surface plasmons establish a nonradiative evanescent wave on the interface and its amplitude which is maximum for $\mathrm{z}=0$ exponentially decreases into the dielectric [6], as represented in figure 1.3.

The skin depth for metals is around nanometers, this detail is important for the characterization of the surface nanostructure.


FIGURE 1.3. Evanescent electric field associated to surface plasmons at a metal/dielectric interface.

The second point concerning the equation (40) is that the wave vector of a freephoton which is propagating on the dielectric is given by:

$$
\begin{equation*}
k_{f}=\frac{\omega}{c} \sqrt{\varepsilon_{d}} \tag{41}
\end{equation*}
$$

This is always smaller than the wave vector of a surface plasmon $k_{s p}$ which is propagated at an interface between dielectric and metal, as shown in figure 1.4.


FIGURE 1.4 X component of the wave vectors of a incident photon on the dielectric where the surface plasmons are propagated by the dielectric/metal

### 1.3 SURFACE PLASMON EXCITATION

For the coupling of light with the surface plasmon only the x -axis component of the wave vector is of interest. This means that using incident light on the flat interface and by varying the incident angle one can obtain $k_{f x}=k_{f} \cdot \sin \theta_{i}$. (see figure 1.4); it shows us that is not possible to generate a surface plasmon since the wave vector is smaller than that of the surface plasmon, as shown schematically in figure 1.5. Thus, we can see that the plasmon dispersion curve is always below the
dispersion curve of the photon which propagates in the dielectric. Therefore, it is impossible for this light to be coupled to the surface plasmons.


FIGURE 1.5 Dispersion relation for incident photons into a dielectric $d$ (dark line), a prism $p$ of refraction index higher (dot line) and surface plasmon at an interface between a metal and dielectric $d$.

To solve this point, we consider the Kretschmann configuration [7] (Other coupling methods exist too, as example Otto method [8] and Gratings coupling [1]), which allows the coupling of light with plasmon waves. This consists in illuminating a dielectric prism $p$ of higher refraction index $\left(\sqrt[\varepsilon_{p}]{>} \sqrt{\varepsilon_{d}}\right)$, so the dispersion curve of the photons, is shown by the dot-line in figure 1.5. The wave vectors of light are for some frequencies bigger than the wave vectors of the surface plasmon, so that the photons can excite a surface plasmon if they illuminate at a correct angle, as exposed in figure 1.6.


FIGURE 1.6 Component $x$ of the wave vectors of a incident photon up to prism and the surface plasmons are propagated between metal and dielectric of refraction index smaller than the prism.

The resonant coupling can be measured as a function of angle versus the reflection intensity. We check a minimum in the reflectivity for an incident angle, so that one verifies the resonant condition.

$$
\begin{equation*}
k_{s p x}=\frac{\omega}{c} \sqrt{\frac{\varepsilon_{m} \cdot \varepsilon_{d}}{\varepsilon_{m}+\varepsilon_{d}}}=\frac{\omega}{c} \sqrt{\varepsilon_{p} \sin \theta_{s p}=k_{f x}} \tag{4}
\end{equation*}
$$

The resonant condition of equation (42) can be calculated by the incident angle associated with the coupling of the incident light with the surface plasmons, so we obtain:

$$
\begin{equation*}
\theta_{s p}=\arcsin \sqrt{\frac{\varepsilon_{m} \cdot \varepsilon_{d}}{\left(\varepsilon_{m}+\varepsilon_{d}\right) \cdot \varepsilon_{p}}} \tag{43}
\end{equation*}
$$

### 1.4 SYSTEM LAYERS WITH SURFACE PLASMON

In previous sections we showed how the surface plasmons waves may be obtained and we showed that the surface electromagnetic field presents a maximum of intensity at the surface and decreases exponentially inside the dielectric/metal media, so the decreasing amplitude depends on the wavelength used also as the optical features of the metal for different frequency wavelengths (at the visible spectrum and almost IR approximately 630 nm , He-Neon laser) and for thin metal films with values around 200nm. Because of this, any changes on the optic properties of the dielectric films around 200 nm , the conditions excitement for surface plasmons changes. This characteristic allows us to detect the absorption of a film new by means of controlling of resonant angle.

Now, we suppose the system prism/metal/dielectric and insert a new dielectric film on the metal. Taking this into account, we imagine that the new dielectric has a dielectric constant $\varepsilon_{a}$ bigger than the original dielectric medium $\left(\varepsilon_{a}>\varepsilon_{d}\right)$. In this case, the effective refraction index is higher and related with the surface plasmons, so in a similar way (see figure 1.5), the dispersion curve is moved by the surface plasmon up to higher wave vectors, as represented in figure 1.7, the curve $\omega_{s p}$ is the dispersion relation of the surface plasmon in the normal system and the curve $\omega_{s p}$ is the system in which was included the new dielectric layer.

Therefore, for a specified frequency the wave vector of surface plasmons will be higher and, of course, we will need a high incidence angle of for excitement the modes. Therefore, one concludes that the condition for excitement of surface plasmons depends on the effective refraction index of the materials deposited on the metal with thickness around 200 nm . Hence, if a dielectric layer is deposited on the metal, one can calculate the refraction index as a function of the film thickness.


FIGURE 1.7 Dispersion relations for incident photons up to a dielectric $d$ (dark line), a prism p of refraction index higher (dot line), surface plasmon at a interface between a metal $m$ and dielectric $d$ (dark curve $\omega_{s p}$ ) and plasmons at a interface whit a layer more (dot curve $\omega_{s p^{\prime}}$ )

The reflectivity of system layers can be theoretically calculated by the dielectric constants and determined on each layer. These calculations, about reflectivity, can be related to the amplitude/irradiance distribution fields in each media/layer. Hence, the mathematical development can be done using the section 1.3.

### 1.5 OBJETIVES

The objective of the present thesis is to find the general expression for surface plasmon waves, and its applications for the synthesis of self-imaging surface plasmon fields and the generation of surface plasmon focusing. In this last topic, we will show the possibility of generating dynamical features by means of the generation of surface plasmon vortices.

The structure of the thesis is as follows:

In chapter two, we consider the description of surface plasmon fields starting from Helmholtz's equation establishing an analog formalism for diffracting free beams.

In chapter three we shall comment on the surface wave mode for the description of plasmons at the interface, by obtaining the refraction index for interfering plasmons fields also as the description of general plasmon modes.

In chapter four, we shall describe surface plasmon (SP) self-imaging fields where the analysis is performed at the spatial frequency and the obtained condition is matched by Montgomery's condition for homogeneous media.

In chapter five we describe the dynamics of surface plasmon SP and the analysis is applied to the description of vortex for surface plasmon.

Finally in chapter six, we shall present the general conclusions as well as the future research.

## CHAPTER 2

## Description of Surface Plasmon Fields

### 2.1 INTRODUCTION

We obtain the surface mode solution for the Helmholtz equation using formalism analogous to diffracted free beams. By means of a linear superposition of the surface modes we obtain an expression similar to the angular spectrum model which allows us the generation of general surface diffraction fields.

### 2.2 MODES THEORY

A propagation mode is a solution of the Helmholtz equation that we describe in the analysis by the form:

$$
\begin{equation*}
\nabla^{2} \phi+k^{2} \phi=0 \tag{2.1}
\end{equation*}
$$

However, the Helmholtz equation can be described in a reference system which is separable in the coordinates of propagation, without a loss of generality the coordinate is considered as the z-axis, defining the transversal Laplace operator:

$$
\begin{equation*}
\nabla_{\perp}^{2}=\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}} \tag{2.2}
\end{equation*}
$$

Equation (2.1) can be rewritten in the form:

$$
\begin{equation*}
\frac{\partial^{2} \phi}{\partial z^{2}}+\left(\nabla_{\perp}^{2}+k^{2}\right) \phi=0 \tag{2.3}
\end{equation*}
$$

Equation (2.3) is developed for coordinate z and by considering that equation (2.1) has one solution, expressed as:

$$
\begin{equation*}
\phi(x, y, z)=f(x, y) \exp (i z \hat{A}) \tag{2.4}
\end{equation*}
$$

where $\hat{A}$ is an operator that is defined as $\hat{A}=\sqrt{\nabla_{\perp}^{2}+k^{2}}$, and $f(x, y)$ is an arbitrary function. Hence, we can expand equation (2.4) in series given by:

$$
\begin{equation*}
\phi(x, y, z)=\sum_{n=0}^{\infty} \frac{(i z \hat{A})^{n}}{n!} f(x, y) \tag{2.5}
\end{equation*}
$$

Because of function $f(x, y)$, it can assign a solution group, such as an eigenvalue of operator $\hat{A}$, i.e. which satisfies:

$$
\begin{equation*}
\hat{A} f(x, y)=\nsim(x, y) \tag{2.6}
\end{equation*}
$$

by applying $n$ times the operator to the function $f(x, y)$, this yields:

$$
\begin{equation*}
\hat{A}^{n} f(x, y)=\gamma^{n} f(x, y) \tag{2.7}
\end{equation*}
$$

By obtaining as a result a proposed solution (2.5) that is:

$$
\begin{equation*}
\phi(x, y, z)=f(x, y) \exp (i z \gamma) \tag{2.8}
\end{equation*}
$$

where $\gamma$ is a constant and the function $f(x, y)$ describes the profile of the wave. Solutions of this type are known as propagation modes. The eigenvalue equation satisfies:

$$
\begin{equation*}
\nabla^{2} f(x, y)+\left(k^{2}-\gamma^{2}\right) f(x, y)=0 \tag{2.9}
\end{equation*}
$$

Therefore, we have a mode that is a solution of the Helmholtz equation whose profile does not change when it is propagating. In fact, the amplitude function satisfies a two-dimensional Helmholtz equation. This is the reason why mode solutions are known as non-diffracting beams. In general, the prototype solutions of the modes are the plane wave and Bessel waves [9], [10].

### 2.3 BEAMS THEORY

The modes are exact solutions of the Helmholtz equation where we assume that the complex amplitude of any monochromatic optical disturbance propagating in a homogeneous medium must obey such a relation. Since the Helmholtz equation is a homogeneous linear equation the general solution can be represented by a coherent superposition of plane waves. On the other hand, another type of expression representing the propagation light is by using paraxial equation. The paraxial wave equation is an intermediary between the simple concepts of rays
and plane waves and deeper concepts embodied in the wave equation; therefore, the solutions of the paraxial equation are known as beams [11].The Helmholtz equation is a linear equation, then the general solution can be expressed as a superposition of plane waves of the form

$$
\begin{equation*}
\phi(x, y, z)=\iint A(u, v) \exp (i 2 \pi(x u+y v+z \rho)) d u d v \tag{2.10}
\end{equation*}
$$

where $u, v$ and $\rho$ are parameters which must satisfy the following condition

$$
\begin{equation*}
u^{2}+v^{2}+\rho^{2}=1 / \lambda^{2} \tag{2.11}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
\rho=1 / \lambda \sqrt{1-\lambda^{2}\left(u^{2}+v^{2}\right)} \tag{2.12}
\end{equation*}
$$

By considering the condition $1 \gg \lambda^{2}\left(u^{2}+v^{2}\right)$ which is obtained from (2.12) and by using the Taylor series, it yields:

$$
\begin{equation*}
\rho=1 / \lambda-\lambda\left(u^{2}+v^{2}\right) / 2 \tag{2.13}
\end{equation*}
$$

Then the approximate solution to the Helmoholtz equation is

$$
\phi(x, y, z)=\iint A(u, v) \exp (i 2 \pi(x u+y v)) \exp \left(-i \pi \lambda \cdot z\left(u^{2}+v^{2}\right)\right) \exp (i k z) d u d v
$$

Hence, this equation is known as the paraxial solution and can be rewritten in the form:

$$
\begin{equation*}
\phi(x, y, z)=\exp (i k z) \xi(x, y, z) \tag{2.15}
\end{equation*}
$$

The first factor on the right hand represents a plane wave while the second factor is either the modulation function or the monochromatic plane wave amplitude along a coordinate z , which is represented by:

$$
\begin{equation*}
\xi(x, y, z)=\iint A(u, v) \exp \left(-i \pi \lambda z\left(u^{2}+v^{2}\right)\right) \exp (i 2 \pi(x u+y v)) d u d v \tag{2.16}
\end{equation*}
$$

where $A(u, v)$ is the considered beam function. Thus, the differential equation which satisfies the modulation function can be found by taking the Laplace equation from equation (2.15). $\xi(x, y, z)$ varies linearly along coordinate $z$. Thus, we obtain:

$$
\begin{equation*}
\partial \xi / \partial x^{2}+\partial^{2} \xi / \partial y^{2}+2 k i \partial \xi / \partial z=0 \tag{2.17}
\end{equation*}
$$

Equation (2.17) is known as the wave paraxial equation.

### 2.4 MODE EXTREMAL PROPERTIES



FIGURE 2.1 Representation of different electric fields on the surface.

Our notion of representing a mode is extended to flat metal surfaces, as illustrated in Fig. 2.1, where $E_{1}$ and $E_{2}$ are the amplitude fields, $\varepsilon_{1}$ and $\varepsilon_{2}$ are the media permittivity. Hence, we request a mode propagating on the plane surface. We consider that the field's amplitudes $E_{1}$ and $E_{2}$ are expressed by the form:

$$
\begin{align*}
& E_{1}(x, z)=\hat{C}_{1} f_{1}(x) \exp (i \beta z)  \tag{2.18a}\\
& E_{2}(x, z)=\hat{C}_{2} f_{2}(x) \exp (i \beta z) \tag{2.18b}
\end{align*}
$$

We suppose that $E_{1}$ and $E_{2}$ are the electric fields, $\hat{C}_{i}$ can be considered as a vector and $f_{k}(x)$ describes the wave profile whereas $\exp (i \beta z)$ is the propagation wave, where ( $k=1,2$ ). Hence, on the interface one obtains equations by the boundary conditions of the electric field. If the boundary is free of charges and currents, the boundary conditions for the electric field are:

$$
\begin{gather*}
E_{1 T}=E_{2 T}  \tag{2.19a}\\
\varepsilon_{1 n} E_{1 n}=\varepsilon_{2 n} E_{2 n} \tag{2.19b}
\end{gather*}
$$

$E_{k T}$ and $E_{k n}$ are tangential and normal components, $\varepsilon_{i n}$ is the media permittivity where ( $k=1,2$ ), $\hat{C}_{1,2}$ of equation (2.18) can be represented as a vector in two components by:

$$
\begin{equation*}
\hat{C}_{n}=\left(a_{n}, b_{n}\right) \tag{2.20}
\end{equation*}
$$

where $n=1,2$. So, equation (2.18) is expressed by:

$$
\begin{align*}
& E_{1}(x, z)=\left(\hat{i} a_{1}+\hat{k} b_{1}\right) f_{1}(x) \exp (i \beta z)  \tag{2.21a}\\
& E_{2}(x, z)=\left(\hat{i} a_{2}+\hat{k} b_{2}\right) f_{2}(x) \exp (i \beta z) \tag{2.21b}
\end{align*}
$$

where $a_{k}$ and $b_{k}$ are the amplitude components of the electric fields for different angles; for $k=1,2$. Thus, the boundary condition is described by equation (2.19) and is evaluated at the origin. By considering this and since the phase matching condition is $\beta_{1}=\beta_{2}$, one obtains:

$$
\begin{gather*}
b_{1} f_{1}(0)=b_{2} f_{2}(0)  \tag{2.22a}\\
\varepsilon_{1} a_{1} f_{1}(0)=\varepsilon_{2} a_{2} f_{2}(0) \tag{2.22b}
\end{gather*}
$$

By considering a perturbation on the surface of the form

$$
\begin{align*}
& f_{1}(x)=\exp \left(-\alpha_{1} x\right)  \tag{2.23a}\\
& f_{2}(x)=\exp \left(-\alpha_{2} x\right) \tag{2.23b}
\end{align*}
$$

The boundary conditions allow us to find a function relation between the parameters ( $a_{1}, a_{2}, b_{1}$.and.$b_{2}$ ), and by using the equations (2.21), (2.22) and (2.23), we have:

$$
\begin{equation*}
E_{1}(x, z)=\left(\hat{i} a_{1}+\hat{k} b_{1}\right) \exp \left(-\alpha_{1} x\right) \exp (i \beta z), x>0 \tag{2.24a}
\end{equation*}
$$

$$
\begin{equation*}
E_{2}(x, z)=\left(\hat{i} \frac{\varepsilon_{1}}{\varepsilon_{2}} a_{1}+\hat{k} b_{2}\right) \exp \left(-\alpha_{2} x\right) \exp (i \beta z), x<0 \tag{2.24b}
\end{equation*}
$$

Both representations are obtained when we illuminate a surface with P-polarized light and they described the propagation plasmon on the surface in each media. Now the charge-free media conditions and equations (2.24) must satisfy $\vec{\nabla} \cdot E_{1,2}=0$, given by:

$$
\begin{equation*}
\varepsilon_{1} \alpha_{2}=\varepsilon_{2} \alpha_{1} \tag{2.25}
\end{equation*}
$$

Equation (2.25) implies that the attenuation ratio is related by the electric fields in each media. To obtain the possible values for $\beta$ in equation (2.24) each component must satisfy the Helmholtz equation $\vec{\nabla}^{2} \cdot E_{1,2}+K^{2} E_{1,2}=0$.Thus, it follows:

$$
\begin{align*}
& K_{1}=\beta^{2}-\alpha_{1}^{2}  \tag{2.26a}\\
& K_{2}=\beta^{2}-\alpha_{2}^{2} \tag{2.26b}
\end{align*}
$$

Equations (2.25) and (2.26) can be explicitly resolved by the $\beta$ parameter. Therefore, it becomes

$$
\begin{equation*}
\beta=\frac{\omega}{c}\left(\frac{\varepsilon_{1 r} \varepsilon_{2 r}}{\varepsilon_{1 r}+\varepsilon_{2 r}}\right)^{1 / 2} \tag{2.27}
\end{equation*}
$$

where $\varepsilon_{i r}=\varepsilon_{i} / \varepsilon_{0}$ refers to the relative permittivity. It should be noted that the expression for $\beta$ represents the possible values for the phase parameters associated with surface waves which can correspond to evanescent waves if $\beta$ is complex and plasmon waves if $\beta$ is real. For the last case, the dependence of the
wave vector $\beta$ on the frequency of light is known as the dispersion relation and it is related with the plasma oscillation as was discussed by Raether [1].

### 2.5 ANGULAR SPECTRUM

Our idea is to use the mode solution to represent an arbitrary surface field. As a particular case and an example, we suppose two counter-propagating waves, as it is illustrated in Fig. 2.2


FIGURE 2.2 Two similar waves counter-propagating on the surface.

Thus, one may interpret the equation in two different electric fields by representing it in the vector form, which is given by:

$$
\begin{equation*}
E_{T}(x, z)=\left(\hat{i} a_{1}+\hat{k} b_{2}\right) \exp \left(-\alpha_{1} x\right) \exp (i \beta z)+\left(\hat{i} a_{1}-\hat{k} b_{2}\right) \exp \left(-\alpha_{1} x\right) \exp (-i \beta z) \tag{2.28}
\end{equation*}
$$

These waves generate spatial redistribution charges, so by performing the explicit calculus we obtain:

$$
\begin{equation*}
E_{T}(x, z)=\hat{i} 2 a_{1} \exp \left(-\alpha_{1} x\right) \cos (\beta . z)+\hat{k} 2 b_{2} i \exp \left(-\alpha_{1} x\right) \operatorname{sen}(\beta . z) \tag{2.29}
\end{equation*}
$$

In this case, when we evaluate it at the origin, (i.e. $z=0$ ), and by using the Maxwell equation

$$
\begin{equation*}
\nabla \cdot E_{T}=\rho / \varepsilon \tag{2.30}
\end{equation*}
$$

we have:

$$
\begin{equation*}
\rho=-2 \varepsilon\left(a_{1} \alpha_{1}-i b_{1} \beta\right) \exp \left(-\alpha_{1} x\right) \cos (\beta z) \tag{2.31}
\end{equation*}
$$

This equation shows that the electric fields produce a distribution charge and depends on the attenuation coefficient $\alpha_{1}$; however, we can use the modal solution for an analogous structure at angular spectrum, and by using equation (2.24).


FIGURE 2.3 Rotating the x -axis on the plane y -z only.

Thus, once the reference system is rotated along the x-axis, as shown Fig 2.3 the expression for the surface wave becomes:

$$
\begin{align*}
& E_{1}(x, y, z)=\left(\hat{i} a_{1}+\hat{j} b_{2} \cos \theta+\hat{k} b_{3} \operatorname{sen} \theta\right) \exp \left(-\alpha_{1} x\right) e^{i \beta(\operatorname{sen} \theta+y \cos \theta)}  \tag{2.32a}\\
& E_{2}(x, y, z)=\left(\hat{i} \frac{\varepsilon_{1}}{\varepsilon_{2}} a_{1}+\hat{j} b_{2} \cos \theta+\hat{k} b_{3} \operatorname{sen} \theta\right) \exp \left(-\alpha_{2} x\right) e^{i \beta(2 \operatorname{sen} \theta+y \cos \theta)} \tag{2.32b}
\end{align*}
$$

where $z \rightarrow y \cos \theta+z \operatorname{sen} \theta$; then, the representation for arbitrary surface fields can be obtained by a superposition of mode surfaces, by the following equations:

$$
\begin{align*}
& E_{1}(x, y, z)=\int_{-\infty}^{\infty}\left(\hat{i} A\left(u_{1}\right)+\hat{j} B\left(u_{1}\right)+\hat{k} C\left(u_{1}\right)\right) \exp \left(-\alpha_{1} x\right) e^{i \beta\left(y u_{1}+z p_{1}\right)} d u_{1}  \tag{2.33a}\\
& E_{2}(x, y, z)=\int_{-\infty}^{\infty}\left(\widehat{i} A\left(u_{1}\right)+\widehat{j} B\left(u_{1}\right)+\hat{k} C\left(u_{1}\right)\right) \exp \left(-\alpha_{2} x\right) e^{i \beta\left(y u_{1}+z p_{1}\right)} d u_{2} \tag{2.33b}
\end{align*}
$$

$u_{1}=\cos \theta / \lambda_{1,2}$ corresponds to spatial frequencies and $\lambda_{1,2}$ represents the wavelength in each media. The parameters $A(u), B(u)$ and $C(u)$ are related by the expression:

$$
\begin{equation*}
u^{2}+p^{2}=1 / \lambda^{2} \tag{2.34}
\end{equation*}
$$

Essentially equation (2.33) has the same mathematical structure as the angular spectrum model [11].

### 2.6 CONCLUSIONS

We described which modes are known like diffraction free beams and satisfy the Helmholtz equation where the complex amplitude of any monochromatic optical disturbance propagating in a homogeneous medium must obey such a relation. We determined that the solutions of the paraxial wave equation are known as beams too. In fact, the representation of a mode is extended to flat metal surfaces where we obtained the dispersion relation which represents the possible values for the phase parameters associated with surface waves which can correspond to evanescent waves, and finally we used the modal solution for an analogous structure at angular spectrum.

## CHAPTER 3

## Modal Description of a Surface Wave

### 3.1 INTRODUCTION

In this chapter we comment on the general wave mode representation for flat metal surfaces, known as surface plasmons modes, obtaining an analytical expression for the effective refraction index. The general modes are obtained by means of the interference between two non-parallel surface plasmons. We show that the interference between surface plasmons generates charge redistribution. The explicit calculus is obtained by stationary surface waves obtained by interfering two counter-propagating plasmons.

### 3.2 SURFACE WAVE MODAL DESCRIPTION

### 3.2.1 PLASMON DESCRIPTION

In chapter 2 we have obtained the dispersion relation function for surface plasmons (SP) using the charge media-free conditions. Now we can consider a plasmon wave propagating on the interfaces metal- dielectric media, as sketched in figure 3.1.


FIGURE 3.1, Representation of a plasmon wave propagating at the interface between metal and dielectric media.

The expression for the surface plasmon propagating on the metal surface is

$$
\begin{equation*}
E_{1}(x, z)=(\hat{i} A+\hat{k} B) \exp \left(-\alpha_{1} x\right) \exp (i \beta z) \tag{3.1}
\end{equation*}
$$

The expression for the phase function, considering the dispersion relation function is

$$
\begin{equation*}
\exp (i k z)=\exp \left(i w / c n_{e f f} z\right) \tag{3.2}
\end{equation*}
$$

From this equation, the effective refraction index is given by:

$$
\begin{equation*}
n_{e f f}=(\xi+i \eta) \tag{3.3}
\end{equation*}
$$

The refractive index can be obtained from the dispersion relation function, having the form

$$
\begin{equation*}
(\xi+i \eta)=\left(\frac{\varepsilon_{1 r} \varepsilon_{2 r}}{\varepsilon_{1 r}+\varepsilon_{2 r}}\right)^{1 / 2} \tag{3.4}
\end{equation*}
$$

where $\varepsilon_{i r}=\varepsilon_{i} / \varepsilon_{0}$ refers to the relative permittivity, then we have that equation (3.1) takes the form:

$$
\begin{equation*}
E_{1}(x, z)=(\hat{i} A+\widehat{k} B) \exp \left(-\alpha_{1} x\right) \exp \left(i k_{0} \xi \cdot z\right) \exp \left(-k_{0} \eta \cdot z\right) \tag{3.5}
\end{equation*}
$$

This equation describes a plasmon wave propagating into the media as an evanescent wave, so we can generate two surface plasmon waves propagating at the plane $y-z$ with an angle, as represented in the figure 3.2 , and then one obtains:

$$
\begin{align*}
& E_{i}(x, z)=(\hat{i} A+\hat{j} B \cos \theta+\hat{k} B \operatorname{sen} \theta) \exp \left(-\alpha_{1} x\right) \exp (i \beta(z \operatorname{sen} \theta+y \cos \theta))+ \\
& (\hat{i} A-\hat{j} B \cos \theta+\hat{k} B \operatorname{sen} \theta) \exp \left(-\alpha_{1} x\right) \exp (i \beta(z \operatorname{sen} \theta-y \cos \theta)) \tag{3.6}
\end{align*}
$$



FIGURE 3.2 Representation of two plasmon waves propagating along the interface with an angle

Equation (3.6) becomes:

```
\(E_{i}(x, z)=\{2 A \cos (\beta y \cos \theta) \hat{i}+2 i B \cos \theta \operatorname{sen}(\beta y \cos \theta) \hat{j}+2 B \operatorname{sen} \theta \cos (\beta y \cos \theta) \hat{k}\} \times\)
\(\exp \left(-\alpha_{1} x\right) \exp (i \beta z \operatorname{sen} \theta)\)
```

Therefore, we can take the exponential term of equation (3.7) and compare with equation (3.5). Thus we obtain the new expression for the effective refractive index

$$
\begin{equation*}
\left(\frac{\varepsilon_{1 r} \varepsilon_{2 r}}{\varepsilon_{1 r}+\varepsilon_{2 r}}\right)^{1 / 2} \operatorname{sen} \theta=(\xi+i \eta)=n_{e f f} \tag{3.8}
\end{equation*}
$$

From the real and complex parts of the effective refractive index we can obtain the values for the electrical permittivity.

### 3.2.2 SURFACE MODE PLASMONS

The following analysis is applied to SP modes, i.e. we are considering $\beta$ as real. Surface mode solutions given by equation ( 2.33 of chapter 2 ) are the analogous to plane waves for homogeneous media. For this reason they can be implemented to describe arbitrary SP fields of the form
$E_{1}(x, y, z)=\int_{-\infty}^{\infty}\left(\vec{i} A\left(u_{1}\right)+\vec{j} B\left(u_{1}\right)+\vec{k} C\left(u_{1}\right)\right) \exp \left(-\alpha_{1} x\right) \exp \left\{i 2 \pi\left(\frac{\varepsilon_{1} \varepsilon_{2}}{\varepsilon_{1}+\varepsilon_{2}}\right)^{1 / 2}\left(y u_{1}+z p_{1}\right)\right\} d u_{1}$

$$
E_{2}(x, y, z)=\int_{-\infty}^{\infty}\left(\vec{i} A\left(u_{2}\right)+\vec{j} B\left(u_{2}\right)+\vec{k} C\left(u_{2}\right)\right) \exp \left(-\alpha_{2} x\right) \exp \left\{i 2 \pi\left(\frac{\varepsilon_{1} \varepsilon_{2}}{\varepsilon_{1}+\varepsilon_{2}}\right)^{1 / 2}\left(y u_{2}+z p_{2}\right)\right\} d u_{2}
$$

where $u=\frac{\cos \theta}{\lambda_{0}}$ corresponds to spatial frequencies, $\lambda_{0}$ represents the wavelength in vacuum and $A(u), B(u), C(u)$ are the amplitude functions which are related to the transmittance function by means of the Fourier transform. Essentially, equations (3.9) have the same mathematical structure as that of the angular spectrum model [11]. The electric field given by equation (3.9) generates surface charge redistribution; however, the spatial average of the charge must be zero to satisfy the condition of media free charge. This is the manifestation of the interference fringes between the elementary SP modes emerging from the transmittance function.

### 3.2.3 DIFFRACTION FREE BEAMS

The next point of the analysis consists of finding a general expression for SP modes, from which equations ( 2.24 of chapter 2 ) is a particular case. By analogy with diffraction free beams, the general structure of the mode must be of the form:

$$
\begin{equation*}
E_{1}(x, y, z)=(\vec{i} a(y)+\overrightarrow{j b}(y)+\vec{k} c(y)) \exp (-\alpha x) e^{i \Omega z} \tag{3.10}
\end{equation*}
$$

where the problem to be solved consists in finding the expression for the [a(y), $b(y), c(y)]$ functions and phase parameter $\Omega$. By substitution in the Helmholtz' equation these functions satisfy:

$$
\begin{equation*}
\frac{\partial^{2}\left(a_{n}(y), b_{n}(y), c_{n}(y)\right)}{\partial y^{2}}+\left(K^{2}+\alpha^{2}-\Omega^{2}\right)\left(a_{n}(y), b_{n}(y), c_{n}(y)\right)=0 \tag{3.11}
\end{equation*}
$$

and the general expression for mode solution is

$$
\begin{equation*}
E_{1}(x, y, z)=\left(\vec{i} \xi_{1}+\vec{j}_{2}+k \vec{\xi}_{3}\right) \cos (p y+\eta) \exp (-\alpha x) e^{i \Omega z} \tag{3.12}
\end{equation*}
$$

where $p=\sqrt{\left(K^{2}+\alpha^{2}-\Omega^{2}\right)}$ and $\left(\xi_{1,2,3}\right), \eta$ are arbitrary constants.

Equation (3.12) can be obtained by means of the interference between two elementary plasmon modes of equations ( 2.24 of chapter 2 ). The parameter $\Omega$ is related to the dispersion relation function by means of $\Omega=\beta \sin \theta$. In figure 3.3 we show the parameters associated to equation (3.12).


FIGURE 3.3.- Interference between two elementary SP modes and the synthesis of an effective SP mode propagating along z-coordinate, the phase parameter $\Omega$ is smaller than the dispersion relation $\beta$.

Until this point, we have shown that it is possible to generate new modes by interfering elementary SP modes. This construction from the angular spectrum model is analogous to diffraction free beams for free space, a prototype of that being the Bessel beams; for such modes, the spatial frequency representation should be on a circle. For general SP modes the corresponding representation consists of two points as can be deduced from equation (3.12). These two points are generated by the intersection of a frequency circle for homogeneous media with the ( $u, p$ ) plane. More details about the frequency representation for diffracting free beams can be found in [9,10,12]. A natural extension of the previous analysis consists in generating a coherent interaction between two or more modes. This can be obtained by a superposition of modes propagating in different directions and with different phase parameters. An interesting case occurs in some regions where a consonance between the phase parameters occurs.

### 3.3 INTERACTION OF PLASMONS

The electric field given by equation (2.33) chapter 2 generates charge redistribution where the expressions for the charge density functions in both media are given by

$$
\begin{align*}
& \rho_{1 d}=\varepsilon_{1} \nabla \cdot \int_{-\infty}^{\infty}\left(\hat{i} A\left(u_{1}\right)+\hat{j} B\left(u_{1}\right)+\hat{k} C\left(u_{1}\right)\right) \exp \left(-\alpha_{1} x\right) e^{i \beta\left(y u_{1}+z p_{1}\right)} d u_{1}  \tag{3.13a}\\
& \rho_{2 d}=\varepsilon_{2} \nabla \cdot \int_{-\infty}^{\infty}\left(\hat{i} A\left(u_{2}\right)+\hat{j} B\left(u_{2}\right)+\hat{k} C\left(u_{2}\right)\right) \exp \left(-\alpha_{2} x\right) e^{i \beta\left(y u_{2}+z p_{2}\right)} d u_{2} \tag{3.13b}
\end{align*}
$$

In order to understand the physical meaning of equations (3.13) we describe the interaction between two counter-propagating plasmon waves, performing the calculus only for the electric field amplitude on media 1 . We find that the electric field satisfies

$$
\begin{align*}
& \hat{E}_{1}(x, z)=(\vec{i} a+\vec{k} b) \exp \left(-\alpha_{1} x\right) e^{i \beta z}+(\vec{i} a-\vec{k} b) \exp \left(-\alpha_{1} x\right) e^{-i \beta z} \\
& =2(\vec{i} a \cos (\beta z)+\vec{k} i b \sin (\beta z)) \exp \left(-\alpha_{1} x\right) \tag{3.14}
\end{align*}
$$

A similar expression is obtained for media 2. In order to associate a physical meaning to the electric field, the parameters $a, b$ must be imaginary. The expression corresponds to a standing wave. Taking the divergence of equation (3.14) we obtain

$$
\begin{equation*}
\nabla \cdot E_{1}=-4 i e^{-\alpha x}\left(a \alpha_{1} \sin \beta z+b \beta \cos \beta z\right) \tag{3.15}
\end{equation*}
$$

This is in general non-zero. This means that the superposition of two counterpropagating plasmon waves generates a charge redistribution, where the charge density function of media 1 is given by

$$
\begin{equation*}
\rho_{d 1}=4 i \varepsilon_{1} e^{-\alpha x}\left(a \alpha_{1} \sin \beta z+b \beta \cos \beta z\right) \tag{3.16}
\end{equation*}
$$

Again a similar expression is obtained for media 2. It must be noted that a periodical array of charges is generated, the spatial period of which is proportional to the inverse value of the dispersion relation given by

$$
\begin{equation*}
d=\frac{2 \pi}{\beta}=\lambda\left(\frac{\varepsilon_{1 r}+\varepsilon_{2 r}}{\varepsilon_{1 r} \varepsilon_{2 r}}\right)^{1 / 2} \tag{3.17}
\end{equation*}
$$

The possibility of generating charge distributions is a very interesting topic and it has been described by Rather [1].

### 3.4 CONCLUSIONS

We obtained the general expression for surface plasmon modes establishing an analogy with diffraction free beams for homogeneous media. We show that the spatial frequency representation corresponds with two points. From this representation we obtain the expression for the effective refraction index and we show that it is possible modify the dispersion relation function. This kind of beams offers interesting applications as two-dimensional plasmon twisters. Finally we show that the coherent interaction between two or more surface plasmon modes generates charge redistribution. This feature allows us to generate the boundary condition to generate other kind of surface plasmon fields.

## CHAPTER 4

## Description of Diffraction Plasmon Fields and Some of Its applications

### 4.1 INTRODUCTION

In the present chapter, we describe the possibility of synthesis for surface plasmons self-imaging fields, this analysis is performed in the frequency space and the obtained condition is matched with the Montgomery's Rings for homogeneous media [13,14]. The angular spectrum model for SP allows us to describe the plasmonic diffraction field. This model is implemented to describe new features such as the description of surface singularities which occur in focusing regions $[15,16,17]$. The phase function on these regions presents an adiabatic behavior, and it is deeply related to the charge redistribution. Thus, the construction of focusing regions and some of its features is associating a catastrophe relation to the phase function.

In this way, we can start the study by using the equations for the angular spectrum model for both media as follows.

$$
\begin{gathered}
E_{1}(x, y, z)=\int_{-\infty}^{\infty}\left(\hat{i} A\left(u_{1}\right)+\hat{j} B\left(u_{1}\right)+\hat{k} C\left(u_{1}\right)\right) \exp \left(-\alpha_{1} x\right) e^{i \beta\left(y u_{1}+z p_{1}\right)} d u_{1} \\
E_{2}(x, y, z)=\int_{-\infty}^{\infty}\left(\hat{i} A\left(u_{2}\right)+\hat{j} B\left(u_{2}\right)+\hat{k} C\left(u_{2}\right)\right) \exp \left(-\alpha_{2} x\right) e^{i \beta\left(y u_{2}+z p_{2}\right)} d u_{2},
\end{gathered}
$$

where $\left(\hat{i} A\left(u_{i}\right)+\hat{j} B\left(u_{i}\right)+\hat{k} C\left(u_{i}\right)\right)$ is the amplitude vector for a plasmon mode propagating in the direction defined by spatial frequency " $u$ ".

### 4.2 SURFACE PLASMON SELF-IMAGING FIELDS

The self-imaging condition means that the amplitude function for the optical field must be periodic along the propagation coordinate and it can be represented as a Fourier series. By considering the z coordinate as direction of propagation, the vector amplitude distribution (on media 1) can be represented by:

$$
\begin{gather*}
E_{1}(x, y, z)=\exp (-\alpha x) \sum\left(\vec{i} a_{n}+\overrightarrow{j b}_{n}+\overrightarrow{k c_{n}}\right) \exp (i 2 \pi z n / d)  \tag{4.1a}\\
E_{1}(x, y, z)=\int_{-\infty}^{\infty}(\vec{i} A(u)+\vec{j} B(u)+\vec{k} C(u)) \exp (-\alpha x) \exp \left\{i 2 \pi\left(\frac{\varepsilon_{1} \varepsilon_{2}}{\varepsilon_{1}+\varepsilon_{2}}\right)^{1 / 2}(u y+p z)\right\} d u
\end{gather*}
$$

where " $d$ " is the period of self-imaging along the z-coordinate, and the coefficients ( $a_{n}, b_{n}, c_{n}$ ) can be expressed at functions of the $y$-coordinate. To find the general structure of the amplitude frequency functions $(A(u), B(u), C(u))$, one
can substitute expression (4.1) into the Helmholtz equation and by solving for the $y$-variable we obtain that solutions are given by:

$$
\begin{equation*}
\left(a_{n}(y), b_{n}(y), c_{n}(y)\right)=\left(a e^{i \delta_{n} y}, b e^{i \delta_{n} y}, c e^{i \delta_{n} y}\right) \tag{4.2}
\end{equation*}
$$

being ( $a, b, c$ ) arbitrary constants, and the phase term satisfies:

$$
\begin{equation*}
\delta_{n}=\sqrt{\left(\beta^{2}+\alpha^{2}-\frac{4 \pi^{2} n^{2}}{d^{2}}\right)} \tag{4.3}
\end{equation*}
$$

By using this representation in the series of equation (4.1b) and considering $z=0$, we obtain that the amplitude frequency representation can be obtained by means of a Fourier transform having the structure of a Dirac- $\delta$ function of the form:

$$
\begin{equation*}
(A(u), B(u), C(u))=\left(A_{n}(u), B_{n}(u), C_{n}(u)\right) \delta\left(u-\beta-\delta_{n}\right) \tag{4.4}
\end{equation*}
$$

where $\left(A_{n}(u), B_{n}(u), C_{n}(u)\right)$ are arbitrary functions. Equation (4.4) consists of a set of points in frequency space; this result is equivalent to the Montgomery condition for free space [13]. The spatial analysis for the self-imaging fields can be obtained by using the paraxial approximation in equation (4.3). This corresponds to the weak self-imaging condition and by following the Montgomery's treatment it can be rewritten as:

$$
\begin{equation*}
\delta_{n} \approx \frac{\lambda}{\left(\frac{\varepsilon_{1} \varepsilon_{2}}{\varepsilon_{1}+\varepsilon_{2}}\right)^{1 / 2} 2 d^{2}} \sqrt{n} \tag{4.5}
\end{equation*}
$$

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A particular case of equation (4.5) is when $n=0,1,4,9, \ldots$; this is known as Talbot's effect, and the transmittance function corresponds to a periodical object of period " $a$ " which is related to the self-imaging period by means of :

$$
\begin{equation*}
d \approx \frac{2 a^{2}}{\lambda\left(\frac{\varepsilon_{1} \varepsilon_{2}}{\varepsilon_{1}+\varepsilon_{2}}\right)^{1 / 2}} \tag{4.6}
\end{equation*}
$$

Equation (4.5) represents a set of points whose structure is a one-dimensional zone plate, as shown in figure 4.1. They can be obtained by intersecting a zone plate with u-axes.


Figure 4.1. a) Montgomery rings and its intersection with u-axis. The highlighted points are the frequency condition for the plasmon fields to present the "weak self-imaging". b) Schematic set up for the generation of self-imaging by diffraction of SPs.

The self-imaging field is obtained by means of the diffraction field generated by illuminating a transmittance whose Fourier transform corresponds to a set of "points" which must satisfy equation (4.5). In figure 4.2 we show a computational

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simulation of the irradiance distribution on the Y-Z plane metal surface for the SP Talbot's effect.


FIGURE 4.2. Irradiance distribution for a weak SP self-imaging field. The field was generated by interfering 7 elementary SPs, propagating along directions making the following angles with respect to the positive z-axis: $\theta_{s p}=\left( \pm 9^{0}, \pm 6^{0}, \pm 3^{0}, 0\right)$. The wavelength in vacuum is $\lambda=2 \pi / \omega=502 \mathrm{~nm}$. a) On a gold surface for which the SP wavelength is $\lambda_{\text {sp }}=476 \mathrm{~nm}$; b) On the surface of an Au thin film with thickness $d=40 \mathrm{~nm}$, covered by an oil drop of glycerin ( $n=2.1$ ) for $\lambda_{s p}=68 \mathrm{~nm}$.

For a periodic transmittance with a period of 10 microns, and for a wavelength of 476 nm , the period of self-imaging is about 300 microns. For this propagation length, the absorption plays an important role and the self-imaging phenomenon is not possible. However, by considering a reduction of the period about 1.5 microns deposited on a gold film of thickness $d=40 \mathrm{~nm}$ and covered with oil of refractive index $n=2.1$, the effective refractive index is $n_{\text {eff }}=\lambda_{0} / \lambda_{S P}=7$. For these configurations, the self-imaging field has a period of 50 microns, which is feasible to implement experimentally avoiding absorption analysis. These comments are sketched in figures (4.2a) and (4.2b) just to compare the scale reduction in the self-imaging period. Similar configurations have been proposed recently to implement an oil drop as a mirror/lens [18,19].

### 4.3 DESCRIPTION OF PLASMON SINGULARITIES

In order to have a complete description of the surface optical field it is necessary to describe the singular regions. On these regions, the amplitude distribution has adiabatic features; it means that the spatial/temporal amplitude function changes very slowly. These features are deeply connected with charge redistribution.

The simplest case to generate a singularity is by interfering two elementary counter-propagating plasmon waves, as represented in figure (4.3a), the nodes correspond to singular points. Performing the calculus only for the electric field amplitude on media 1 and using the equation (3.14) described in Chapter 3, we find that the electric field satisfies:

$$
\begin{equation*}
E_{1}(x, z)=2 \exp \left(-\alpha_{1} x\right)(\vec{i} a \cos \beta z+\vec{k} i b \sin \beta z) \tag{4.7}
\end{equation*}
$$

A similar expression is obtained for the medium 2. To associate a physical meaning to the electric field, the parameters $(a, b)$ must be pure imaginary. The expression corresponds to a standing wave.

It must be noted that the period "d" (equation 3.17 of chapter 3) of the charge array is proportional to the inverse value of the dispersion relation function given by

$$
\begin{equation*}
d=\frac{2 \pi}{\beta}=\lambda\left(\frac{\varepsilon_{1 r}+\varepsilon_{2 r}}{\varepsilon_{1 r} \varepsilon_{2 r}}\right)^{1 / 2} \tag{4.8}
\end{equation*}
$$

For the SP fields, the charge period satisfies $d>\lambda$. Expression (4.7) has associated a discrete set of singular points; the continuous case is generated by a
set of elementary mode solutions. The geometrical construction is analogous to the envelope region for homogeneous media.


FIGURE 4.3. Geometrical description of singular regions: a) The generation of discrete singular points by means of the interference between two contra-propagating elementary SPs. b) The singular region by means of the envelope of elementary SPs. The modes emerge in a perpendicular direction to the curve.

The singular regions associated to the diffraction field emerging from a transmittance function of the form $t(y, z)=\delta(y-g(z))$ means that parameterized amplitude function can be represented by

$$
\begin{equation*}
E(x, y, u)=(\vec{i} A(u)+\vec{j} B(u)+\vec{k} C(u)) \exp \left(-\alpha_{1} x\right) e^{i \beta(y u+g(y) p)} \tag{4.9}
\end{equation*}
$$

where $u$ is considered as a parameter. From this representation, the extremal features of the mode trajectories whose phase function is $L(x, y, u)=\beta(y u+g(y) p)$ are given by $\frac{\partial L}{\partial y}=0$ and the calculus of the

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envelope implies that the phase function satisfies $\frac{\partial g}{\partial y}=\frac{p}{u}=\tan \theta$. This is the tangential condition for envelope regions of mode trajectories. Hence, the geometrical point of view means that the trajectories are tangent to the singular regions, as represented in figure (4.3b). In figure 4.4 one shows the computational simulations for the focusing region for the case when the geometry of the boundary condition corresponds to a Gaussian profile.


FIGURE 4.4.. Focusing region obtained by illuminating with an elementary surface plasmon wave a reflecting surface with Gaussian shape with $\sigma=10 \lambda_{s p}$ and depth $5 \lambda_{s p}$.

Due to the tangent property, generic features for the phase function can be implemented. This means that a catastrophe function for the phase function can be used [15,17]. By considering a curve where the trajectories emerge in a perpendicular way, it is easy to show that singularities correspond with the envelope of the curvature centers, this curve is known as evolute curve. Thus, on this region adiabatic features are presented, which means that the phase function changes very slowly and charge re-distribution is generated. More details concerning this representation can be found within [17]. By associating to the set

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of surface modes a parametric representation as a catastrophe function the surface plasmon field has a structure of the form:

$$
\begin{equation*}
E_{1}(x, z)=(\vec{i} a+\vec{j} b+\vec{k} c) \exp \left(-\alpha_{1} x\right) \exp (i(\text { catastrophe function }(y, z, u))) \tag{4.10}
\end{equation*}
$$

### 4.4 EIKONAL EQUATION FOR PLASMON FIELDS

A very important topic can be incorporated in order to improve the understanding of the physical characteristics of the plasmon fields. This can be obtained by incorporating a geometrical analysis. From the fact that a plasmonic field can be described by a superposition of plasmon modes, we can describe the evolution of the surface field using a model similar to geometric optics, i.e. we are incorporating extremal features to trajectories of the surface plasmon mode. We can propose the evolution of the optical field through a solution of the Helmholtz equation of the form:

$$
\begin{equation*}
\hat{\mathrm{E}}_{i}(x, y, z)=\hat{A}_{i} \exp \left(-\alpha_{i} x\right) \exp \left(i k L_{i}(y, z)\right) \tag{4.24}
\end{equation*}
$$

where $k$ is the wave number in free space. By substituting equation (4.24) in the set of Helmholtz equations (4.10) and comparing real and complex parts, we obtain a set of eikonal equations

$$
\begin{align*}
& k^{2} \left\lvert\, \nabla L_{1}(y, z)^{2}=\frac{w^{2}}{v_{1}^{2}}+\alpha_{1}^{2}\right.  \tag{4.25a}\\
& k^{2} \left\lvert\, \nabla L_{2}(y, z)^{2}=\frac{w^{2}}{v_{2}^{2}}+\alpha_{2}^{2}\right. \tag{4.25b}
\end{align*}
$$

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When we approach the surface through media 1 or through media 2 , the optical path length must have the same value. This is possible only if

$$
\begin{equation*}
\frac{w^{2}}{v_{1}^{2}}+\alpha_{1}^{2}=\frac{w^{2}}{v_{2}^{2}}+\alpha_{2}^{2} \tag{4.26}
\end{equation*}
$$

The physical interpretation is that once a plasmon mode is generated, it propagates in a media with a plasmonic refractive index, given by

$$
\begin{equation*}
N_{p}^{2}=n_{1}^{2}+\frac{\alpha_{1}^{2}}{k^{2}}=n_{2}^{2}+\frac{\alpha_{2}^{2}}{k^{2}} \tag{4.27}
\end{equation*}
$$

The plasmonic refractive index depends on the attenuation rate in each media, which is a fundamental difference with the effective index method as described by Marcuse [20]. As a principal result of this section, we have that homogeneous light arriving at the surface detects two refractive indices: $n_{1}$ and $n_{2}$, however the plasmon mode "feels" a single refractive index $N_{p}$. From equation (4.27), we have that the decreasing rate in each media is related to the homogeneous refractive index according to

$$
\begin{equation*}
n_{1}^{2}-n_{2}^{2}=\frac{1}{k^{2}}\left(\alpha_{2}^{2}-\alpha_{1}^{2}\right) \tag{4.28}
\end{equation*}
$$

Using the expression for the plasmonic refractive index, we can associate extremal features with the optical path length. For this case, the functional is given by

$$
\begin{equation*}
L(y, z)=\int_{a}^{b} N_{p} d s \tag{4.29}
\end{equation*}
$$

and the extremal behavior can be considered as Fermat's principle for plasmonic fields. The proposition of a plasmonic refractive index and its extremal analysis
has deep physical consequences, because each set of extremal trajectories associated with the surface modes generates a "flow of extremals", also known as geodesic flow.

The flow has generic features: It is ergodic and structurally stable [21, 22, 23]. The first property allows us to analyze the possibility for generating partially coherent plasmons. This can be pursued using the correlation function obtained from equation (4.7), given by

$$
\left.W\left(P_{0}, \gamma\right)=\left\langle E_{i}(X) \cdot E_{i}^{*}\left(X^{`}\right)\right\rangle=\left.e^{-\alpha_{i} x_{0}} \int_{-\infty}^{\infty}\langle | A\left(u_{i}\right)\right|^{2}+\left|B\left(u_{i}\right)\right|^{2}+\left|C\left(u_{i}\right)\right|^{2}\right\rangle e^{i \beta\left(u_{i} y_{0}+p_{i} z_{0}\right)} d u_{i}
$$

where $P_{o}=\left(y-y_{1}, z-z_{1}\right)$ and $W$ is the cross spectral density function. It has been shown that this function satisfies the Helmholtz equation [11]. By the same approach we can also associate an eikonal equation for the cross spectral density function, which means that along surface modes the correlation function is an extremal. More details can be found in [24-28].The structurally stable features of the flow allow us to analyze the structural stability of the surface optical field. We can analyze the behavior of the plasmonic field when the refractive index $N$ is changing. The changes must be manifested by the fluctuations of the optical path length, given by

$$
\begin{equation*}
L(z(y))=\int_{a}^{b}\left(N_{p}+\xi\right) d s \tag{4.31}
\end{equation*}
$$

where $\xi$ is a function which may depend on spatial coordinates and/or time. It describes the fluctuations of the refractive index. If this parameter has only a temporal dependence, then equation (4.31) has the form:

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$$
\begin{equation*}
L(y(t), z(t))=\int_{a}^{b}\left(N_{p}+\xi(t)\right) \sqrt{\left(y^{\prime}\right)^{2}+\left(z^{\prime}\right)^{2}} d t \tag{4.32}
\end{equation*}
$$

This can occur when surface parameters are changing with time, for example if the surface has a different temperature than the environment and the refractive index is changing due to the process of thermal equilibrium. Another interesting situation occurs when we have a rough surface. For this case, Eq. (4.31) takes the form

$$
\begin{equation*}
L(y(x), z(x))=\int_{A}^{B}\left(N_{p}+\xi(x, y, z)\right) \sqrt{d x^{2}+d y^{2}+d z^{2}} \tag{4.33}
\end{equation*}
$$

This extremal model is very interesting, because we can compare the trajectories obtained following (4.33) by the trajectory obtained by a flat surface when $\xi(x, y, z)=0$. When the correlation function between these two trajectories diverges, the two electromagnetic features must be dramatically different, and this occurs when plasmons are coupled to homogeneous media, generating light propagating in space. Equation (4.26) allows us to determine the range of variability in the refractive index and thus determine the stability of the plasmonic field. From the expression for plasmonic refractive index associated to roughness, we find that at points where $\left\lvert\, \xi(x, y, z)^{2} \approx \frac{\alpha_{1}^{2}}{k^{2}}\right.$ the plasmon mode generates light propagating back into space. This is because $N_{p}^{2} \approx n_{1}^{2}$ which means that the plasmonic refractive index tends to the refractive index for homogeneous media. This is a very interesting result because it allows us to identify the scattering points for plasmonic fields, avoiding statistical treatments, i.e., we can identify points where plasmons are converted to light propagating in space.

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### 4.5 CONCLUSIONS

We find that the sufficient condition for self-imaging is analogous to the Montgomery's condition for homogenous media. We showed that singular points can be generated experimentally by means of the interference between two plane plasmon modes propagating in opposite directions. Finally, we have shown that the surface optical field can be described using an extremal analysis where a refractive index for plasmon modes is obtained. The optical path length was calculated for a flat surface; however expression (4.29) is more general and can be applied to curved geometries, where the trajectories correspond to a set of geodesics.

Radiometric Features of Surface Plasmons Fields and Synthesis of Plasmon Vortex

## CHAPTER 5

## Radiometric features of Surface Plasmon Fields and synthesis of Plasmon Vortex

### 5.1 INTRODUCTION

In this chapter we describe the radiometric features for scalar optical fields by means of the spectral density function and the spectral flux function. The analysis is transferred to surface plasmon fields. After this, the definition of Surface plasmon optical singularity is presented and we will show that in these regions the phase function for SP has adiabatic features. Also in these regions the front wave concept is not valid. These regions are formed by the envelope of general plasmon modes, which justifies the name of singularities. Thus, in these regions the curvature of the front wave is reversed, i.e. front waves with negative curvature after this region acquire a positive curvature. For this reason, they represent regions of organization for the surface optical field. These regions are implemented to describe the generation of surface plasmon vortex.

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### 5.2 DESCRIPTION OF THE ANALYSIS OF OPTICAL VORTICES FOR SURFACE PLASMON

In fact we know that structural features of light such as polarization and coherence can be obtained by the phase function. Also, the momentum transfer and the radiometric behavior can be obtained by the amplitude scalar function [29,30]. In this sense, a very interesting topic consists in describing the radiometric and momentum behavior of surface plasmons. The staring point consists in describing two functions known as spectral density of energy and spectral flux of energy, defined respectively as

$$
\begin{align*}
& H(X, t)=\frac{1}{v^{2}} \frac{\partial \phi}{\partial t} \frac{\partial \phi^{*}}{\partial t}+\nabla \phi \nabla \phi^{*}  \tag{5.1}\\
& F(X, t)=-\frac{\partial \phi}{\partial t} \nabla \phi^{*}-\frac{\partial \phi^{*}}{\partial t} \nabla \phi \tag{5.2}
\end{align*}
$$

These two functions satisfy the continuity equation

$$
\begin{equation*}
\nabla \cdot F+\frac{\partial H}{\partial t}=0 \tag{5.3}
\end{equation*}
$$

This is known as the law of conservation of energy. The scalar function $\phi(X, t)$ satisfies the wave equation $\nabla^{2} \phi=\frac{1}{v^{2}} \frac{\partial^{2} \phi}{\partial t^{2}}$.

In order to obtain a physical meaning for the $F$ function, the definition is applied to a plane wave of the form

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$$
\begin{equation*}
\phi(X, t)=A(u, v, \gamma) e^{i 2 \pi(x u+y v+z p-\gamma)} \tag{5.4}
\end{equation*}
$$

Substituting this equation into equation (5.2) we obtain the vector function

$$
\begin{equation*}
F(x, t)=-8 \pi^{2} \gamma \mid A(u, v, \gamma)(u, v, \rho) \tag{5.5}
\end{equation*}
$$

This function can be interpreted as the energy transported by a plane wave in a direction given by the spatial frequencies at a given temporary frequency. By analogy, it is called vector power spectrum. From this equation the information about energy and linear momentum can be obtained, because power spectrum must be proportional to number of photons. The mathematical generalizations can be obtained, using the angular spectrum model, associating an arbitrary optical field. Thus, the angular spectrum model is given by

$$
\begin{equation*}
\phi(X, t)=\iiint A(u, v, \gamma) e^{i 2 \pi(x u+y v+z p-\gamma)} d u d v d \gamma \tag{5.6}
\end{equation*}
$$

Consequently the flux spectral density function spectrum takes the form:

$$
\begin{equation*}
\vec{F}(X, t)=4 \pi^{2}\left\{\iiint\left(\gamma A^{*}(U) A\left(U^{\prime}\right) e^{i 2 \pi(X U)} e^{-i 2 \pi\left(X U U^{\prime}\right)} \vec{U}+\gamma^{\prime} A^{*}(U) A\left(U^{\prime}\right) e^{-i 2 \pi(X U)} e^{i 2 \pi\left(X U U^{\prime}\right)} \vec{U}^{\prime}\right) d U d U\right\} \tag{5.7}
\end{equation*}
$$

This expression corresponds to completely coherent fields. Partially coherent effects can be associated when the power spectrum is fluctuating, having the form

$$
\begin{equation*}
F(X, t)=\iiint 8 \pi^{2} \gamma\left(| |^{2} e^{i 2 \pi(x u+y v+z p-\gamma)}(u, v, p) d u d v d \gamma\right. \tag{5.8}
\end{equation*}
$$

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where brackets mean an average value. The points that remain to be solved consist in the analysis of angular momentum. For this case, in general we have that the optical fields have an optical representation given by

$$
\begin{equation*}
\phi(k, r)=g(x, y, z) \exp i K_{o}(L(x, y, z, \nsucc)) \tag{5.9}
\end{equation*}
$$

where $g$ and $L$ are real functions. The $H$ function has the form

$$
\begin{equation*}
H(X)=|\nabla g|^{2}+|g|^{2}|\nabla L|^{2}+2 g \nabla L \cdot \nabla g+\frac{\gamma^{2}}{v^{2}} K_{0}^{2} \tag{5.10}
\end{equation*}
$$

It should be noted that if $g$ is constant, equation (5.9) represents the eikonal equation.

Expression (5.6) has other physical implications. Some of them are described as follows. If $v=V(u), p=P(u)$, then the vector function depends only on a single variable and takes the form:

$$
\begin{align*}
& \phi(k, r)=\operatorname{expi}(\overrightarrow{f(u)} \cdot g(x, \vec{y}, z))  \tag{5.11}\\
& =\operatorname{expi}(f \overrightarrow{f(u)} \cdot g(x, \vec{f}(x), z))
\end{align*}
$$

So, the front wave is given by

$$
\begin{equation*}
\overrightarrow{f(u)} \cdot g(x, y, z)=\text { cons } \tan t \tag{5.12}
\end{equation*}
$$

The evolution of the front wave is given by its gradient function

$$
\begin{equation*}
\nabla(f \overrightarrow{f(u)} \cdot g(x, y, z))=(f(u) \cdot \nabla) g(x, y, z)+f(u) \times \nabla \times g(x, y, z) \tag{5.13}
\end{equation*}
$$

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From the latter representation, we have that regions exist where the curl is different of zero.

$$
\begin{equation*}
\nabla \times g(x, y, z)=h(x, \overrightarrow{y, z}) \tag{5.14}
\end{equation*}
$$

Very interesting features can be obtained. One of them is generating a closed path and calculating the flux of the curl. For this case, we have that

$$
\begin{equation*}
\iint_{D} \nabla \times g(x, y, z) \cdot n d s=\int_{L} g(x, y, z) \cdot d l \tag{5.15}
\end{equation*}
$$



Figure 5.1 Envelope of the critical points

The angular momentum transfer can be obtained through the collective effects of the envelope of critical points. The understanding of these features can be obtained from figure 5.1. It must be clear that in a region with a single value in the phase function; only linear momentum can be generated. However in multivalue phase functions, it is possible to obtain angular moment transfer.

The region must satisfy the following equation:

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$$
\begin{equation*}
\frac{\partial^{2} g}{\partial x^{2}} \frac{\partial^{2} g}{\partial y^{2}}-\left[\frac{\partial^{2} g}{\partial x \partial y}\right]^{2}=0 \tag{5.16}
\end{equation*}
$$

whose solution is given by:

$$
\begin{equation*}
g(x, y)=x^{\alpha} y^{1-\alpha} \tag{5.17}
\end{equation*}
$$

### 5.3 RADIOMETRIC FEATURES OF SURFACE PLASMON FIELDS.

The plasmon is a vector wave, however each scalar component has associated a function $F(X, t), N(X, t)$. The structure of these functions is very simple so that the scalar components for the electric field present a common phase. This fact is interesting because the singularities of the plasmon surface can be obtained from the phase function. In this way a parallelism with homogeneous fields may be implemented. Hence, it is possible to predict focal plasmon regions that one elaborated in chapter 4 . Thus, the dynamic behavior is obtained by differential partial equation for focal regions through scaling and rotating. Hence, we can use an expression for the angular spectrum model described in chapter 2 given by:

$$
\begin{align*}
& E_{1}(x, y, z)=\int_{-\infty}^{\infty}\left(\hat{i} A\left(u_{1}\right)+\hat{j} B\left(u_{1}\right)+\hat{k} C\left(u_{1}\right)\right) \exp \left(-\alpha_{1} x\right) e^{i \beta\left(y u_{1}+z p_{1}\right)} d u_{1}  \tag{5.18}\\
& E_{2}(x, y, z)=\int_{-\infty}^{\infty}\left(\hat{i} A\left(u_{1}\right)+\hat{j} B\left(u_{1}\right)+\hat{k} C\left(u_{1}\right)\right) \exp \left(-\alpha_{2} x\right) e^{i \beta\left(y u_{1}+z p_{1}\right)} d u_{2} \tag{5.19}
\end{align*}
$$

where each component for the electric field $E=\left(\phi_{1}, \phi_{2}, \phi_{3}\right)$ is represented by:

$$
\begin{equation*}
\phi(x, y, z)=\int_{-\infty}^{\infty} \hat{i} A\left(u_{1}\right) \exp \left(-\alpha_{2} x\right) e^{i \beta\left(y u_{1}+z p_{1}-\omega t\right)} d u \tag{5.20}
\end{equation*}
$$

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We can proceed in the same way for other components. Thus, equation (5.20) is an example to be substituted into equations (5.1) and (5.2), so that we take the sum of all components. Hence, equation $H$ represents the flux density of energy and its equation is given by:

$$
\begin{equation*}
H_{\text {Totally }}=\sum_{i=1}^{3} H_{i} \tag{5.21}
\end{equation*}
$$

so that

$$
\begin{equation*}
H_{i}=\frac{1}{v^{2}} \frac{\partial \phi}{\partial t} \frac{\partial \phi^{*}}{\partial t}+\nabla \phi \nabla \phi^{*} \tag{5.22}
\end{equation*}
$$

Equation (5.20) only expresses one component, so that one can take all components. Hence, the flux density of phase $F$ becomes.

$$
\begin{equation*}
F_{\text {totally }}=(A(u)+B(u)+C(u))\left(-\frac{\partial \phi}{\partial t} \nabla \phi^{*}-\frac{\partial \phi^{*}}{\partial t} \nabla \phi\right) \tag{5.22}
\end{equation*}
$$

This equation means that the flux density of energy is along the gradient of the phase function. In this way, the F-function allows to describe the moment transfer for surface plasmon fields.

### 5.4 VORTEX OF SURFACE PLASMONS

The previous analysis was described for scalar fields; however it may be extended to the description of surface plasmon fields. This can be done from the analysis of the structure of equation (5.16). After long but straightforward calculations, it is easy to show that equation (5.17) remains non-variable under the changes of variable

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$$
\begin{equation*}
\xi=a x+b y+c, \eta=a x-b y+d \tag{5.23}
\end{equation*}
$$

So equation (5.16) becomes:

$$
\begin{equation*}
\frac{\partial^{2} L}{\partial \xi^{2}} \frac{\partial^{2} L}{\partial \eta^{2}}-\left[\frac{\partial^{2} L}{\partial \xi \partial \eta}\right]^{2}=0 \tag{5.24}
\end{equation*}
$$

The changes of variable, for real values of ( $a, b, c, d$ ) parameters may represent scaling, rotations, translations or combinations of them. We have that a solution for Equation (5.24) takes the form

$$
\begin{equation*}
L(\xi, \eta)=\xi^{\alpha} \eta^{1-\alpha} \tag{5.25}
\end{equation*}
$$

or in the plane $x-y$ the solution is given by

$$
\begin{equation*}
L(x, y)=(a x+b y+c)^{\alpha}(a x-b y+d)^{1-\alpha} \tag{5.26}
\end{equation*}
$$

The first transformation means that the equation is non-variant with a scale change, and the second condition means that it is non-variable under a rotation. By describing a dynamics motion for equation (5.17) we make a transformation to the cylindrical coordinate system. Thus, the equation becomes:

$$
\begin{equation*}
L(x, y)=(r \cos \theta)^{\alpha}(r \operatorname{sen} \theta)^{1-\alpha} \tag{5.27}
\end{equation*}
$$

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where $L(r, \theta)$ is the evolution of the curve which is changing along of the variables $r$ and $\theta, x=r \cos \theta$ and $y=r \operatorname{sen} \theta$. Figure 5.2 shows a graph of the curve in space $x, y, z$ for equation (5.17), and figure 5.3 shows the projection on the plane $x, y$ represented by equation (5.17). The value of parameter $\alpha=0.5$.


FIGURE 5.2 Evolution of the curve for a value alfa= 0.1


FIGURE 5.3 Projection of the curve on the plane $\mathrm{x}, \mathrm{y}$ for a value alfa=0.1.

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Then, if we take another value of $\alpha=0.1$ the curves are presented in figures 5.4 and 5.5


FIGURE 5.4 Evolution of the curve for a value alfa $=0.01$


FIGURE 5.5 Projection of the curve on the plane $\mathrm{x}, \mathrm{y}$ for a value alfa $=0.01$

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### 5.5 CONCLUSIONS

We described the radiometric features for scalar optical fields by means of the spectral density function and the spectral flux function, so we have demonstrated that energy and linear momentum can be obtained from angular spectrum model representation associating an arbitrary optical field. The angular momentum transfer can be obtained through the collective effects of the envelope of critical points, taking this into account, the singularities of the plasmon surface can be obtained from the phase function, and it is possible to predict focal plasmon regions. Finally, we showed that on multivalue phase functions, angular moment transfer can be obtained which satisfies in the region the differential equation (5.16), this differential equation is non-variant with a scale change, and is nonvariable under a rotation. Hence, we have shown by different figures under a transformation the generation of surface plasmon vortex.

## CHAPTER 6

## Conclusions

In this Thesis we described the treatment for the coupling of light with surface plasmon modes; the analysis was obtained by establishing an analogy with homogeneous modes known as diffraction free beams.

We generate a coherent superposition of elementary surface plasmon modes in order to describe arbitrary plasmon fields. The expression obtained corresponds with the angular spectrum model. This representation allows us to describe diffraction features from which we can expect novel surface plasmon fields.

We showed that a physical manifestation of the electromagnetic field associated to plasmon fields consists in the generation of charge distribution. The simplest case occurs for two counter-propagating plasmon modes. This simple interaction allows the establishment of the boundary condition to generate more complex plasmon fields such as the self-imaging plasmon fields.

The general case offers interesting technological applications, for example ion trapping, generation of quantum dots, generation of tunable photonic crystals, etc.

We obtained a more general mode solution for surface plasmon fields having the property that the phase function along the coordinate of propagation is less small than the one determined by the dispersion relation function $\beta>\Omega$. This behavior of the phase function allows one to generate surface plasmon modes of large trajectory and the self-imaging phenomenon can be generated experimentally.

We found that the sufficient condition for plasmon self-imaging is analogous to the Montgomery's condition for homogenous media.

We showed that singular points can be generated experimentally by means of the interference between two plane plasmon modes propagating in opposite directions. The nodes correspond to stationary charge distribution, having a periodic representation. General singular regions were described using a parameterization for the phase function by means of a catastrophe function.

The mode analysis presented allows incorporation of other interesting features related to self imaging, such as the Lau effect as well as the generation of partially coherent effects. Recall that the plasmon self-imaging reproduces periodical patterns of relatively intense surface electromagnetic fields, which could be exploited to generate surface optical lattices and surface optical tweezers. The associated arrays of surface charge distributions can be also of fundamental interest in a number of fields.

We have shown that the surface optical field can be described using an extremal analysis where a refractive index for plasmon modes is obtained. The optical path length was calculated for a flat surface; however, the equation for
extremal features with the optical path length is more general and can be applied to curved geometries, where the trajectories correspond to a set of geodesics.

We described the radiometric features for scalar optical fields by means of the spectral density function and the spectral flux function, this model was used to describe the radiometric features of surface plasmon fields. We demonstrated that energy and linear momentum can be obtained from the angular spectrum model for an arbitrary surface optical field. The angular momentum transfer can be obtained through the collective effects for the envelope of critical points. The singularities of the plasmon surface can be obtained from the phase function and it makes possible the prediction of focusing plasmon regions.

Finally, we showed that the singularities for the phase functions allow us to describe the angular transfer moment in the neighborhood of the focusing regions. This is possible because the differential equation associated to the singularities remain non-variable under linear transformations which represent scaling or rotating. These features are the support for the study of dynamical surface plasmon behavior which explains the conditions to generate surface plasmon vortex.

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## Appendices

## APPENDIX 1

## i INTRODUCTION

In this thesis work we use techniques concerning excitement of the surface plasmons and the interaction of the evanescent electric field with thin films. This is the reason why one must know the theory of the electromagnetic modes.

In this appendix we show the development of the electromagnetic theory, in particular we study their ground and the mode excitements.

## ii ELECTROMAGNETIC THEORY

The description of electromagnetic theory is represented by the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ for describing the electromagnetic fields at the matter with the need to add a second set of vectors: the density current $\boldsymbol{j}$, the electric displacement $\boldsymbol{D}$ and the magnetic induction $\boldsymbol{B}$. The spatial and temporal derivates of these vectors are related by the Maxwell equations, which are accepted at all
point where the physical properties are continuous in their environment. They are given by:

$$
\begin{gather*}
\nabla \cdot D=4 \pi \rho  \tag{1}\\
\nabla \cdot B=0  \tag{2}\\
\nabla \times E+\frac{1}{c} \frac{\partial B}{\partial t}=0  \tag{3}\\
\nabla \times H-\frac{1}{c} \frac{\partial D}{\partial t}=\frac{4 \pi}{c} j \tag{4}
\end{gather*}
$$

In this case, the Maxwell equations are written in the form of the Gaussian system (CGS). The constant $c$ relates the electric and the magnetic quantities and represents the speed of light in vacuum, and $\rho$ is the volume density of charge. To determine the form of the vector fields in particular conditions of charges and electric currents the Maxwell equations must be complemented with equations described by the process with the electromagnetic fields. These equations are called constitutive equations or material equations.

In general, resolution of Maxwell's equations is complicated, but if the bodies are static (or are put in motion) and the materials are isotropic, the material equations or constitutive equations are given by:

$$
\begin{align*}
& j=\sigma \cdot E  \tag{5}\\
& D=\varepsilon \cdot E  \tag{6}\\
& B=\mu \cdot H \tag{7}
\end{align*}
$$

where $\sigma$ is the electric conductivity, $\varepsilon$ the dielectric constant and $\mu$ the magnetic permittivity. Maxwell's equations are related by vector fields through a system of differential equations. Thus, if one manipulates mathematically we can obtain differential equations that will satisfy the electric and magnetic vectors.

Taking this into account, the space part does not contain charges nor currents and is a homogenous medium, i.e., $\rho=0, j=0$; in addition, $\varepsilon$ and $\mu$ are independent of the position by substituting the material equation (7) into the Maxwell equation (3) and by calculating the rotation in both parts. Hence, one obtains:

$$
\begin{equation*}
\nabla \times(\nabla \times E)+\nabla \times\left(\frac{\mu}{c} \frac{\partial H}{\partial t}\right)=0 \tag{8}
\end{equation*}
$$

By taking material equations (6) into Maxwell equation (4) and by deriving with respect to time one obtains:

$$
\begin{equation*}
\nabla \times\left(\frac{\partial H}{\partial t}\right)=\frac{\varepsilon}{c} \frac{\partial^{2} E}{\partial t^{2}}=\frac{\varepsilon}{c} \ddot{E} \tag{9}
\end{equation*}
$$

Substituting equation (9) into (8) we find:

$$
\begin{equation*}
\nabla \times(\nabla \times E)+\frac{\varepsilon \mu}{c^{2}} \ddot{E}=0 \tag{10}
\end{equation*}
$$

Now, by doing use of the identity $\nabla \times(\nabla \times v)=\nabla \nabla \cdot v-\nabla^{2} v$ and by taking into account that there is no charge, i.e., $\nabla \cdot E=0$, one obtains a wave equation similar to equation (10) for the magnetic field $\boldsymbol{H}$ :

$$
\begin{align*}
& \nabla^{2} E-\frac{\varepsilon \mu}{c^{2}} \ddot{E}=0  \tag{11}\\
& \nabla^{2} H-\frac{\varepsilon \mu}{c^{2}} H=0 \tag{11a}
\end{align*}
$$

These equations indicate the existence of electromagnetic waves with velocity of propagation given by:

$$
\begin{equation*}
v=\frac{c}{\sqrt{\varepsilon \mu}} \tag{12}
\end{equation*}
$$

This concept of the velocity of an electromagnetic wave is only represented when it refers to simple waves, for example plane waves. Equation (12) does not represent the velocity of propagation for any solution of equation (11), if one takes into account that stationary waves are solutions too.

For the common substances used the relative dielectric constant is larger than unit, and the magnetic permittivity is equal to the unit; thus, in agreement with equation (12) the velocity $v$ is almost always smaller than the speed of light in vacuum $c$.

## iii BOUNDARY CONDITIONS

Maxwell's equations have been sketched for regions of space upon which the properties of the medium, like $\varepsilon$ and $\mu$, are continuous. However, sometimes one needs to deal with physical situations whereupon the properties of medium abruptly change through one or more surfaces; in fact, one may expect that the vectors $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}$ and $\boldsymbol{B}$ change abruptly at the surfaces while $\rho$ and $j$ give surface quantities correspondingly.

Next, we derive the relation that describes the transition at an interface. First it is necessary to think about the surface of discontinuity, i.e., as not only a surface but also a thin film where the values of $\varepsilon$ and $\mu$ will vary constantly from the values $\varepsilon_{1}, \mu_{1}$ on one side of the film to $\varepsilon_{2}, \mu_{2}$ which represents the other side of the film. So into the film one considers a little cylinder whose main axis is normal to the interface, and its bases have an area $\delta A$, as represented in figure A1.


Figure A1 Interface considered as a volume of transition by the optics properties

The magnetic vector induction $\boldsymbol{B}$ and its derivates can be assumed continuous in the transition film. In this case, one can apply Gauss's theorem up to the divergence integral $\boldsymbol{B}$ into the volume of the cylinder, yielding:

$$
\begin{equation*}
\int \operatorname{div} B d V=\int B \cdot n d s=0 \tag{13}
\end{equation*}
$$

The second integral is realized at the surface of the cylinder, and $\boldsymbol{n}$ is the normal vector. Because of the consideration that the areas are very small, we can suppose that $\boldsymbol{B}$ is constant. Then, the integral of surface (13) can be changed into:

$$
\begin{equation*}
B_{1} \cdot n_{1} \delta A+B_{2} \cdot n_{2} \delta A+\text { the.participation.of the.surfaces }=0 \tag{14}
\end{equation*}
$$

Taking the limit for $\delta h$ approaching zero and neglecting the contribution of the walls, we obtain:

$$
\begin{equation*}
n \cdot\left(B_{2}-B_{1}\right)=0 \tag{15}
\end{equation*}
$$

This equation assigns the first boundary condition; the normal component of the magnetic induction is continuous at the surface discontinuity, so for writing equation (14) we take the condition $n_{1}=-n_{2}$. The electric displacement is treated in a similar way but one considers an additional term if charges are present as:

$$
\begin{equation*}
\int d i v D d V=\int D \cdot n d S=4 \pi \int \rho d V \tag{16}
\end{equation*}
$$

We take the limit for the thickness of the film approaching zero, so we must make a change from volume density to surface charge density which is given by:

$$
\begin{equation*}
\lim _{\delta h \rightarrow 0} \int \rho d V=\int \rho^{*} d A \tag{17}
\end{equation*}
$$

So

$$
\begin{equation*}
n \cdot\left(D_{2}-D_{1}\right)=4 \pi \rho^{*} \tag{18}
\end{equation*}
$$

Equation (18) assigns a second boundary condition: for a surface charge with surface density $\rho^{*}$ the normal component of the electric displacement changes quickly through the surface by a value $4 \pi \rho^{*} / \mathrm{n}$. Next, we study the tangential components by considering a rectangular area perpendicular to the film, as shown in figure A2. Let $\boldsymbol{b}$ be the unit vector perpendicular to the rectangle; then, by using Stokes' relation, Maxwell's equation (3) becomes:

$$
\begin{equation*}
\int \nabla \times E \cdot b d S=\int E \cdot d r=-\frac{1}{c} \int \dot{B} \cdot b d S \tag{19}
\end{equation*}
$$



Figure A2 Transversal cutting of the interface considered as a space of transition by the optics properties.

The first and the third integral can be realized on the area of the rectangle and the second integral along the rectangle edge. If the length $P_{1} Q_{1}$ or $P_{2} Q_{2}$ are very $\operatorname{small}(\delta \delta)$, one can accept that the electric field takes constant values $E_{1}$ and $E_{2}$ along the respective sides of the rectangle, and one can similarly consider that $B$ is constant along these paths. Taking into account these considerations, equation (19) can be expressed by:

$$
\begin{equation*}
E_{1} \cdot t_{1} \delta s+E_{2} \cdot t_{2} \delta s+\text { the.contributive.sides }=-\frac{1}{c} \dot{B} \cdot b \delta s \delta h \tag{20}
\end{equation*}
$$

Taking the limit for $\delta h$ approaching zero and assuming that $E$ presents no singularity and $\dot{B}$ is finite, then the contribution of sides $P_{1} P_{2}$ and $Q_{1} Q_{2}$ disappear:

$$
\begin{equation*}
t \cdot\left(E_{1}-E_{2}\right)=b \cdot\left[n \times\left(E_{2}-E_{1}\right)\right]=0 \tag{21}
\end{equation*}
$$

The position of the rectangle is generally arbitrary as well as the unit vector $\boldsymbol{b}$. Therefore, we have:

$$
\begin{equation*}
n \times\left(E_{1}-E_{2}\right)=0 \tag{22}
\end{equation*}
$$

Equation (22) defines a third boundary condition, i.e., the tangential component of the electric vector is continuous at the interface. One can examine the case of magnetic field, where the only difference is that a term appears when electric currents exist; thus, instead of equation (20):

$$
\begin{equation*}
H_{1} \cdot t_{1} \delta s+H_{2} \cdot t_{2} \delta s+\text { the.contributive.sides }=-\frac{1}{c} \dot{D} \cdot b \delta s \delta h+\frac{4 \pi}{c} j^{*} \cdot b \delta s \tag{23}
\end{equation*}
$$

The vector $j^{*}$ represents the surface current density in a form similar to the surface density of charge, equation (17). Hence, taking the limit of Sh approaching zero:

$$
\begin{equation*}
n \times\left(H_{1}-H_{2}\right)=\frac{4 \pi}{c} j^{*} \tag{24}
\end{equation*}
$$

Finally, equation (24) assigns a fourth boundary condition, i.e., in the presence of a surface current density $j^{*}$ the magnetic field vector undergoes a sudden change across the interface, and the value of the continuity can be written as $4 \pi / j^{*} \times n$.

## iV ELECTORMAGNETIC WAVES

In an homogeneous medium free of charge and without currents the electric and magnetic vector must satisfy the wave equation (11). The easy solution is to use the electromagnetic field of a plane wave so we will study this case.

Let $r(x, y, z)$ be the position vector of a point $P$ in space and $s\left(s_{x}, s_{y}, s_{z}\right)$ be a unit vector in the direction of propagation of the wave; thus, any solution of the wave equation takes the form:

$$
\begin{align*}
E & =E(r \cdot s-v t)  \tag{25a}\\
H & =H(r \cdot s-v t) \tag{25b}
\end{align*}
$$

These equations represent a plane wave because at each instant $t, E$ or $H$ are constants on the plane given by $r \cdot s=$ constant.

Showing with a point the derivative with respect to time $t$ and with an accent the derivative with respect to the variable $u=r \cdot s-v t$, one can write:

$$
\begin{gather*}
E=-v E^{\prime}  \tag{26}\\
(\nabla \times E)_{x}=\frac{\partial E_{z}}{\partial y}-\frac{\partial E_{y}}{\partial z}=E_{z}^{\prime} s_{y}-E_{y}^{\prime} s_{z}=\left(s \times E^{\prime}\right)_{x} \tag{27}
\end{gather*}
$$

In an analogous way one can obtain equations for the magnetic field $H$ and for the components $y$ and $z$ of the vector rotors. Substituting these equations into Maxwell's equations (3) and (4):

$$
\begin{align*}
& s \times E^{\prime}+\frac{\mu v}{c} H^{\prime}=0  \tag{28a}\\
& s \times E^{\prime}+\frac{\mu v}{c} H^{\prime}=0 \tag{28b}
\end{align*}
$$

Integrating equations (28) (discarding any constant field in space) and considering $\nu / c=1 / \sqrt{\varepsilon \mu}$, one obtains:

$$
\begin{align*}
& E=-\sqrt{\frac{\mu}{\varepsilon}} s \times H  \tag{29a}\\
& H=-\sqrt{\frac{\varepsilon}{\mu}} s \times E \tag{29b}
\end{align*}
$$

Taking the scalar product with $s$ in equations (29) one gets:

$$
\begin{equation*}
E \cdot s=H \cdot s=0 \tag{30}
\end{equation*}
$$

This equation shows the transversal electric and magnetic vectors, i.e., the field vectors are in planes perpendicular to the propagation direction. In addition, it can be concluded from equations (29) and (30) that $E, H$ and $s$ are orthogonal vectors and satisfy:

$$
\begin{equation*}
\mu H=\sqrt{\varepsilon E} \tag{31}
\end{equation*}
$$

An important particular case, is the harmonic plane waves; each Cartesian component of $E$ and $H$ has the form:

$$
\begin{align*}
& a \cos (\tau+\delta)=\operatorname{Re}\left\{a e^{-i(\tau+\delta)}\right\}  \tag{32}\\
& \tau=\omega\left(t-\frac{r \cdot s}{v}\right)=\omega t-k \cdot r \tag{33}
\end{align*}
$$

## V POLARIZATION

In previous sections, we showed how Maxwell's electromagnetic theory predicts the electromagnetic waves and could observe that for the electromagnetic waves the electric and magnetic vectors and the propagation vector establish an orthogonal system, i.e., if one takes the z -axis parallel to the direction of propagation $s$, only the components $x$ and $y$ of the vectors $E$ and $H$ will be not void.

Also one is interested to know the curve described by the vector $E$ along time at a specific point of space, so this curve is called state polarization. We do not develop those equations in this section, because the most important is the geometric concept. Generally, the electric vector $E$ will describe an ellipse in the plane XY and it is called elliptically polarized light. However this ellipse can be changed to a circle or line form, these cases are respectively called circular and lineal polarizations.

We consider a plane wave which influences a surface where the plane of incidence is defined by the direction of propagation of the wave and the normal component up to the surface. If the wave is linearly polarized then there are two cases to stand out:
a) The vector $E$ is perpendicular to the plane of incidence, in which case the wave is called Transverse Electric (TE);

In the case of a plane wave one can obtain the polarization state determining the components $x$ and $y$ of the wave, according to equations 32 and 33 , with independent amplitudes for each component. If one use the identity $\cos (\tau+\delta)=\cos \tau \cos \delta-\sin \tau \sin \delta$, so one can eliminate the variable part which contain $\tau$ and one obtain the curve. Thus the curve would be a rotated ellipse but for in particular cases can be obtained as a circle or as a line.
b) The vector $H$ is perpendicular to the plane of incidence, in which case the wave is called Transverse magnetic (TM).

We note that a polarized wave in the arbitrary form is important and it can be decomposed in two waves, i.e., a TE and a TM wave.

The TE waves are called S-polarized while the TM waves are named P polarized.

## Vi REFLECTION AND REFRACTION

When a plane wave strikes at the interface between two homogenous media having different optical properties the wave is separated in two forms: one wave enters the second medium (refracted wave) and the other is reflected and goes on propagating into the first medium. The problem of reflected and refracted waves can be resolved by means of the boundary conditions at the interface, i.e., these conditions cannot be fulfilled without both waves.

A plane wave propagates in the direction of unit vector $s_{i}$, so it can be completely defined when one knows a particular point in space. For example $F(t)$ represents the temporal state of a field at a point specified, the temporal state at another point whose position from the first point is $r$ is $F(t-r \cdot s / v)$. At the interface between two media the temporal variation of the secondary field must be equal to the primary field; then, if the unit vectors representing the directions of propagation of the reflected and refracted waves are given by $s_{r}$ and $s_{t}$, respectively, one can make the analysis of the waves at a point $r$ of the interface:

$$
\begin{equation*}
t-\frac{r \cdot s_{i}}{v_{1}}=t-\frac{r \cdot s_{r}}{v_{1}}=t-\frac{r \cdot t_{t}}{v_{2}} \tag{34}
\end{equation*}
$$

where $v_{1}$ and $v_{2}$ are the velocities of propagation in the media.

We now consider a plane wave which propagates in the plane XZ at the interface $\mathbf{z}=0$. Then, $r=(x, y, 0)$ and equation (34) becomes:

$$
\begin{equation*}
\frac{x s_{i x}+y s_{i y}}{v_{1}}=\frac{x s_{r x}+y s_{r y}}{v_{1}}=\frac{x s_{t x}+y s_{t y}}{v_{2}} \tag{35}
\end{equation*}
$$

Equation (35) must be satisfied at any point on the interface, i.e., any value of $x$ and y given by:

$$
\begin{align*}
& \frac{s_{i x}}{v_{1}}=\frac{s_{r x}}{v_{1}}=\frac{s_{t x}}{v_{2}}  \tag{36a}\\
& \frac{s_{i y}}{v_{1}}=\frac{s_{r y}}{v_{1}}=\frac{s_{t y}}{v_{2}} \tag{36b}
\end{align*}
$$

The plane defined by $s_{i}$ and the normal vector is known as the plane of incidence; in our case it is taken as the plane XZ , equation (36) shows that the incident, reflected and refracted waves are in the plane of incidence. Defining $\theta_{i}, \theta_{r}$ and $\theta_{t}$ as the angles that the vectors $s_{i}, s_{r}$ and $s_{t}$ make with the z -axis, they can be written in the form:

$$
\begin{array}{lll}
s_{i x}=\sin \theta_{i} & s_{i y}=0 & s_{i z}=\cos \theta_{i} \\
s_{i r}=\sin \theta_{r} & s_{i r}=0 & s_{i z}=\cos \theta_{r} \\
s_{i t}=\sin \theta_{t} & s_{i t}=0 & s_{i z}=\cos \theta_{t} \tag{37}
\end{array}
$$

The subscript $i$, r and $t$ are to denote the magnitudes of the incident, reflected and transmitted waves.

Equation (37) is substituted by the components x into equation (36):

$$
\begin{equation*}
\frac{\sin \theta_{i}}{v_{1}}=\frac{\sin \theta_{r}}{v_{1}}=\frac{\sin \theta_{t}}{v_{2}} \tag{38}
\end{equation*}
$$

Then, $\sin \theta_{i}=\sin \theta_{r}$ and because the reflected wave propagates in the same plane than the incident wave $\cos \theta_{i}=\cos \theta_{r}$, one obtains:

$$
\begin{equation*}
\theta_{r}=2 \pi-\theta_{i}=-\theta_{i} \tag{39}
\end{equation*}
$$

This equation together with the fact that the reflected wave is in the same incident plane, establish the Reflection Law. Moreover, from equation (38) one can see:

$$
\begin{equation*}
\frac{\sin \theta_{i}}{\sin \theta_{t}}=\frac{v_{1}}{v_{2}} \tag{40}
\end{equation*}
$$

So that using equation (12) and defining the refraction index of a medium $n$ as the ratio between the velocity in vacuum and the velocity of propagation in the medium, we obtain:

$$
\begin{gather*}
n_{k}=\frac{v_{k}}{c}=\frac{\sqrt{\varepsilon_{0} \mu_{0}}}{\sqrt{\varepsilon_{k} \mu_{k}}}  \tag{41}\\
\frac{\sin \theta_{i}}{\sin \theta_{t}}=\frac{n_{1}}{n_{2}}=\sqrt{\frac{\varepsilon_{2} \mu_{2}}{\varepsilon_{1} \mu_{1}}}=n_{12} \tag{42}
\end{gather*}
$$

Equation (42) along with the fact that the refracted wave is in the incident plane establish the Refraction law. $n_{12}=\frac{n_{1}}{n_{2}}$ is as the ratio between indexes of refraction both media (see figure A3).


FIGURE A3 Direction and paths of the waves and incident, reflected and refracted electric fields at an interface on the plane XY

## Vii FRESNEL'S FORMULA

In previous sections we presented the laws of reflection and refraction which indicate the direction of the electromagnetic waves in each case. Both are functions of the incident wave and of the optical properties of the medium. However, we have not considered the amplitudes and intensities of the reflected and refracted fields. Fresnel`s theory shows us how one can calculate the amplitude of the fields.

We consider the media as homogenous and isotropic, having no conductivity and being transparent, i.e., $\left(\varepsilon=\varepsilon^{\prime}, \varepsilon^{\prime \prime}=0\right)^{*}$; moreover, their magnetic permeability is equal to the unit $\left(\mu_{1}=\mu_{2}=1\right)$.

Let $E_{i}$ be the amplitude of electric incident field, so $E_{i}$ is a complex number with its phase equal to wave function $\delta$ (See equation (32)) yields:

$$
\begin{equation*}
\tau_{i}=\omega\left(t-\frac{r \cdot s_{i}}{v_{1}}\right)=\omega\left(t-\frac{x \sin \theta_{i}+z \cos \theta_{i}}{v_{i}}\right) \tag{43}
\end{equation*}
$$

We separate the vectors in the form of components parallel (subscript $E_{/ /}$) and perpendicular (subscript $E_{\perp}$ ) to the plane of incidence. In Figure A3 one can see the positive direction of the components, so that the Cartesian components of the electric incident field are given by:

$$
\begin{gather*}
E_{i x}=-E_{i / /} \cos \theta_{i} \exp \left(-i \tau_{i} \omega\right)  \tag{44a}\\
E_{i y}=-E_{i \perp} \exp \left(-i \tau_{i} \omega\right)  \tag{44b}\\
E_{i z}=E_{i / /} \sin \theta_{i} \exp \left(-i \tau_{i} \omega\right) \tag{44c}
\end{gather*}
$$

The components of the magnetic field can be calculated by equation (41) with $\mu=1$ :

$$
\begin{equation*}
H=\sqrt{\varepsilon} . . s \times E \tag{45}
\end{equation*}
$$

[^3]\[

$$
\begin{gather*}
H_{i x}=-E_{i \perp} \cos \theta_{i} \sqrt{\varepsilon_{1}} \exp \left(-i \tau_{i} \omega\right)  \tag{46a}\\
H_{i y}=-E_{i / /} \sqrt{\varepsilon_{1}} \exp \left(-i \tau_{i} \omega\right)  \tag{46b}\\
H_{i z}=E_{i \perp} \sin \theta_{i} \sqrt{\varepsilon_{1}} \exp \left(-i \tau_{i} \omega\right) \tag{46c}
\end{gather*}
$$
\]

Taking the reflection and refraction laws of previous section and by following a development analogous to the above one can develop the reflected and refracted components of the electric and magnetic fields.

The boundary conditions can be obtained from the continuity of the tangential components of the fields, hence, they yield:

$$
\begin{array}{ll}
E_{i x}+E_{r x}=E_{t x} & E_{i y}+E_{r y}=E_{t y} \\
H_{i x}+H_{r x}=H_{t x} & H_{i y}+H_{r y}=H_{t y} \tag{47a}
\end{array}
$$

The conditions (15) and (18) for the normal components of $\boldsymbol{B}$ and $\boldsymbol{D}$ satisfy automatically. Substituting equation (47) into the components of the incident reflected and refracted field and taking into account $\cos \theta_{r}=-\cos \theta_{i}$, one obtains:

$$
\begin{gather*}
\cos \theta_{i}\left(E_{i / /}-E_{r / /}\right)=\cos \theta_{t} E_{t / /}  \tag{48a}\\
E_{i \perp}+E_{r \perp}=E_{r \perp}  \tag{48b}\\
\sqrt{\varepsilon_{1}} \cos \theta_{i}\left(E_{i \perp}-E_{r \perp}\right)=\sqrt{\varepsilon_{2}} \cos \theta_{t} E_{t \perp}  \tag{48c}\\
\sqrt{\varepsilon_{1}}\left(E_{i / /}+E_{r / /}\right)=\sqrt{\varepsilon_{2}} E_{t / /} \tag{48d}
\end{gather*}
$$

These equations consist of two groups of two equations: a group contains the parallel components while the other group contains the perpendicular components to the plane of incidence, therefore these two kinds of waves are independent.

According to the relation $n=\sqrt{\varepsilon}$, one can resolve the system of equations (48) for the reflected and refracted components in terms of the incident wave:

$$
\begin{align*}
& E_{t / /}=\frac{2 n_{1} \cos \theta_{i}}{n_{2} \cos \theta_{i}+n_{1} \cos \theta_{t}} E_{i / /}  \tag{49a}\\
& E_{t \perp}=\frac{2 n_{1} \cos \theta_{i}}{n_{1} \cos \theta_{i}+n_{2} \cos \theta_{t}} E_{i \perp}  \tag{49b}\\
& E_{r / /}=\frac{n_{2} \cos \theta_{i}-n_{1} \cos \theta_{t}}{n_{2} \cos \theta_{i}+n_{1} \cos \theta_{t}} E_{i / /}  \tag{49c}\\
& E_{r \perp}=\frac{n_{1} \cos \theta_{i}-n_{2} \cos \theta_{t}}{n_{1} \cos \theta_{i}+n_{2} \cos \theta_{t}} E_{i \perp} \tag{49d}
\end{align*}
$$

Equations (49) are called Fresnel's formulas. By using these equations one can calculate the reflectivity and refractivity, thus these developments can be expressed for multilayer systems including film of conductive materials (i.e. nontransparent materials with $\left.\varepsilon=\varepsilon^{\prime}+i \varepsilon^{\prime \prime}\right)$.

## Viii TOTAL INTERNAL REFLECTION

We have excluded up to now the case where the refraction obtains an imaginary value for the angle of refraction $\theta_{t}$. It happens when the light is propagated from a dense medium to another less dense medium so that:

$$
\begin{equation*}
n_{12}=\frac{n_{2}}{n_{1}}=\sqrt{\frac{\varepsilon_{2} \mu_{2}}{\varepsilon_{1} \mu_{1}}}<1 \tag{50}
\end{equation*}
$$

and its incident angle is larger than critic angle $\theta_{c}$ given by $\sin \theta_{c}=n_{12}$.

When $\theta_{i}=\theta_{c}, \sin \theta_{t}=1$, then $\theta_{t}=90^{\circ}$ and the reflected light emerges in a direction tangential to the interface. If $\theta_{i}$ is higher than $\theta_{c}$ all the incident light is reflected into the first medium, a phenomenon known as total internal reflection.

Taking into account that $\theta_{t}$ is a complex number $\left(\sin \theta_{t}>1\right)$, one can write:

$$
\begin{gather*}
\sin \theta_{t}=\frac{\sin \theta_{i}}{n_{12}}  \tag{51a}\\
\cos \theta_{t}= \pm i \sqrt{\frac{\sin ^{2} \theta_{i}}{n_{12}^{2}}-1} \tag{51b}
\end{gather*}
$$

Substituting these equations into Fresnel`s equations, one can see for each component that the intensity of light is totally reflected:

$$
\begin{equation*}
\left|E_{r / /}\right|=\left|E_{i / /}\right| \quad\left|E_{r \perp}\right|=\left|E_{i \perp}\right| \tag{52}
\end{equation*}
$$

However the electromagnetic field in the second medium does not disappear, it only implies that there is no net flux of energy through the interface; thus one can remember the phase that is the part variable written by the refracted and reflected wave (equation (43) written by the refracted wave)

$$
\begin{equation*}
\tau_{t}=\omega\left(t-\frac{r \cdot s_{t}}{v_{2}}\right)=\omega\left(t-\frac{x \sin \theta_{t}+z \cos \theta_{t}}{v_{2}}\right) \tag{53}
\end{equation*}
$$

Substituting equations (52) into the phase, the wave equation becomes:

$$
\begin{equation*}
\exp \left(-i \omega \tau_{t}\right)=\exp \left(-i\left(t-\frac{x \sin \theta_{t}}{n v_{2}}\right)\right) \exp \left( \pm \frac{\omega z}{v_{2}} \sqrt{\frac{\sin ^{2} \theta_{i}}{n_{2}}-1}\right) \tag{54}
\end{equation*}
$$

This equation shows an aspect of the phenomenon of total internal reflection, i.e., that an electromagnetic field in the incident plane extends beyond the interface, this electromagnetic field decays exponentially in amplitude in the depth of the second medium ${ }^{*}$; the length of the amplitude decay at $1 / e$ is given by:

$$
\begin{equation*}
L=\frac{v_{2}}{\omega}\left(\frac{\sin ^{2} \theta_{i}}{n_{12}^{2}}-1\right)^{-1 / 2}=\frac{\lambda_{2}}{2 \pi}\left(\frac{\sin \theta_{i}}{n_{12}^{2}}-1\right)^{-1 / 2} \tag{54}
\end{equation*}
$$

[^4]
## APPENDIX 2

## Electrons in Movement in a Metal According to Drude`s theory

## iX INTRODUCTION

The discovery by J. J. Thomson, of the electron in 1897, has caused an impact in the theories of the structure of matter and has proposed a structure of the conductivity of the metals. After three years of the discovery by Thomson, Drude`s theory was constructed for the electric and thermal conduction which is the kinetic theory of the gas applied to the metals considered as an electron gas.

Drude`s theory [31] supposes that the electrons are immersed in a positive uniform potential imposed by the immobile ions at the crystalline-lattice. By considering that only during the collisions (with the ions or with other electrons) are crated power on the electrons and the duration of the collisions are insignificant.

## X EQUATION OF MOVEMENT OF THE ELECTRONS

The movement of the electrons is caused by a uniform power (electric or magnetic) in the Drude model.

An electron taken at random at the instant $t$ will cause a collision at the instant $t+d t$ with probability $d t / \tau$, where $\tau$ is the relaxation time, conversely it will pass a time $d t$ without causing collision with probability $1-d t / \tau$. If the electron does not undergo a collision it will evolve by the action of the uniform power which is caused on him, because of spatially electric and magnetic uniform field, and it will acquire a quantity of movement $f(t) d t+O(d t)^{2}$. Let $x(t)$ be as the displacement of the electron at the instant $t$. The contribution to the displacement of all the electrons that have not undergone a collision between $t$ and $t+d t$ is represented by the product between the fraction $(1-d t / \tau)$ and the average displacement $\left\lfloor x(t)+f(t) d t+O(d t)^{2}\right\rfloor$ of the electrons.

Then, if one eliminates the contribution to $x(t+d t)$ of the electrons that are caused by a collision between $t$ and $t+d t$, one obtains:

$$
\begin{align*}
& x(t+d t)=\left(1-\frac{d t}{\tau}\right)\left(x(t)+f(t) d t+O(d t)^{2}\right)  \tag{1a}\\
& x(t+d t)=x(t)-\left(\frac{d t}{\tau}\right) x(t)+f(t) d t+O(d t)^{2} \tag{1b}
\end{align*}
$$

The adjustment equation (1), because of the electrons which cause a collision, is of second order in $d t$. To see this one must first note that the electrons are composed by a fraction $d t / \tau$. In addition, as the velocity and displacement has
taken a random direction after each collision, each electron will contribute to the average of the displacement $x(t+d t)$ only if it has acquired a displacement through the power action $f$ from the last collision. Then the quantity must be acquired by a time no higher than $d t$, thus it is approximately $f(t) d t$, thus the correction is about $(d t / \tau) f(t) d t$ which does not affect the lineal terms at $d t$. Therefore, one can write:

$$
\begin{equation*}
x(t+d t)-x(t)=-\left(\frac{d t}{\tau}\right) x(t)+f(t) d t+O(d t)^{2} \tag{2}
\end{equation*}
$$

where the contribution has been taken by all the electrons. By dividing $d t$ and taking the limit when $d t$ approaches to zero, it becomes:

$$
\begin{equation*}
\frac{d x(t)}{d t}=-\frac{x(t)}{\tau}+f(t) \tag{3}
\end{equation*}
$$

This is an equation of movement of the electrons driven by a uniform power according to Drude`s model, the equation establishes that the electrons impact the ions and insert a term of attenuation in the equation of movement.

## Conferences

## 1. Participation in National and International Conferences

### 1.1 Participation in National Conferences

1. Hèctor Hugo Sánchez Hernández, Gabriel Martínez Niconoff, Javier Muñoz Lopez, "Descripción Modal en Interfaces Conductoras", XVLIII Congreso Nacional de Física SMF, Guadalajara, Jalisco, del 17 al 21 de Octubre 2005.
2. Daniel Rojano Guido, Gabriel Martínez Niconoff, Hèctor Hugo Sánchez Hernández, Graciela Hernández y Orduña, "Convergencia Incoherente de Campos Ópticos Libres de Difracción", XVLIII Congreso Nacional de Física SMF, Guadalajara, Jalisco, del 17al 21 de Octubre 2005.
3. Hèctor Hugo Sánchez Hernández, Gabriel Martínez Niconoff, Nicolás Grijalva y Ortiz, "Análisis extremal de plasmones superficiales", XLIX Congreso Nacional de Física SMF, San Luís Potosí, del 16 al 20 de Octubre del 2006.
4. Hèctor Hugo Sánchez Hernández, Gabriel Martínez Niconoff, Daniel Rojano Guido,"Autoimágenes con ondas plasmonicas", XLIX Congreso Nacional de Física SMF, San Luís Potosí, del 16 al 20 de Octubre del 2006.

## Participation in International Conferences

2. Héctor Hugo Sánchez Hdez. and Gabriel Martínez Niconoff, "Angular Spectrum Model for Plasmon Fields", SPIE Optics Photonics, 13-17 August 2006.

## 2. Publications

### 2.1 National Conference Proceedings

1. Hèctor Hugo Sánchez Hernández, Gabriel Martínez Niconoff, Daniel Rojano Guido,"Autoimágenes con ondas plasmonicas", XLIX Congreso Nacional de Física SMF, XIX Anual AMO, San Luís Potosí, SLP. Óptica HO01, 2006.
2. Hèctor Hugo Sánchez Hernández, Gabriel Martínez Niconoff, Daniel Rojano Guido, "Análisis extremal de plasmones superficiales", XLIX Congreso Nacional de Física SMF, XIX Anual AMO, San Luís Potosí, SLP. Óptica OE-02 2006
3. Héctor Hugo Sánchez Hdez. and Gabriel Martínez Niconoff, "Angular Spectrum Model for Plasmon Fields", SPIE Optics Photonics, Proceedings of SPIE, Volume 6323, Agu. 30, 2006.

### 2.2 International Publications

1. Gabriel Martinez Niconoff, J. A: Sanchez-Gil, Hector Hugo Sanchez and A Perez Leija, "Self-Imaging and Caustics in Two-Dimentional Surface Plasmons Optics", Optics Communications, accepted [10 december 2007].

[^0]:    * Equation (2) can be found in appendix 1, where $e$ is the electron charge and $\tau$ is the relaxation time, i.e. the average time during which an electron can move without having a collision with an ion.

[^1]:    ${ }^{*}$ Taking into account Lorentz law $F=-e(E+1 / c v \times H)$

[^2]:    *This condition is not satisfied and it is necessary to tackle the problem with nonlocal theories.
    ! We consider an electromagnetic wave where there is no induced charge density, farther on we consider the case of oscillations at the charge density.

[^3]:    *The dielectric constant is in general a complex number $\varepsilon=\varepsilon^{\prime}+i \varepsilon^{\prime \prime}$. The imagine part $\varepsilon^{\prime \prime}$ is related with the losses by absorption. When a metal is transparent i.e., there is not absorption of the light $\varepsilon^{\prime \prime}=0$ the Fresnel calculates for transparent materials making easy the understanding. Farther on, one will take the problem the surface plasmon excitation, but it is necessary of a film metal which it is not transparent. In this case one would see the fundamental roll of imagine part of the dielectric constant

[^4]:    *Only the sing negative in front of the square root is the signified physics. The positive sing mean which the amplitude of the wave would infinitely increase in the form exponential while it enter at the second medium.
    ${ }^{+}$The interaction of the electromagnetic evanescent field with the medium start a number of techniques known as Attenuated Total Internal Reflection (ATR) .

