Semi-classical Model of a Pair of Coupled Quantum Dots.

By:

Phys. Julio César García Melgarejo

INAOE

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Advisors:

Dr. José Javier Sánchez Mondragón

Dr. Ponciano Rodríguez Montero

INAOE researchers

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Abstract

Quantum dots are semiconductor structures whose size is in the order of 10 nm. These structures can confine inside themselves electrons. The study of these structures is interesting because of their application range is quite wide. They are used in the manufacture of lasers with a small bandwidth. In medicine they are used to obtain highly-contrasted medical images. Also, they can be applied in the manufacture of efficient solar cells.

Among quantum dots there is an interaction, called Foerster interaction; it consists on the exciton transfer from a quantum dot to another in a non-radiative transfer mechanism.

This thesis work consist of two sections, the first is devoted to develop a semi-classical model of the interaction between a single quantum dot and a classical electric field. Analytic expressions for the single QD-population inversion and complex amplitude electric dipole are given. Later this analysis is generalized into a semiclassical study of a pair of coupled quantum dots through their Foerster interaction; each quantum dot is within its own cavity interacting with its own classical electric field. We give analytic expressions for their inversions and for their respective complex electric dipole.

Finally we emphasize the weak coupling regime and in addition, we point out the characteristics of that system in comparison to the single quantum dot. The observation of those characteristics is a proof of the coupling and its analysis allows obtaining information on the strength of the coupling.

Resumen

Los puntos cuánticos son estructuras semiconductoras cuyo tamaño es del orden de 10 nm. Estas estructuras tienen la propiedad de confinar en su interior electrones. El estudio de estas estructuras es interesante, ya que su campo de aplicación es muy amplio. Son utilizados en la fabricación de láseres con un ancho de banda muy pequeño. En la medicina son utilizados para obtener imágenes médicas de alto contraste. También pueden ser aplicados en la construcción de celdas solares eficientes.

Entre los puntos cuánticos se presenta una interacción, denominada interacción de Foerster, la cual consiste en la transferencia de un excitón de un punto cuántico al otro. Este es un mecanismo de transferencia de energía no radiativo.

Este trabajo de tesis consta de dos secciones, una dedicada a desarrollar el modelo semiclásico de la interacción de un punto cuántico con un campo eléctrico clásico. Se proporciona la expresión analítica para la inversión del punto cuántico; de la misma forma se proporcionan expresiones analíticas para las oscilaciones de la amplitud compleja del dipolo eléctrico.

Posteriormente este análisis es generalizado al estudio de un par de puntos cuánticos acoplados mediante una interacción de Foerster; cada punto cuántico está en su propia cavidad interactuando con su propio campo eléctrico clásico. Proporcionamos expresiones analíticas para la inversión y para las oscilaciones complejas del dipolo eléctrico.

Finalmente se pone énfasis en el régimen de acoplamiento débil comparado con la Frecuencia de Rabi y se señalan las características que el sistema acoplado presenta respecto al sistema de un punto cuántico. La observación de estas características es una prueba del acoplamiento y su análisis permite obtener información acerca de su fuerza.

Contents

Chapter 1

General Introduction

1.1 Historical introduction

For a long period of time the scientific research in electronic systems was limited to systems as isolated atoms or particles, metals or semiconductor crystals, or beams of beta radiation; most of those are three-dimensional systems.

In the early 1970s, the research on semiconductor structures introduced an important development [13]. They were the quantum wells; structures build as very thin flat layers of semiconductor with high conduction-band energies [1]. The motion of bound electrons in a layer, as thin as several crystalline monolayers, is two-dimensional; and the excitations in the perpendicular direction are strongly quantized. Nowadays, the characteristics of these quasi-two dimensional systems are well understood; quantum wells has been produced and implemented in devices as numerous as common, such is the case of CD players or microwave receivers used in satellite television.

At the beginning of the 1980s the progress in lithographic techniques allowed to confine electrons in a quasi one-dimensional structure, the so called quantum wire [14]. They were produced in the form of a miniature strip, etched in a sample containing a quantum well.

A complete trapping of the electrons in a quasi-zero dimensional structure was reported by scientists from Texas Instrument Incorporated. Their dimensions were of the order of 250 nm [2]. Subsequent publications reported structures of the order of 30-45 nm at the AT&T Bell Laboratories and Bell Communication Research Incorporated [8, 11]; these structures are the quantum dots. The quantum dots (QD) are solid-state structures on the 10 nanometers range; they are often called artificial atoms because they have quantum properties similar to those of individual atoms at the 0.1 nm scale [16]. These nanometer-scale material structures provide a potential energy well in a size similar to that of the deBroglie wavelength, trapping the carriers in discrete energy levels, and resulting in objects with atom-like optical properties.

1.2 Quantum dots relevance in optics

QD have important and varied scientific and technological applications. The use of quantum dots to produce solar cells allows to increase the maximum attainable thermodynamic conversion efficiency of solar photon conversion up to 66% [9]. On the other hand QDs have applications in biological areas, fluorescence imaging and tumor imaging [20] among others. Also has been explored the possibility to use the quantum dots as active medium in the construction of lasers with a high spectral purity.

1.3 Interaction among quantum dots

When a quantum dot is in the presence of an electric field, there is a dipolar interaction between them, and a well expected dynamics like the one with a Two Level Atom. However, if more than one QD is nearby, there is an additional quantum and non radiative coupling between the QDs, produced by the exchange of an exciton. Therefore the marriage of both interactions introduces quite an interesting dynamics that is the object of this thesis.

When two quantum dots are sufficiently close, about 10nm, a resonant energy-transfer process is originated. This process is called Foerster interaction and is fundamental in biologic processes, and organic systems, such the photosynthesis. In this process there is an excitation exchange with the neighbor QD, in other words, an excited QD decays into the ground state while its neighbor change to the excited state without the emission of a photon [6].

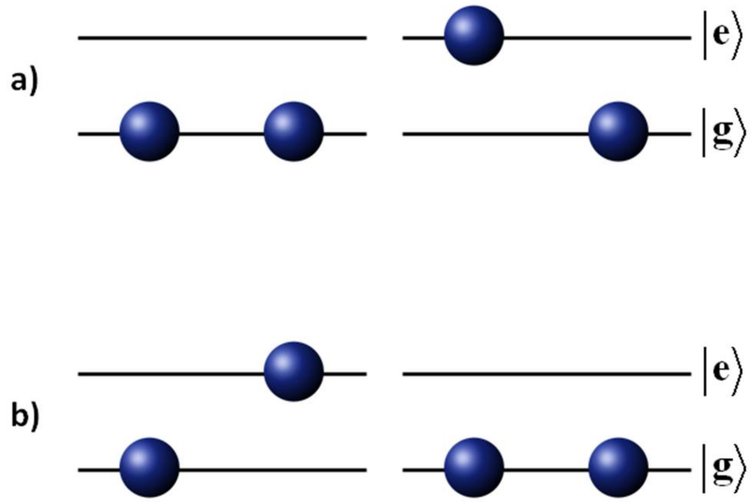


Figure 1.1: Forster interaction. *a*) Initial configuration. *b*) Final configuration. An electron in the first system is excited while the second has decayed.

We study a system of N, identical quantum and equally spaced, QDs with a null net charge and illuminated by a classical light. If $N = 2$ then the QDs are in a line, when $N = 3$ the QD are in the vertices of an equilateral triangle and when $N = 4$ the QDs are at the vertices of a pyramid.

The Hamiltonian that gives the interaction of these quantum dots [12] is

$$\begin{aligned} H(t) &= \frac{1}{2}\hbar\epsilon \sum_{n=1}^{N} \left(e_n^\dagger e_n - h_n h_n^\dagger\right) - \frac{1}{2}\hbar W \sum_{n,n'}^{N} \left(e_n^\dagger h_{n'} e_{n'} h_n^\dagger + h_n e_{n'}^\dagger h_{n'}^\dagger e_n\right) \\ &\quad -d(t)\cdot E(t) \sum_{n=1}^{N} e_n^\dagger h_n^\dagger - d^*(t)\cdot E^*(t) \sum_{n=1}^{N} e_n h_n \end{aligned} \tag{1.1}$$

Here ε is the band gap energy, W gives the strength of the Foerster interaction, $e_n^\dagger (h_n^\dagger)$ is the electron (hole) creation operator in the nth QD. Let us notice the presence of a common electric field to the system of N quantum dots.

1.4 Structure of this thesis

The aim of this thesis work is to develop an analysis of the optical control of a pair of quantum dots; each one interacting with its own classical electric field. To achieve this aim, the thesis is organized as follows.

Chapter 2 gives a detailed study of a single quantum dot interacting with a classical electric field. This interaction is given in exact resonance, i.e., when the exciting frequency is equal to the QD frequency. We carry on a derivation along the lines of a well know derivation for a Two Level Atom (TLA), that we could summarize as deriving the equations of motion from the wave function. This is, we derive the corresponding wave function and from it, the density matrix and therein the TLA Bloch equations, instead of the common approach of deriving the Heisenberg Equations. A detailed development in the Heisenberg picture can be found in Optical Resonance and Two-Level Atoms [3], but soon we will realize that the Foerster dipole-dipole interaction introduces an unnecessary complexity. We have preferred to carry out a study in the Schrödinger picture because of the simplicity of the Pauli matrices acting on the eigenstates of the QD. We will produce analytical expressions for the single QD-population inversion and we will analyze the resonance fluorescent spectrum.

In chapter 3, we will introduce the study of a pair of quantum dots, each one within its own micro-cavity and interacting with its own classical electric field, but still coupled trough the Foerster interaction. The aim is to develop a similar analysis to that done in chapter 2 and to produce the corresponding results for the inversion and the resonance fluorescent spectra of each one of the QDs. However, such a simple idea shows the convenience of our approach. First, it allows to conveniently deal with the Foerster term and then by introducing the entanglement description in a straightforward manner.

Chapter 2

Semiclassical model of a single quantum dot

2.1 Introduction

The Hamiltonian for the study of a set of N quantum dots interacting with the same electric field has been developed by Quiroga et al, and their Hamiltonian will be the starting point of our discussion; we will analyze a single quantum dot interacting with its classical electric field.

This development is relevant because will give us the mathematical frame that we will purpose in the following chapter when we will study the coupling between a pair of coupled QDs; the results that we will obtain in this chapter shall allow us to determinate the additional features because of the coupling.

2.2 Model

The Hamiltonian that describes a set of N quantum dots interacting with an electric field is given by

$$H(t) = \frac{1}{2}\hbar\epsilon \sum_{n=1}^{N} \left(e_n^\dagger e_n - h_n h_n^\dagger\right) - \frac{1}{2}\hbar W \sum_{n,n'}^{N} \left(e_n^\dagger h_{n'} e_{n'} h_n^\dagger + h_n e_{n'}^\dagger h_{n'}^\dagger e_n\right) - d(t) \cdot E(t) \sum_{n=1}^{N} e_n^\dagger h_n^\dagger - d^*(t) \cdot E^*(t) \sum_{n=1}^{N} e_n h_n. \tag{2.1}$$

Our interest in this chapter is studying the case of a single QD, i. e., $N = 1$. In that case the Hamiltonian 2.1 becomes,

$$H(t) = \frac{\epsilon}{2}\left(e^\dagger e - hh^\dagger\right) - \frac{1}{2}W\left(e^\dagger h e h^\dagger + h e^\dagger h^\dagger e\right) - d(t) \cdot E(t) e^\dagger h^\dagger - d^*(t) \cdot E^*(t) he,$$

This can be rewritten in a simplified form by using the pseudo spin operators:

$$H(t) = \hbar\epsilon J_z - \hbar W\left(J^2 - J_z^2\right) - d(t) \cdot E(t) J_+ - d^*(t) \cdot E^*(t) J_-,$$

Where

$$\begin{aligned} J_+ &= e^\dagger h^\dagger, \\ J_- &= he, \\ J_z &= \frac{1}{2}\left(e^\dagger e - hh^\dagger\right). \end{aligned} \tag{2.2}$$

These operators satisfy the usual commutation relations

$$[J_+, J_-] = 2J_z,$$
$$[J_\pm, J_z] = \mp J_\pm.$$

The use of two convenient expressions

$$J_{\pm} = J_x \pm i J_y,$$

and

$$J^2 - J_z = J_+ J_- - J_z,$$

allows to rewrite the Hamiltonian in a more convenient form in terms of the electric field and the electric dipole, both as real quantities

$$H(t) = \frac{1}{2}\hbar\left(\epsilon - W\right)\sigma_z - d(t)\cdot E(t)\sigma_x - \hbar W \sigma_+ \sigma_-, \tag{2.3}$$

where J is related with the Pauli's matrices through $J = \sigma/2$.

2.2.1 Non interacting system

The Hamiltonian without interaction is given by:

$$H_0 = \frac{1}{2}\hbar\left(\epsilon - W\right)\sigma_z - \hbar W \sigma_+ \sigma_-$$

For a single spinor $\sigma_+\sigma_- = (1+\sigma_z)/2$, therefore

$$H_0 = \frac{1}{2}\hbar\left(\epsilon - 2W\right) - \hbar\frac{W}{2}.$$

Therefore the two eigenstates, for the ground $|g\rangle$ and the excited state $|e\rangle$, are the well known spinor eigenvalues of σ_z, with eigenvalues $E_0^g = -\hbar(\epsilon - W)/2$ and $E_0^e = \hbar(\epsilon - 3W)/2$ respectively. The time dependent wavefunction for the free quantum dot is obtained from the Schrödinger's equation:

$$i\hbar\left|\dot{\psi}_0(t)\right\rangle = H_0\left|\psi_0(t)\right\rangle,$$

with solution $\left|\psi_0^l(t)\right\rangle = |l\rangle \exp\left(-iE_0^l t/\hbar\right)$ for $l = g, e$. The general wavefunction is given by:

$$|\psi_0(t)\rangle = C_g(0)\exp\left(-iE_0^g t/\hbar\right)|g\rangle + C_e(0)\exp\left(-iE_0^e t/\hbar\right)|e\rangle \tag{2.4}$$

where $C_g(0)$ and $C_e(0)$ are constants, but we will use the parenthesis for convenience in the next chapter.

2.3 Interaction with a classical electric field

The interaction Hamiltonian, in addition of the specified non-interacting Hamiltonian H_0, has a term of the form $H_0 = -d(t) \cdot E(t)\sigma_x$. In the Schrödinger's picture, the dynamics of the interaction system is completely contained by the wave function, and to take into account the dipole interaction in the wave function the C coefficients must be functions depending on time:

$$|\psi(t)\rangle = C_g(t)\exp\left(-iE_0^g t/\hbar\right)|g\rangle + C_e(t)\exp\left(-iE_0^e t/\hbar\right)|e\rangle \tag{2.5}$$

and satisfy the Schrödinger's equation given by the interaction Hamiltonian:

$$i\hbar\left|\dot{\psi}(t)\right\rangle = (H_0 + H_{\text{int}})\,|\psi(t)\rangle \tag{2.6}$$

The left hand side of is

$$\begin{aligned} i\hbar\left|\dot{\psi}(t)\right\rangle = &\left[\dot{C}_g(t)\exp\left(-iE_0^g t/\hbar\right) - iE_0^g C_g(t)\exp\left(-iE_0^g t/\hbar\right)/\hbar\right]|g\rangle \\ &+ \left[\dot{C}_e(t)\exp\left(-iE_0^e t/\hbar\right) - iE_0^e C_e(t)\exp\left(-iE_0^e t/\hbar\right)/\hbar\right]|e\rangle \end{aligned}$$

To calculate the right hand side of Eq it is necessary to obtain expressions for the σ_x acting over the ground and excited states. This can be obtained by considering the expressions for the σ_+ and σ_- operators

$$\begin{aligned}\sigma_+ &= \frac{1}{2}\left(\sigma_x + i\sigma_y\right)\\ \sigma_- &= \frac{1}{2}\left(\sigma_x - i\sigma_y\right)\end{aligned}$$

From here, $\sigma_x = \sigma_+ + \sigma_-$ and therefore the action of σ_x on the ground and excited state is

$$\begin{aligned}\sigma_x\left|e\right\rangle &= \left(\sigma_+ + \sigma_-\right)\left|e\right\rangle = \left|g\right\rangle\\ \sigma_x\left|g\right\rangle &= \left(\sigma_+ + \sigma_-\right)\left|g\right\rangle = \left|e\right\rangle\end{aligned}$$

With these expressions we can determinate the right hand side of the equation in which the dipole interaction is proportional to the σ_x Pauli matrix

$$\begin{aligned}(H_0 + H_{\text{int}}) &= \left[H_0 - d(t)\cdot E(t)\sigma_x\right]C_g(t)\exp\left(-iE_0^g t/\hbar\right)\left|g\right\rangle\\ &\quad + \left[H_0 - d(t)\cdot E(t)\sigma_x\right]C_e(t)\exp\left(-iE_0^e t/\hbar\right)\left|e\right\rangle\\ &= C_g(t)\exp\left(-iE_0^g t/\hbar\right)E_0^g\left|g\right\rangle - C_g(t)\exp\left(-iE_0^g t/\hbar\right)d(t)\cdot E(t)\left|e\right\rangle\\ &\quad + C_e(t)\exp\left(-iE_0^e t/\hbar\right)E_0^e\left|e\right\rangle - C_e(t)\exp\left(-iE_0^e t/\hbar\right)d(t)\cdot E(t)\left|g\right\rangle\\ &= \left[C_g(t)\exp\left(-iE_0^g t/\hbar\right)E_0^g - C_e(t)\exp\left(-iE_0^e t/\hbar\right)d(t)\cdot E(t)\right]\left|g\right\rangle\\ &\quad + \left[C_e(t)\exp\left(-iE_0^e t/\hbar\right)E_0^e - C_g(t)\exp\left(-iE_0^g t/\hbar\right)d(t)\cdot E(t)\right]\left|e\right\rangle\end{aligned}$$

From here, we obtain

$$\begin{aligned}\dot{C}_g(t)\exp\left(-iE_0^g t/\hbar\right) - i\frac{E_0^g}{\hbar}C_g(t)\exp\left(-iE_0^g t/\hbar\right) &= -iC_g(t)\exp\left(-iE_0^g t/\hbar\right)\frac{E_0^g}{\hbar}\\ &\quad + i\frac{d(t)\cdot E(t)}{\hbar}C_e(t)\exp\left(-iE_0^e t/\hbar\right)\\ \dot{C}_e(t)\exp\left(-iE_0^e t/\hbar\right) - i\frac{E_0^e}{\hbar}C_e(t)\exp\left(-iE_0^e t/\hbar\right) &= -iC_e(t)\exp\left(-iE_0^e t/\hbar\right)\frac{E_0^e}{\hbar}\\ &\quad + i\frac{d(t)\cdot E(t)}{\hbar}C_g(t)\exp\left(-iE_0^g t/\hbar\right)\end{aligned}$$

Or

$$\begin{aligned}\dot{C}_g(t) &= i\frac{d(t)\cdot E(t)}{\hbar}C_e(t)\exp\left(-iE_0^e t/\hbar\right)\exp\left(iE_0^g t/\hbar\right)\\ \dot{C}_e(t) &= i\frac{d(t)\cdot E(t)}{\hbar}C_g(t)\exp\left(-iE_0^g t/\hbar\right)\exp\left(iE_0^e t/\hbar\right)\end{aligned}$$

2.4 Rotating wave approximation

We will call quantum dot frequency to the quantity

$$\omega_{QD} = \frac{E_0^e - E_0^g}{\hbar} \tag{2.7}$$

Then, equations becomes

$$\begin{aligned}\dot{C}_g(t) &= i\frac{d(t)\cdot E(t)}{\hbar}C_e(t)\exp\left(-i\omega_{QD}t\right)\\ \dot{C}_e(t) &= i\frac{d(t)\cdot E(t)}{\hbar}C_g(t)\exp\left(i\omega_{QD}t\right)\end{aligned}$$

The applied electric field $E(t)$ is close to resonance with the transition QD frequency in the cases of interest to us. Explicitly it is

$$E(t) = e(t)\cos\left(\nu t\right) = \frac{e(t)}{2}\left[\exp\left(i\nu t\right) + \exp\left(-i\nu t\right)\right] \tag{2.8}$$

Here $e(t)$ is the amplitude of the electric field and ν is its frequency. If we ignore the counter-rotating terms, i.e. terms proportional to $\exp\left[\pm i\left(\omega_{QD}+\nu\right)t\right]$ then reduces to

$$
\begin{aligned}
\dot{C}_g(t) &= i\frac{\Omega_R(t)}{2}C_e(t)\exp\left[-i\left(\omega_{QD}-\nu\right)t\right] \\
\dot{C}_e(t) &= i\frac{\Omega_R(t)}{2}C_g(t)\exp\left[i\left(\omega_{QD}-\nu\right)t\right]
\end{aligned}
$$

This approximation is the so called Rotating Wave Approximation (RWA) and it is common to define $\Delta = \omega_{QD} - \nu$ as the detuning. The Rabi frequency is defined as $\Omega_R(t) = d(t)\cdot e(t)/\hbar$.

$$
\dot{C}_g(t) = i\frac{\Omega_R(t)}{2}C_e(t)\exp\left(-i\Delta t\right). \tag{2.9}
$$

$$
\dot{C}_e(t) = i\frac{\Omega_R(t)}{2}C_g(t)\exp\left(i\Delta t\right). \tag{2.10}
$$

To solve equation is convenient to define the quantities

$$
\begin{aligned}
c_g(t) &= C_g(t)\exp\left(i\frac{\Delta}{2}t\right) \\
c_e(t) &= C_e(t)\exp\left(-i\frac{\Delta}{2}t\right)
\end{aligned}
$$

The coefficient $c_g(t)$ satisfies the differential equation

$$
\begin{aligned}
\dot{c}_g(t) &= \dot{C}_g(t)\exp\left(i\frac{\Delta}{2}t\right) + i\frac{\Delta}{2}C_g(t)\exp\left(i\frac{\Delta}{2}t\right) \\
&= i\frac{\Omega_R(t)}{2}C_e(t)\exp\left(-i\Delta t\right)\exp\left(i\frac{\Delta}{2}t\right) + i\frac{\Delta}{2}C_g(t)\exp\left(i\frac{\Delta}{2}t\right) \\
&= i\frac{\Omega_R(t)}{2}C_e(t)\exp\left(-i\frac{\Delta}{2}t\right) + i\frac{\Delta}{2}C_g(t)\exp\left(i\frac{\Delta}{2}t\right)
\end{aligned}
$$

While $c_e(t)$

$$\begin{aligned}
\dot{c}_e(t) &= \dot{C}_e(t)\exp\left(-i\frac{\Delta}{2}t\right) - i\frac{\Delta}{2}C_e(t)\exp\left(-i\frac{\Delta}{2}t\right) \\
&= i\frac{\Omega_R(t)}{2}C_g(t)\exp\left(i\Delta t\right)\exp\left(-i\frac{\Delta}{2}t\right) - i\frac{\Delta}{2}C_e(t)\exp\left(-i\frac{\Delta}{2}t\right) \\
&= i\frac{\Omega_R(t)}{2}C_g(t)\exp\left(i\frac{\Delta}{2}t\right) - i\frac{\Delta}{2}C_e(t)\exp\left(-i\frac{\Delta}{2}t\right)
\end{aligned}$$

In this way we have found the set of equations

$$\begin{aligned}
\dot{c}_g(t) &= i\frac{\Omega_R(t)}{2}c_e(t) + i\frac{\Delta}{2}c_g(t) \\
\dot{c}_e(t) &= i\frac{\Omega_R(t)}{2}c_g(t) - i\frac{\Delta}{2}c_e(t)
\end{aligned}$$

Which can be solved exactly by using the usual methods to solve differential equations, for example taking Laplace transform, and easily because there are not complex exponential terms left. By considering an monochromatic electric field with constant amplitude we take Laplace transform for the system and we obtain

$$\begin{aligned}
s\tilde{c}_g(s) - c_g(0) &= i\frac{\Omega_R}{2}\tilde{c}_e(s) + i\frac{\Delta}{2}\tilde{c}_g(s) \\
s\tilde{c}_e(s) - c_e(0) &= i\frac{\Omega_R}{2}\tilde{c}_g(s) - i\frac{\Delta}{2}\tilde{c}_e(s)
\end{aligned}$$

After adequately arranging the terms, it can be written in a matrix form

$$\begin{pmatrix} s - i\frac{\Delta}{2} & -i\frac{\Omega_R}{2} \\ -i\frac{\Omega_R}{2} & s + i\frac{\Delta}{2} \end{pmatrix} \begin{pmatrix} \tilde{c}_g(s) \\ \tilde{c}_e(s) \end{pmatrix} = \begin{pmatrix} c_g(0) \\ c_e(0) \end{pmatrix}$$

The solution is obtained by multiplying on the left side by the inverse matrix and taking the inverse Laplace transform (see appendix A) to obtain

$$
\begin{aligned}
c_g(t) &= c_g(0)\left[\cos\left(\frac{1}{2}\Omega_R t\right) + i\frac{\Delta}{\Omega_R}\sin\left(\frac{1}{2}\Omega_R t\right)\right] + ic_e(0)\frac{\Omega_R}{\Omega}\sin\left(\frac{1}{2}\Omega_R t\right) \\
c_e(t) &= ic_g(0)\frac{\Omega_R}{\Omega}\sin\left(\frac{1}{2}\Omega_R t\right) + c_e(0)\left[\cos\left(\frac{1}{2}\Omega_R t\right) - i\frac{\Delta}{\Omega}\sin\left(\frac{1}{2}\Omega_R t\right)\right]
\end{aligned}
$$

Where Ω_R is the off resonance Rabi frequency. These quantities are the probability amplitude coefficients of the wave function in the rotating frame. They allow us to determinate the $C_g(t)$ and $C_e(t)$ coefficients in the non-rotating by inverting

$$
\begin{aligned}
C_g(t) &= c_g(t)\exp\left(-i\frac{\Delta}{2}t\right) \\
C_e(t) &= c_e(t)\exp\left(i\frac{\Delta}{2}t\right)
\end{aligned}
$$

2.5 Vector representation

2.5.1 Bloch's vector

The equations and allow us to know the wave function of the single quantum dot.

$$
|\psi(t)\rangle = c_g(t)\exp\left(-i\frac{\Delta}{2}t\right)\exp\left(-iE_0^g t/\hbar\right)|g\rangle + c_e(t)\exp\left(i\frac{\Delta}{2}t\right)\exp\left(-iE_0^e t/\hbar\right)|e\rangle
$$

In order to simplify the notation, let's consider the variables

$$
\begin{aligned}
\alpha_g(t) &= c_g(t)\exp\left(-i\frac{\Delta}{2}t\right)\exp\left(-iE_0^g t/\hbar\right) \\
\alpha_e(t) &= c_e(t)\exp\left(i\frac{\Delta}{2}t\right)\exp\left(-iE_0^e t/\hbar\right)
\end{aligned}
$$

By taking the expectation values of the Pauli matrices we can define the real quantities

$$\begin{aligned}
U(t) &= \langle\psi(t)|\,\sigma_x\,|\psi(t)\rangle = \alpha_g^*(t)\alpha_e(t) + \alpha_g(t)\alpha_e^*(t) \\
V(t) &= \langle\psi(t)|\,\sigma_y\,|\psi(t)\rangle = -i\alpha_g(t)\alpha_e^* + i\alpha_e(t)\alpha_g^*(t) \\
W(t) &= \langle\psi(t)|\,\sigma_z\,|\psi(t)\rangle = |\alpha_g(t)|^2 - |\alpha_e(t)|^2
\end{aligned}$$

They are the components of the called Bloch's vector, i.e., $R(t) = (U(t), V(t), W(t))$. The expected value of the σ_x Pauli's matrix is

$$\begin{aligned}
U(t) &= \alpha_g^*(t)\alpha_e(t) + \alpha_g(t)\alpha_e^*(t) \\
&= c_g^*(t)c_e(t)\exp(i\Delta t/2)\exp(i\Delta t/2)\exp(iE_0^g t/\hbar)\exp(-iE_0^e t/\hbar) \\
&\quad + c_g(t)c_e^*(t)\exp(-i\Delta t/2)\exp(-i\Delta t/2)\exp(-iE_0^g t/\hbar)\exp(iE_0^e t/\hbar) \\
&= c_g^*(t)c_e(t)\exp(-i\nu t) + c_g(t)c_e^*(t)\exp(i\nu t) \\
&= c_g^*(t)c_e(t)\cos(\nu t) - ic_g^*(t)c_e(t)\sin(\nu t) + c_g(t)c_e^*(t)\cos(\nu t) + ic_g(t)c_e^*(t)\sin(\nu t) \\
&= \left[c_g^*(t)c_e(t) + c_g(t)c_e^*(t)\right]\cos(\nu t) - \left[ic_g^*(t)c_e(t) - ic_g(t)c_e^*(t)\right]\sin(\nu t)
\end{aligned}$$

The expected value of the σ_y Pauli's matrix is

$$\begin{aligned}
V(t) &= -i\alpha_g(t)\alpha_e^* + i\alpha_e(t)\alpha_g^*(t) \\
&= -ic_g(t)\exp(-i\Delta t/2t)\exp(-iE_0^g t/\hbar)\,c_e^*(t)\exp(-i\Delta t/2)\exp(iE_0^e t/\hbar) \\
&\quad + ic_e(t)\exp(i\Delta t/2)\exp(-iE_0^e t/\hbar)\,c_g^*(t)\exp(i\Delta t/2)\exp(iE_0^g t/\hbar) \\
&= -ic_g(t)c_e^*(t)\exp(i\nu t) + ic_e(t)c_g^*(t)\exp(-i\nu t) \\
&= -ic_g(t)c_e^*(t)\cos(\nu t) + c_g(t)c_e^*(t)\sin(\nu t) + ic_e(t)c_g^*(t)\cos(\nu t) + c_e(t)c_g^*(t)\sin(\nu t) \\
&= \left[-ic_g(t)c_e^*(t) + ic_e(t)c_g^*(t)\right]\cos(\nu t) + \left[c_g(t)c_e^*(t) + c_e(t)c_g^*(t)\right]\sin(\nu t)
\end{aligned}$$

And the expected value of the σ_z

$$W(t) = |\alpha_g(t)|^2 - |\alpha_e(t)|^2 = |c_g(t)|^2 - |c_e(t)|^2$$

As we have said, the expectation values of the Pauli's matrices are the components of the Bloch's vector and are given in terms of the coefficients $c_g(t)$ and $c_e(t)$ as

$$\begin{pmatrix} U(t) \\ V(t) \\ W(t) \end{pmatrix} = \begin{pmatrix} \cos(\nu t) & -\sin(\nu t) & 0 \\ \sin(\nu t) & \cos(\nu t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_g^*(t)c_e(t) + c_g(t)c_e^*(t) \\ -ic_g(t)c_e^*(t) + ic_e(t)c_g^*(t) \\ |c_g(t)|^2 - |c_e(t)|^2 \end{pmatrix}$$

This equation allows us to give an interpretation to $c_g(t)$ and$c_e(t)$ coefficients: they are the amplitude probability coefficients of the wave function in the rotating frame. The Bloch's vector in the rotating frame is given by

$$\begin{aligned} u(t) &= c_g^*(t)c_e(t) + c_g(t)c_e^*(t) \\ v(t) &= -ic_g(t)c_e^* + ic_e(t)c_g^*(t) \\ w(t) &= |c_g(t)|^2 - |c_e(t)|^2 \end{aligned}$$

The pure unexcited state $|g\rangle$ has the vector $r = (0, 0, -1)$ while the pure excited state $|e\rangle$ has the unit Bloch-vector $(0, 0, 1)$. Intermediate states have Bloch-vectors pointing in other directions, and any state that is in a mixture of upper and lower states has a Bloch-vector pointing in another direction, but conserving unit norm.

The quantities $u(t)$ and $v(t)$ are related with the complex amplitude electric dipole through

$$d(t) = u(t) + iv(t). \tag{2.11}$$

And the quantity $W(t) = w(t)$ is called inversion. When it is positive the probability to find the QD in the upper state is higher than in the ground state and vice versa.

2.5.2 Bloch's equations

In this section we find the time evolution of the Bloch's vector in the rotating frame, i.e., we will take the derivative of equation and we find

$$\begin{aligned}\dot{u}(t) &= \dot{c}_g^*(t)c_e(t) + c_g^*(t)\dot{c}_e(t) + \dot{c}_g(t)c_e^*(t) + c_g(t)\dot{c}_e^*(t)\\ \dot{v}(t) &= -i\dot{c}_g(t)c_e^*(t) - ic_g(t)\dot{c}_e^*(t) + i\dot{c}_e(t)c_g^*(t) + ic_e(t)\dot{c}_g^*(t)\\ \dot{w}(t) &= \dot{c}_g^*(t)c_g(t) + c_g^*(t)\dot{c}_g(t) - \dot{c}_e(t)c_e^*(t) - c_e(t)\dot{c}_e^*(t)\end{aligned}$$

But we already know the differential equation satisfied by $c_g(t)$ and $c_e(t)$

$$\begin{aligned}\dot{c}_g(t) &= i\frac{\Omega_R}{2}c_e(t) + i\frac{\Delta}{2}c_g(t)\\ \dot{c}_e(t) &= i\frac{\Omega_R}{2}c_g(t) - i\frac{\Delta}{2}c_e(t)\end{aligned}$$

For $\dot{u}(t)$

$$\begin{aligned}\dot{u}(t) &= \left[-\frac{i\Delta}{2}c_g^*(t) - i\frac{\Omega_R}{2}c_e^*(t)\right]c_g(t) + c_g^*(t)\left[-\frac{i\Delta}{2}c_e(t) + i\frac{\Omega_R}{2}c_g(t)\right]\\ &\quad + \left[\frac{i\Delta}{2}c_g(t) + i\frac{\Omega_R}{2}c_e(t)\right]c_e^*(t) + c_g(t)\left[\frac{i\Delta}{2}c_e^*(t) - i\frac{\Omega_R}{2}c_g^*(t)\right]\\ &= -i\Delta c_g^*(t)c_e(t) + i\Delta c_g(t)c_e^*(t)\\ &= i\Delta\left[c_g(t)c_e^*(t) - c_g^*(t)c_e(t)\right]\\ &= \Delta v(t)\end{aligned}$$

For $\dot{v}(t)$

$$
\begin{aligned}
\dot{v}(t) &= -i\left\{\left[-\frac{i\Delta}{2}c_g^*(t) - i\frac{\Omega_R}{2}c_e^*(t)\right]c_e(t) + c_g^*(t)\left[-\frac{i\Delta}{2}c_e(t) + i\frac{\Omega_R}{2}c_g(t)\right]\right. \\
&\quad \left. - \left[\frac{i\Delta}{2}c_g(t) + i\frac{\Omega_R}{2}c_e(t)\right]c_e^*(t) - c_g(t)\left[\frac{i\Delta}{2}c_e^*(t) - i\frac{\Omega_R}{2}c_g^*(t)\right]\right\} \\
&= \left\{-\frac{\Delta}{2}c_g^*(t)c_e(t) - \frac{\Omega_R}{2}c_e^*(t)c_e(t) - \frac{\Delta}{2}c_g^*(t)c_e(t) + \frac{\Omega_R}{2}c_g^*(t)c_g(t)\right. \\
&\quad \left. -\frac{\Delta}{2}c_g(t)c_e^*(t) - \frac{\Omega_R}{2}c_e(t)c_e^*(t) - \frac{\Delta}{2}c_g(t)c_e^*(t) + \frac{\Omega_R}{2}c_g(t)c_g^*(t)\right\} \\
&= \left\{-\frac{\Delta}{2}c_g^*(t)c_e(t) - \frac{\Omega_R}{2}c_e^*(t)c_e(t) - \frac{\Delta}{2}c_g^*(t)c_e(t) + \frac{\Omega_R}{2}c_g^*(t)c_g(t)\right. \\
&\quad \left. -\frac{\Delta}{2}c_1(t)c_2^*(t) - \frac{\Omega_R}{2}c_2(t)c_2^*(t) - \frac{\Delta}{2}c_1(t)c_2^*(t) + \frac{\Omega_R}{2}c_1(t)c_1^*(t)\right\} \\
&= -\Delta u(t) + \Omega_R w(t)
\end{aligned}
$$

And for $\dot{w}(t)$

$$
\begin{aligned}
\dot{w}(t) &= \left\{\left[\frac{i\Delta}{2}c_1(t) + i\frac{\Omega_R}{2}c_2(t)\right]c_1^*(t) + c_1(t)\left[-\frac{i\Delta}{2}c_1^*(t) - i\frac{\Omega_R}{2}c_2^*(t)\right]\right. \\
&\quad \left. - \left[-\frac{i\Delta}{2}c_2(t) + i\frac{\Omega_R}{2}c_1(t)\right]c_2^*(t) - c_2(t)\left[\frac{i\Delta}{2}c_2^*(t) - i\frac{\Omega_R}{2}c_1^*(t)\right]\right\} \\
&= \left\{\frac{i\Delta}{2}c_1^*(t)c_1(t) + i\frac{\Omega_R}{2}c_1^*(t)c_2(t) - \frac{i\Delta}{2}c_1(t)c_1^*(t) - i\frac{\Omega_R}{2}c_1(t)c_2^*(t)\right. \\
&\quad \left. +\frac{i\Delta}{2}c_2^*(t)c_2(t) - i\frac{\Omega_R}{2}c_2^*(t)c_1(t) - \frac{i\Delta}{2}c_2(t)c_2^*(t) + i\frac{\Omega_R}{2}c_2(t)c_1^*(t)\right\} \\
&= i\Omega_R\left[c_1^*(t)c_2(t) - c_1(t)c_2^*(t)\right] \\
&= -\Omega_R v(t)
\end{aligned}
$$

In this way, they are obtained the well known optical Bloch equations

$$\begin{aligned}\dot{u}(t) &= \Delta v(t)\\ \dot{v}(t) &= -\Delta u(t) + \Omega_R w(t)\\ \dot{w}(t) &= -\Omega_R v(t)\end{aligned}$$

They have been obtained for a constant electric field. Equations can be written as a single vector equation

$$\dot{r} = \Omega \times r$$

r is the Bloch's vector and Ω is the torque vector:

$$\Omega = -\Omega_R \hat{x} - \Delta \hat{z}.$$

Above equations show that in the rotating frame al variables change slowly, because of there are not optical frequencies left. Equation has a conservation law associated:

$$\dot{u}(t)u(t) = \Delta v(t)u(t), \tag{2.12}$$

$$\dot{v}(t)v(t) = -\Delta u(t)v(t) + \Omega_R w(t)v(t), \tag{2.13}$$

$$\dot{w}(t)w(t) = -\Omega_R v(t)w(t). \tag{2.14}$$

Adding these equations it follows

$$u(t)^2 + v(t)^2 + w(t)^2 = 1. \tag{2.15}$$

The norm of the Bloch's vector is constant and because of the conservation of probability its norm is the unit.

2.6 Resonance case

In last section we have introduced the Bloch's vector and the Bloch's equations in a specific form: the electric field is not depending on time and there is a difference between the field frequency and the frequency of the quantum dot, this fact is characterized by the detuning.

This section is devoted to analyze solutions for the equations when the exciting frequency is equal to the QD-frequency and the electric field amplitude is steady. In this case the solutions become

$$\begin{aligned} c_g(s) &= c_g(0)\cos\left(\frac{1}{2}\Omega_0 t\right) + ic_e(0)\sin\left(\frac{1}{2}\Omega_0 t\right) \\ c_e(s) &= ic_g(0)\sin\left(\frac{1}{2}\Omega_0 t\right) + c_e(0)\cos\left(\frac{1}{2}\Omega_0 t\right) \end{aligned}$$

Additionally, if the QD is initially in the excited state then the solutions are

$$\begin{aligned} c_g(s) &= i\sin\left(\Omega_0 t/2\right) \\ c_e(s) &= \cos\left(\Omega_0 t/2\right) \end{aligned}$$

2.6.1 Bloch's vector

By substituting in

$$\begin{aligned} u(t) &= i\cos\left(\Omega_0 t/2\right)\sin\left(\Omega_0 t/2\right) - i\sin\left(\Omega_0 t/2\right)\cos\left(\Omega_0 t/2\right) = 0, && (2.16) \\ v(t) &= -i\left[i\sin\left(\Omega_0 t/2\right)\cos\left(\Omega_0 t/2\right) + i\sin\left(\Omega_0 t/2\right)\cos\left(\Omega_0 t/2\right)\right], \\ &= 2\sin\left(\Omega_0 t/2\right)\cos\left(\Omega_0 t/2\right), && (2.17) \\ w(t) &= \cos^2\left(\Omega_0 t/2\right) - \sin^2\left(\Omega_0 t/2\right). && (2.18) \end{aligned}$$

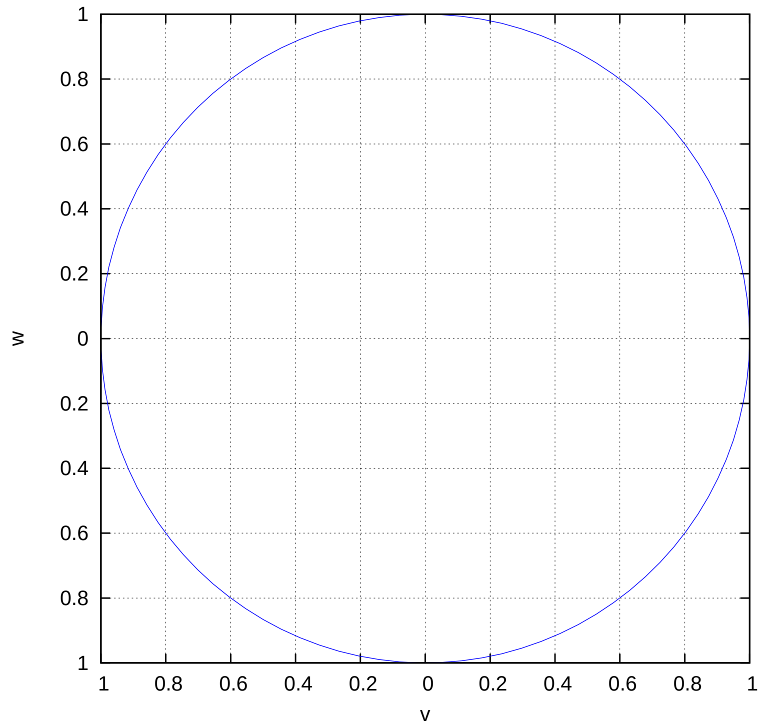


Figure 2.1: Is shown the behavior of the Bloch's vector when the excitation frequency is equal to the QD frequency, i.e., in resonance. Notice the Bloch's vector traces out a circumference in the plane $v - w$

And can be rewritten as

$$\begin{aligned} u(t) &= 0 \\ v(t) &= \sin(\Omega_0 t) \\ w(t) &= \cos(\Omega_0 t) \end{aligned}$$

They can be interpreted as the single vector equation

$$\dot{r}(t) = \Omega \times r(t)$$

Where

$$\Omega = (-\Omega_0, 0, 0)$$

This show us the time evolution of the Bloch's vector as if it were the equations of a solid body acted on by a torque Ω, its precessing is only about the x-axis. Notice the in absence of electric field, there is not time evolution of the Bloch's vector.

2.6.2 Inversion

The single population inversion shows an oscillatory behavior between the values -1 and 1. The applied field has the effect of repeatedly exciting and de-exciting the QD.

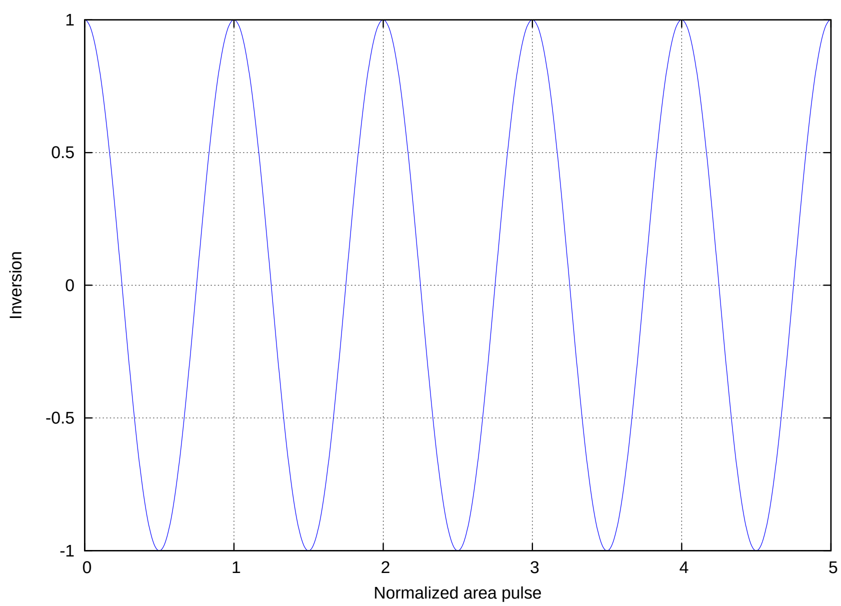


Figure 2.2: Single population inversion of the QD. It initially is in the excited state. The normalized area of the pulse is defined as $A = \Omega_0 (t_2 - t_1)/2\pi = 1$. After a pulse with duration 1 the system again is in the initial state.

2.6.3 Resonance fluorescent spectrum

When the components of the Bloch's vector are known we can determinate the electric dipole oscillations, its Fourier transform will give a close resemblance of its the resonance fluorescent spectrum.

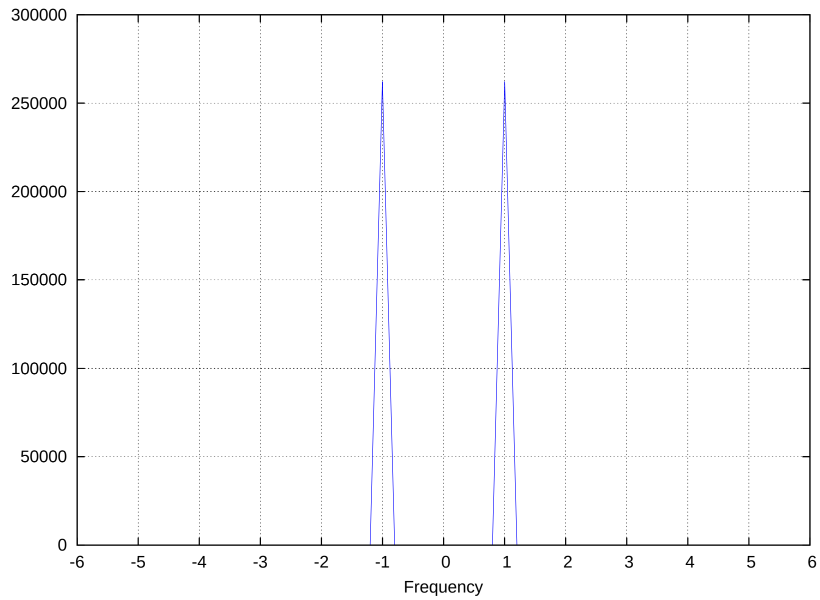


Figure 2.3: Spectrum of the resonance fluorescence. Has been consider the system in resonance, the peaks correspond to the normalized Rabi frequency.

Chapter 3

Semmiclassical model of a pair of coupled quantum dots

3.1 Introduction

In this chapter we will study a pair of quantum dots, each one in its own cavity. Both of them are coupled trough the Foerster interaction between them. In addition, each QD is interacting with its own classical electric field through a dipole interaction. We will develop a model based on the Schrödinger picture, similar to that done in the previous chapter that describes the dynamics of the coupled system.

In this chapter the systems will be labeled as system 1 and system 2. In order to distinguish the operators corresponding to each system we will use the notation

$$\sigma_1^x,\ \ \sigma_1^y,\ \ \sigma_1^z$$

To distinguish the Pauli's matrices of the system 1 from the Pauliâs matrices of the system 2

$$\sigma_2^x,\ \ \sigma_2^y,\ \ \sigma_2^z$$

The assignation of that notation is different to the used in the previous chapter; however it has been introduced in order to avoid confusions between the Pauli's matrices of the system 2 and the square of the Pauli's matrices.

3.2 Model

The Hamiltonian of N quantum dots is

$$\begin{aligned} H(t) &= \frac{\varepsilon}{2}\hbar \sum_{n=1}^{N} \left(e_n^\dagger e_n - h_n h_n^\dagger\right) - \frac{1}{2}\hbar W \sum_{n,n'}^{N} \left(e_n^\dagger h_{n'} e_{n'} h_n^\dagger + h_n e_{n'}^\dagger h_{n'}^\dagger e_n\right) \\ &\quad - d(t) \cdot E(t) \sum_{n=1}^{N} e_n^\dagger h_n^\dagger - d^*(t) \cdot E^*(t) \sum_{n=1}^{N} h_n e_n. \end{aligned} \tag{3.1}$$

This Hamiltonian describes N quantum dots interacting with the same electric field.

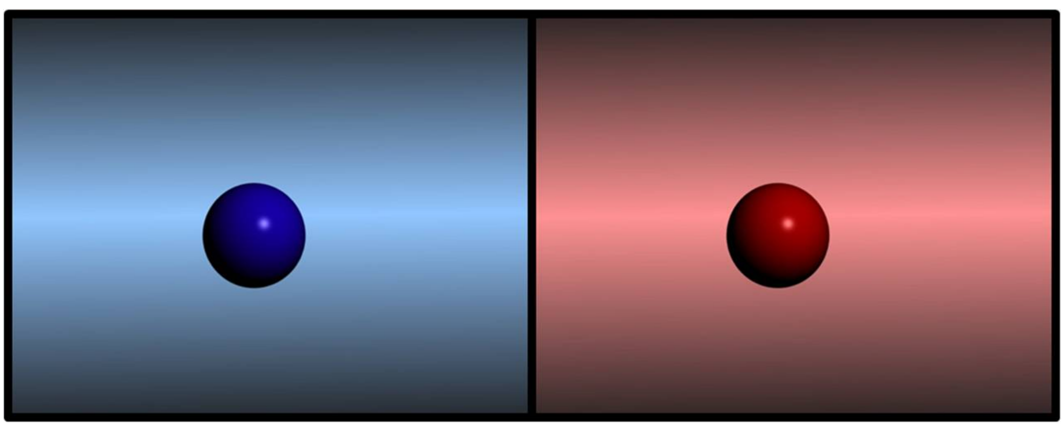


Figure 3.1: This picture shows physical system studyied. Each circle represents a quantum dot in a cavity. The different color used for the cavities suggest they are interacting with different electric fiels

Our aim is to study a pair of quantum dots, so $N = 2$, each quantum dot is inside its own micro cavity interacting with their local electric field. The Hamiltonian that describes this situation is

$$
\begin{aligned}
H(t) &= \frac{\epsilon_1}{2}\hbar\left(e_1^\dagger e_1 - h_1 h_1^\dagger\right) + \frac{\epsilon_2}{2}\hbar\left(e_2^\dagger e_2 - h_2 h_2^\dagger\right) - d_1(t)\cdot E_1(t) e_1^\dagger h_1^\dagger \\
&\quad - d_1^*(t)\cdot E_1^*(t) h_1 e_1 - d_2(t)\cdot E_2(t) e_2^\dagger h_2^\dagger - d_2^*(t)\cdot E_2^*(t) h_2 e_2 \\
&\quad - \frac{1}{2}\hbar W\left(e_1^\dagger h_1 e_1 h_1^\dagger + h_1 e_1^\dagger h_1^\dagger e_1 + e_1^\dagger h_2 e_2 h_1^\dagger + h_1 e_2^\dagger h_2^\dagger e_1\right) \\
&\quad - \frac{1}{2}\hbar W\left(e_2^\dagger h_1 e_1 h_2^\dagger + h_2 e_1^\dagger h_1^\dagger e_2 + e_2^\dagger h_2 e_2 h_2^\dagger + h_2 e_2^\dagger h_2^\dagger e_2\right).
\end{aligned}
$$

The above Hamiltonian is rewritten by using the pseudo spin operators in each cavity

$$
\begin{aligned}
J_1^+ = e_1^\dagger h_1^\dagger,\ J_1^- = h_1 e_1,\ J_1^z = \frac{1}{2}\left(e_1^\dagger e_1 - h_1 h_1^\dagger\right), \\
J_2^+ = e_2^\dagger h_2^\dagger,\ J_2^- = h_2 e_2,\ J_2^z = \frac{1}{2}\left(e_2^\dagger e_2 - h_2 h_2^\dagger\right).
\end{aligned}
$$

In the following form

$$
\begin{aligned}
H(t) &= \hbar\epsilon_1 J_1^z - d_1(t)\cdot E_1(t) J_1^+ - d_1^*(t)\cdot E_1^*(t) J_1^- \\
&\quad + \hbar\epsilon_2 J_2^z - d_2(t)\cdot E_2(t) J_2^+ - d_2^*(t)\cdot E_2^*(t) J_2^- \\
&\quad - \frac{1}{2}\hbar W\left[J_1^2 - (J_1^z)^2 + J_1^+ J_2^- + J_1^- J_2^+ + J_2^+ J_1^- + J_2^- J_1^+ + J_2^2 - (J_2^z)^2\right].
\end{aligned}
$$

Considering real both electric fields and both electric dipoles of each QD; using the identities $J_1^2 - (J_1^z)^2 = J_1^+ J_1^- - J_1^z$ and $J_2^2 - (J_2^z)^2 = J_2^+ J_2^- - J_2^z$ the above Hamiltonian becomes:

$$
\begin{aligned}
H(t) &= \hbar\epsilon_1 J_1^z - d_1(t)\cdot E_1(t) J_1^+ - d_1^*(t)\cdot E_1^*(t) J_1^- \\
&\quad + \hbar\epsilon_2 J_2^z - d_2(t)\cdot E_2(t) J_2^+ - d_2^*(t)\cdot E_2^*(t) J_2^- \\
&\quad - \frac{1}{2}\hbar W\left[J_1^2 - (J_1^z)^2 + J_1^+ J_2^- + J_1^- J_2^+ + J_2^+ J_1^- + J_2^- J_1^+ + J_2^2 - (J_2^z)^2\right].
\end{aligned}
$$

Because $J_1 = \sigma_1/2$ and $J_2 = \sigma_2/2$.

$$\begin{aligned} H(t) &= \frac{1}{2}\hbar\left(\epsilon_1 - W\right)\sigma_1^z - \hbar W\left(\sigma_1^+\sigma_1^-\right) + \frac{1}{2}\hbar\left(\epsilon_2 - W\right)\sigma_2^z - \hbar W\left(\sigma_2^+\sigma_2^-\right) \\ &\quad -\hbar W\left(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+\right) - d_1(t)\cdot E_1(t)\sigma_1^x - d_2(t)\cdot E_2(t)\sigma_2^x \end{aligned}$$

3.2.1 Non interacting Hamiltonian

In this Hamiltonian we can distinguish the terms that describe the model of quantum dots, we will call it non-interacting Hamiltonian H_0, and the terms corresponding to the interaction with the electric field, interaction Hamiltonian H_{int}.

The non-interacting Hamiltonian consists of three terms

$$H_0 = H_0^1 + H_0^2 + H_0^{12} \tag{3.2}$$

The Hamiltonians

$$H_0^i(t) = \frac{1}{2}\hbar\left(\varepsilon_i - W\right)\sigma_i^z - \hbar W\left(\sigma_i^+\sigma_i^-\right) = \frac{1}{2}\hbar\left(\varepsilon_i - 2W\right)\sigma_i^z - \hbar\frac{W}{2}$$

corresponds to the free Hamiltonian of each quantum dot and the Hamiltonian

$$H_0^{12}(t) = -\hbar W\left(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+\right)$$

gives the Foerster interaction.

As we have seen in the last chapter, the single quantum dot 1 has associated a two-dimensional Hilbert space S_1. Similarly the single quantum dot 2 has a two-dimensional Hilbert space S_2. We will designate by $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$ the observables of the quantum dots 1 and 2 respectively. In S_1 (or inS_2) we choose a basis the eigenvectors of σ_1^z (or σ_2^z) which we will denote by $|g(1)\rangle$ and $|e(1)\rangle$ (or $|g(2)\rangle$ and $|e(2)\rangle$). The most general state of the quantum dot 1 in the absence of electromagnetic interaction can be written as:

$$|\psi_1(t)\rangle = C_{1g}(0)\exp\left(-iE_0^{1g}t/\hbar\right)|g_1\rangle + C_{1e}(0)\exp\left(-iE_0^{1e}t/\hbar\right)|e_1\rangle \tag{3.3}$$

And the most general state of the quantum dot 2 is

$$|\psi_2(t)\rangle = C_{2g}(0)\exp\left(-iE_0^{2g}t/\hbar\right)|g_2\rangle + C_{2e}(0)\exp\left(-iE_0^{2e}t/\hbar\right)|e_2\rangle \tag{3.4}$$

Those probability amplitudes have been already found in the chapter 2 and

$$\begin{aligned} E_0^{ig} &= -\hbar\left(\varepsilon_1 - W\right)/2 \\ E_0^{ie} &= \hbar\left(\varepsilon_2 - 3W\right)/2 \end{aligned}$$

Because of the Foerster coupling the two systems are joined making a single system which state space is the tensor product $S = S_1 \otimes S_2$ of the two preceding spaces . This means that a basis of S can be obtained by tensor multiplication the two basis defined for S_1 and S_2

$$\begin{aligned} |g(1), g(2)\rangle &= |g(1)\rangle \otimes |g(2)\rangle \\ |g(1), e(2)\rangle &= |g(1)\rangle \otimes |e(2)\rangle \\ |e(1), g(2)\rangle &= |e(1)\rangle \otimes |g(2)\rangle \\ |e(1), e(2)\rangle &= |e(1)\rangle \otimes |e(2)\rangle \end{aligned}$$

The vector space S is therefore four-dimensional.

Let's start by observing that the Hamiltonian contains terms corresponding to the non-interacting QDs, Eq. establishing a parallel development with the Hamiltonian of the single quantum dot. The single quantum dot does have a free term whose eigenfunctions are the two-level basis of the problem, and the time evolution is due to the action of a non-diagonal term. The Hamiltonian Eq.) is similar in that sense; it has the diagonal terms H_0^1 and H_0^2 and its eigenvectors are the basis Eq. ; also it has the Foerster interaction which is a non-diagonal term. From this observation we can say, even before to solve the problem, that the system will have a time evolution in absence of electric field. The Foerster interaction strength W plays the role of Rabi frequency and the difference on the QDs frequencies, the detuning.

The wave function of the quantum dots, in the absence of Foerster interaction, is

$$\begin{aligned} |\psi_0(t)\rangle &= C_0^{gg}(0)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar}|g(1),g(2)\rangle + C_0^{ge}(0)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar}|g(1),e(2)\rangle \\ &+C_0^{eg}(0)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar}|e(1),g(2)\rangle + C_0^{ee}(0)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar}|e(1),e(2)\rangle \end{aligned}$$

where it has been used the eigenvalues relation for $H_0^1 + H_0^2$

$$\begin{aligned} \left(H_0^1+H_0^2\right)|g(1),g(2)\rangle &= \left(E_0^{1g}+E_0^{2g}\right)|g(1),g(2)\rangle \\ \left(H_0^1+H_0^2\right)|g(1),e(2)\rangle &= \left(E_0^{1g}+E_0^{2e}\right)|g(1),e(2)\rangle \\ \left(H_0^1+H_0^2\right)|e(1),g(2)\rangle &= \left(E_0^{1e}+E_0^{2g}\right)|e(1),g(2)\rangle \\ \left(H_0^1+H_0^2\right)|e(1),e(2)\rangle &= \left(E_0^{1e}+E_0^{2e}\right)|e(1),e(2)\rangle \end{aligned}$$

To take into count the Foerster interaction, we let the coefficients of the wave function Eq.) is time-dependent:

$$\begin{aligned} \left|\psi_0^{12}(t)\right\rangle &= C_0^{gg}(t)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar}|g(1),g(2)\rangle + C_0^{ge}(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar}|g(1),e(2)\rangle \\ &+C_0^{eg}(t)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar}|e(1),g(2)\rangle + C_0^{ee}(t)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar}|e(1),e(2)\rangle \end{aligned}$$

This satisfies the Schrödinger equation

$$i\hbar\left|\dot{\psi}_0^{12}(t)\right\rangle = \left(H_0^1+H_0^2+H_0^{12}\right)\left|\psi_0^{12}(t)\right\rangle \tag{3.5}$$

The left hand side of)

$$
\begin{aligned}
i\hbar\frac{d}{dt}\left\langle g(1),g(2)|\psi_0^{12}\right\rangle &= \dot{C}_0^{gg}(t)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} - i\frac{\left(E_0^{1g}+E_0^{2g}\right)}{\hbar}C_0^{gg}(t)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} \\
i\hbar\frac{d}{dt}\left\langle g(1),e(2)|\psi_0^{12}\right\rangle &= \dot{C}_0^{ge}(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} - i\frac{\left(E_0^{1g}+E_0^{2e}\right)}{\hbar}C_0^{ge}(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} \\
i\hbar\frac{d}{dt}\left\langle e(1),g(2)|\psi_0^{12}\right\rangle &= \dot{C}_0^{eg}(t)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar} - i\frac{\left(E_0^{1e}+E_0^{2g}\right)}{\hbar}C_0^{eg}(t)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar} \\
i\hbar\frac{d}{dt}\left\langle e(1),e(2)|\psi_0^{12}\right\rangle &= \dot{C}_0^{ee}(t)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar} - i\frac{(E_0^{1e}+E_0^{2e})}{\hbar}C_0^{ee}(t)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar}
\end{aligned}
$$

And the right hand side of) gives the following four expressions. Multiplying by $\langle g(1), g(2)|$ on the left:

$$
\begin{aligned}
\frac{d}{dt}\left\langle g(1),g(2)|H_0|\psi_0^{12}\right\rangle &= \langle g(1),g(2)|\left[H_0^1+H_0^2-\hbar W\left(\sigma_1^+\sigma_2^-+\sigma_1^-\sigma_2^+\right)\right]\left|\psi_0^{12}(t)\right\rangle \\
&= \left[\left(E_0^{1g}+E_0^{2g}\right)\langle g(1),g(2)|\right]\left|\psi_0^{12}(t)\right\rangle \\
&= \left(E_0^{1g}+E_0^{2g}\right)C_0^{gg}(t)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar}
\end{aligned}
$$

Multiplying by $\langle g(1), e(2)|$ on the left:

$$
\begin{aligned}
\frac{d}{dt}\left\langle g(1),e(2)|H_0|\psi_0^{12}\right\rangle &= \langle g(1),e(2)|\left[H_0^1+H_0^2-\hbar W\left(\sigma_1^+\sigma_2^-+\sigma_1^-\sigma_2^+\right)\right]\left|\psi_0^{12}(t)\right\rangle \\
&= \left[\left(E_0^{1g}+E_0^{2e}\right)\langle g(1),e(2)|-\hbar W\langle e(1),g(2)|\right]\left|\psi_0^{12}(t)\right\rangle \\
&= \left(E_0^{1g}+E_0^{2e}\right)C_0^{ge}(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar}-\hbar W C_0^{eg}(t)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar}
\end{aligned}
$$

Multiplying by $\langle e(1), g(2)|$ on the left:

$$
\begin{aligned}
\frac{d}{dt}\left\langle e(1),g(2)|H_0|\psi_0^{12}\right\rangle &= \langle e(1),g(2)|\left[H_0^1+H_0^2-\hbar W\left(\sigma_1^+\sigma_2^-+\sigma_1^-\sigma_2^+\right)\right]\left|\psi_0^{12}(t)\right\rangle \\
&= \left[\left(E_0^{1e}+E_0^{2g}\right)\langle e(1),g(2)|-\hbar W\langle g(1),e(2)|\right]\left|\psi_0^{12}(t)\right\rangle \\
&= \left(E_0^{1g}+E_0^{2e}\right)C_0^{eg}(t)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar}-\hbar W C_0^{ge}(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar}
\end{aligned}
$$

And multiplying by $\langle e(1), e(2)|$ on the left:

$$\begin{aligned}
\frac{d}{dt}\left\langle e(1), e(2)|H_0|\psi_0^{12}\right\rangle &= \langle e(1), e(2)|\left[H_0^1 + H_0^2 - \hbar W\left(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+\right)\right]\left|\psi_0^{12}(t)\right\rangle \\
&= \left[\left(E_0^{1e} + E_0^{2e}\right)\langle e(1), e(2)|\right]\left|\psi_0^{12}(t)\right\rangle \\
&= \left(E_0^{1e} + E_0^{2e}\right)C_0^{ee}(t)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar}
\end{aligned}$$

From the identification of the left and the right side of), it is obtained the set of equations

$$\begin{aligned}
\dot{C}_0^{gg}(t) &= 0 \\
\dot{C}_0^{ge}(t) &= iWC_0^{eg}(t)e^{-i\left(E_0^{1e}-E_0^{1g}\right)t/\hbar}e^{i\left(E_0^{2e}-E_0^{2g}\right)t/\hbar} \\
\dot{C}_0^{eg}(t) &= iWC_0^{ge}(t)e^{-i\left(E_0^{1g}-E_0^{1e}\right)t/\hbar}e^{-i\left(E_0^{2e}-E_0^{2g}\right)t/\hbar} \\
\dot{C}_0^{ee}(t) &= 0
\end{aligned}$$

The above equations contain oscillating exponential terms depending on the QD-frequencies

$$\omega_{QD1} = \frac{E_0^{1e} - E_0^{1g}}{\hbar}$$
$$\omega_{QD2} = \frac{E_0^{2e} - E_0^{2g}}{\hbar}$$

Also, we define the quantity

$$\delta = \omega_{QD1} - \omega_{QD2}$$

The amplitude probability coefficients of the non-interacting system obey the differential equation

$$
\begin{aligned}
\dot{C}_0^{gg}(t) &= 0 \\
\dot{C}_0^{ge}(t) &= iWC_0^{eg}(t)e^{-i\delta t} \\
\dot{C}_0^{eg}(t) &= iWC_0^{ge}(t)e^{i\delta t} \\
\dot{C}_0^{ee}(t) &= 0
\end{aligned}
$$

The set of Equations) has the same form that equation)

$$
\begin{aligned}
\dot{C}_g(t) &= i\frac{\Omega_R(t)}{2}C_e(t)\exp\left(-i\Delta t\right) \\
\dot{C}_e(t) &= i\frac{\Omega_R(t)}{2}C_g(t)\exp\left(i\Delta t\right)
\end{aligned}
$$

But we already know its solutions; therefore we know the solutions for the system):

$$
\begin{aligned}
C_0^{gg}(t) &= C_0^{gg}(0) \\
C_0^{ge}(t) &= C_0^{ge}(0)\left[\cos\left(\frac{1}{2}\Omega_{QD}t\right) + i\frac{\delta}{\Omega_{QD}}\sin\left(\frac{1}{2}\Omega_{QD}t\right)\right] + iC_0^{eg}(0)\frac{2W}{\Omega_{QD}}\sin\left(\frac{1}{2}\Omega_{QD}t\right) \\
C_0^{eg}(t) &= iC_0^{ge}(0)\frac{2W}{\Omega_{QD}}\sin\left(\frac{1}{2}\Omega_{QD}t\right) + C_0^{eg}(0)\left[\cos\left(\frac{1}{2}\Omega_{QD}t\right) - i\frac{\delta}{\Omega_{QD}}\sin\left(\frac{1}{2}\Omega_{QD}t\right)\right] \\
C_0^{ee}(t) &= C_0^{ee}(0)
\end{aligned}
$$

here $\Omega_{QD} = \sqrt{4W^2 + \delta^2}$. In this section, we are interested in pure states for the initial states of the system. Therefore, only one coefficient will be initially equal to 1. An important aspect for the time evolution of that system is the initial condition. If both QDs are initially in the ground state or in the excited state, then the system will not present time evolution.

On other hand, if one QD is initially excited and the other is initially in the ground state, then the system will have time evolution because of the Foerster interaction. What happen with the expected value of the Pauliâs matrices of this system? Their expectation values are:

$$
\begin{aligned}
U_1^0(\tau) &= 0 \\
U_2^0(\tau) &= 0 \\
V_1^0(\tau) &= 0 \\
V_2^0(\tau) &= 0 \\
W_1^0(\tau) &= -\left|C_0^{ge}(t)\right|^2 + \left|C_0^{eg}(t)\right|^2 \\
W_2^0(\tau) &= \left|C_0^{ge}(t)\right|^2 - \left|C_0^{eg}(t)\right|^2
\end{aligned}
$$

The expectation value of the σ^z Pauli's matrix is the single quantum dot population inversion. Their frequency is $2W$ and the quantity δ acts as a detuning. This result also has been confirmed by numerical calculation.

The time evolution of the expected value of the Pauli's matrices of the non-interacting system depends not only of δ and W, also depend on the initial conditions they can't be interpreted as a set of Bloch's equations

3.3 QDs interacting with electric fields

We have studied the non-interacting system formed by two QDs in absence of interactions such as electric fields. Now, we will take in count the interaction with an electric field.

$$
\begin{aligned}
|\psi(t)\rangle = C_1(t)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar}\,|g(1),g(2)\rangle + C_2(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar}\,|g(1),e(2)\rangle \\
+C_3(t)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar}\,|e(1),g(2)\rangle + C_4(t)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar}\,|e(1),e(2)\rangle
\end{aligned}
$$

The wave function satisfies the Schrödinger equation

$$
i\hbar\left|\dot{\psi}(t)\right\rangle = H\,|\psi(t)\rangle
$$

The left hand side of gives the equations

$$\begin{aligned}
i\hbar\frac{d}{dt}\langle g(1),g(2)|\psi(t)\rangle &= \dot{C}_1(t)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} - i\frac{\left(E_0^{1g}+E_0^{2g}\right)}{\hbar}C_1(t)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} \\
i\hbar\frac{d}{dt}\langle g(1),e(2)|\psi(t)\rangle &= \dot{C}_2(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} - i\frac{\left(E_0^{1g}+E_0^{2e}\right)}{\hbar}C_2(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} \\
i\hbar\frac{d}{dt}\langle e(1),g(2)|\psi(t)\rangle &= \dot{C}_3(t)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar} - i\frac{\left(E_0^{1e}+E_0^{2g}\right)}{\hbar}C_3(t)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar} \\
i\hbar\frac{d}{dt}\langle e(1),e(2)|\psi(t)\rangle &= \dot{C}_4(t)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar} - i\frac{(E_0^{1e}+E_0^{2e})}{\hbar}C_4(t)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar}
\end{aligned}$$

The right side of) gives the following four expressions. Multiplying by $\langle g(1),g(2)|$ on the left:

$$\begin{aligned}
\langle g(1),g(2)|H|\psi(t)\rangle &= \langle g(1),g(2)|\left[H_0 - d_1(t)\cdot E_1(t)\sigma_1^x - d_2(t)\cdot E_2(t)\sigma_2^x\right]|\psi(t)\rangle \\
&= \Big[\left(E_0^{1g}+E_0^{2g}\right)\langle g(1),g(2)| \\
&\quad -d_1(t)\cdot E_1(t)\langle e(1),g(2)| - d_2(t)\cdot E_2(t)\langle g(1),e(2)|\Big]|\psi(t)\rangle \\
&= \left(E_0^{1g}+E_0^{2g}\right)C_1(t)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} \\
&\quad -d_1(t)\cdot E_1(t)C_3(t)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar} - d_2(t)\cdot E_2(t)C_2(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar}
\end{aligned}$$

Multiplying by $\langle g(1),e(2)|$ on the left:

$$\begin{aligned}
\langle g(1),e(2)|H|\psi(t)\rangle &= \langle g(1),e(2)|\left[H_0 - d_1(t)\cdot E_1(t)\sigma_1^x - d_2(t)\cdot E_2(t)\sigma_2^x\right]|\psi(t)\rangle \\
&= \Big[\left(E_0^{1g}+E_0^{2e}\right)\langle g(1),e(2)| - d_1(t)\cdot E_1(t)\langle e(1),e(2)| \\
&\quad -d_2(t)\cdot E_2(t)\langle g(1),g(2)| - \hbar W\langle e(1),g(2)|\Big]|\psi(t)\rangle \\
&= \left(E_0^{1g}+E_0^{2e}\right)C_2(t)e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} - d_1(t)\cdot E_1(t)C_4(t)e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar} \\
&\quad -d_2(t)\cdot E_2(t)C_1(t)e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} - \hbar W C_3(t)e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar}
\end{aligned}$$

Multiplying by $\langle e(1),g(2)|$ on the left:

$$\begin{aligned}
\langle e(1), g(2)| H |\psi(t)\rangle &= \langle e(1), g(2)| [H_0 - d_1(t)\cdot E_1(t)\sigma_1^x - d_2(t)\cdot E_2(t)\sigma_2^x] |\psi(t)\rangle \\
&= \left(E_0^{1e} + E_0^{2g}\right) C_3(t) e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar} - d_1(t)\cdot E_1(t) C_1(t) e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} \\
&\quad - d_2(t)\cdot E_2(t) C_4(t) e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar} - \hbar W C_2(t) e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar}
\end{aligned}$$

And, multiplying by $\langle e(1), e(2)|$ on the left:

$$\begin{aligned}
\langle e(1), e(2)| H |\psi(t)\rangle &= \langle e(1), e(2)| [H_0 - d_1(t)\cdot E_1(t)\sigma_1^x - d_2(t)\cdot E_2(t)\sigma_2^x] |\psi(t)\rangle \\
&= [E_0^{ee} \langle e(1), e(2)| - d_1(t)\cdot E_1(t) \langle g(1), e(2)| \\
&\quad - d_2(t)\cdot E_2(t) \langle e(1), g(2)|] |\psi(t)\rangle \\
&= e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar} C_4(t) e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar} - d_1(t)\cdot E_1(t) C_2(t) e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} \\
&\quad - d_2(t)\cdot E_2(t) C_3(t) e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar}
\end{aligned}$$

By equating the right and the right side of equation) is obtained the set of equations

$$\begin{aligned}
\dot{C}_1(t) e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} &= i\frac{d_1(t)\cdot E_1(t)}{\hbar} C_3(t) e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar} + i\frac{d_2(t)\cdot E_2(t)}{\hbar} C_2(t) e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} \\
\dot{C}_2(t) e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} &= i\frac{d_1(t)\cdot E_1(t)}{\hbar} C_4(t) e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar} + i\frac{d_2(t)\cdot E_2(t)}{\hbar} C_1(t) e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} \\
&\quad + iW C_3(t) e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar} \\
\dot{C}_3(t) e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar} &= i\frac{d_1(t)\cdot E_1(t)}{\hbar} C_1(t) e^{-i\left(E_0^{1g}+E_0^{2g}\right)t/\hbar} + i\frac{d_2(t)\cdot E_2(t)}{\hbar} C_4(t) e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar} \\
&\quad + iW C_2(t) e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} \\
\dot{C}_4(t) e^{-i\left(E_0^{1e}+E_0^{2e}\right)t/\hbar} &= i\frac{d_1(t)\cdot E_1(t)}{\hbar} C_2(t) e^{-i\left(E_0^{1g}+E_0^{2e}\right)t/\hbar} + i\frac{d_2(t)\cdot E_2(t)}{\hbar} C_3(t) e^{-i\left(E_0^{1e}+E_0^{2g}\right)t/\hbar}
\end{aligned}$$

Then set of coupled differential equations for the amplitude probability is

$$
\begin{aligned}
\dot{C}_1(t) &= i\frac{d_2 \cdot E_2}{\hbar}C_2(t)\exp(-i\omega_{QD2}t) + i\frac{d_1 \cdot E_1}{\hbar}C_3(t)\exp(-i\omega_{QD1}t) \\
\dot{C}_2(t) &= i\frac{d_2 \cdot E_2}{\hbar}C_1(t)\exp(i\omega_{QD2}t) + i\frac{d_1 \cdot E_1}{\hbar}C_4(t)\exp(-i\omega_{QD1}t) \\
&\quad +iWC_3(t)\exp\left[-i(\omega_{QD1} - \omega_{QD2})t\right] \\
\dot{C}_3(t) &= i\frac{d_1 \cdot E_1}{\hbar}C_1(t)\exp(i\omega_{QD1}t) + i\frac{d_2 \cdot E_2}{\hbar}C_4(t)\exp(-i\omega_{QD2}t) \\
&\quad +iWC_2(t)\exp\left[i(\omega_{QD1} - \omega_{QD2})t\right] \\
\dot{C}_4(t) &= i\frac{d_1 \cdot E_1}{\hbar}C_2(t)\exp(i\omega_{QD1}t) + i\frac{d_2 \cdot E_2}{\hbar}C_3(t)\exp(i\omega_{QD2}t)
\end{aligned}
$$

To write in this form the differential equation allow us to apply the rotating wave approximation

3.4 Rotating wave approximation

Equation depends on the frequencies ω_{QD1} and ω_{QD2}, to solve this fact we will follow the same analysis done in the chapter 2. We assume the electric fields in the cavities on the form

$$E_1(t) = e_1(t)\cos(\nu_1 t) = \frac{e_1(t)}{2}\left[\exp(i\nu_1 t) + \exp(-i\nu_1 t)\right], \tag{3.6}$$

$$E_2(t) = e_2(t)\cos(\nu_2 t) = \frac{e_2(t)}{2}\left[\exp(i\nu_2 t) + \exp(-i\nu_2 t)\right]. \tag{3.7}$$

Then the equation become

$$\begin{aligned}
\dot{C}_1(t) &= i\frac{\Omega_2}{2}C_2(t)\exp\left(-i\Delta_2 t\right) + i\frac{\Omega_1}{2}C_3(t)\exp\left(-i\Delta_1 t\right), \\
\dot{C}_2(t) &= i\frac{\Omega_2}{2}C_1(t)\exp\left(i\Delta_2 t\right) + i\frac{\Omega_1}{2}C_4(t)\exp\left(-i\Delta_1\right) + iWC_3(t)\exp\left(-i\delta t\right), \\
\dot{C}_3(t) &= i\frac{\Omega_1}{2}C_1(t)\exp\left(i\Delta_1 t\right) + i\frac{\Omega_2}{2}C_4(t)\exp\left(-i\Delta_2 t\right) + iWC_2(t)\exp\left(i\delta t\right), \\
\dot{C}_4(t) &= i\frac{\Omega_1}{2}C_2(t)\exp\left(i\Delta_1 t\right) + i\frac{\Omega_2}{2}C_3(t)\exp\left(i\Delta_2 t\right).
\end{aligned}$$

Where has been introduced the detunings

$$\delta = \omega_{QD1} - \omega_{QD2}$$

$$\Delta_1 = \omega_{QD1} - \nu_1,$$

$$\Delta_2 = \omega_{QD2} - \nu_2.$$

And the Rabi's frequencies

$$\begin{aligned}
\Omega_1(t) &= \frac{d_1(t)\cdot E_1(t)}{\hbar} \\
\Omega_2(t) &= \frac{d_2(t)\cdot E_2(t)}{\hbar}
\end{aligned}$$

An ideal Foerster interaction requires that the frequencies of both QDs are the same, then the frequencies ω_{QD1} and ω_{QD2} must be the same. This implies that for resonance, where the detunings are equal to zero equation becomes

$$\dot{C}_1(t) = i\frac{\Omega_2(t)}{2}C_2(t) + i\frac{\Omega_1(t)}{2}C_3(t), \tag{3.8}$$

$$\dot{C}_2(t) = i\frac{\Omega_2(t)}{2}C_1(t) + i\frac{\Omega_1(t)}{2}C_4(t) + iWC_3(t), \tag{3.9}$$

$$\dot{C}_3(t) = i\frac{\Omega_1(t)}{2}C_1(t) + i\frac{\Omega_2(t)}{2}C_4(t) + iWC_2(t), \tag{3.10}$$

$$\dot{C}_4(t) = i\frac{\Omega_1(t)}{2}C_2(t) + i\frac{\Omega_2(t)}{2}C_3(t). \tag{3.11}$$

We will focus our attention in one of the QDs, in other words, to understand the dynamics of the system in terms of the dynamics of one of the constituents of the system. So, we will introduce the normalizations

$$W = A\Omega_1 \tag{3.12}$$

$$\Omega_2 = B\Omega_1 \tag{3.13}$$

The time unit now will be expressed as

$$\tau = \Omega_1 t$$

Now the set of equations to be solved will be

$$\dot{\beta}_1(\tau) = i\frac{B}{2}\beta_2(\tau) + i\frac{1}{2}\beta_3(\tau) \tag{3.14}$$

$$\dot{\beta}_2(\tau) = i\frac{B}{2}\beta_1(\tau) + iA\beta_3(\tau) + i\frac{1}{2}\beta_4(\tau) \tag{3.15}$$

$$\dot{\beta}_3(\tau) = i\frac{1}{2}\beta_1(\tau) + iA\beta_2(\tau) + i\frac{B}{2}\beta_4(\tau) \tag{3.16}$$

$$\dot{\beta}_4(\tau) = i\frac{1}{2}\beta_2(\tau) + i\frac{B}{2}\beta_3(\tau) \tag{3.17}$$

Where has been introduced the normalizations.

$$\beta_i(\tau) = \Omega_i C_i$$

Obviously, it has been considered and steady electric field.

3.5 Steady electric field amplitude

Our aim in this section is to find an analytical solution for the system. We can construct an alternative set of equations which solution is easier for finding.

$$\frac{d}{d\tau}\left[\beta_1(\tau) + \beta_4(\tau)\right] = i\frac{1}{2}\left[B+1\right]\left[\beta_2(\tau) + \beta_3(\tau)\right], \tag{3.18}$$

$$\frac{d}{d\tau}\left[\beta_1(t) - \beta_4(\tau)\right] = i\frac{1}{2}\left[B-1\right]\left[\beta_2(\tau) - \beta_3(\tau)\right], \tag{3.19}$$

$$\frac{d}{d\tau}\left[\beta_2(\tau) + \beta_3(\tau)\right] = i\frac{1}{2}\left[B+1\right]\left[\beta_1(\tau) + \beta_4(\tau)\right] + iA\left[\beta_3(\tau) + \beta_2(\tau)\right], \tag{3.20}$$

$$\frac{d}{d\tau}\left[\beta_2(\tau) - \beta_3(\tau)\right] = i\frac{1}{2}\left[B-1\right]\left[\beta_1(\tau) - \beta_4(\tau)\right] - iA\left[\beta_2(\tau) - \beta_3(\tau)\right]. \tag{3.21}$$

Basically we have a pair of equations on the form

$$\begin{aligned}
\frac{d}{d\tau}\xi_\pm &= i\Omega_\pm\chi_\pm \\
\frac{d}{d\tau}\chi_\pm &= i\Omega_\pm\xi_\pm \pm iA\chi_\pm
\end{aligned}$$

Where

$$\begin{aligned}
\xi_\pm(\tau) &= \beta_1(\tau) \pm \beta_4(\tau) \\
\chi_\pm(\tau) &= \beta_2(\tau) \pm \beta_3(\tau) \\
\Omega_\pm &= \frac{1}{2}\left(B \pm 1\right)
\end{aligned}$$

So that

$$\begin{aligned}
\beta_1(\tau) &= \frac{1}{2}\left[\xi_+(\tau)+\xi_-(\tau)\right], \\
\beta_4(\tau) &= \frac{1}{2}\left[\xi_+(\tau)-\xi_-(\tau)\right], \\
\beta_2(\tau) &= \frac{1}{2}\left[\chi_+(\tau)+\chi_-(\tau)\right], \\
\beta_3(\tau) &= \frac{1}{2}\left[\chi_+(\tau)-\chi_-(\tau)\right].
\end{aligned}$$

We will solve

$$\begin{aligned}
\frac{d}{dt}\xi &= i\Omega\chi, \\
\frac{d}{dt}\chi &= i\Omega\xi \pm iA\chi.
\end{aligned}$$

Here ξ, χ and Ω are either ξ_+, χ_+ and Ω_+ or ξ_-, χ_- and Ω_-. By taking Laplace transform

$$\begin{aligned}
s\xi(s) - i\Omega\chi(s) &= \xi(0), \\
s\chi(s) - i\Omega\xi(s) \mp iA\chi(s) &= \chi(0).
\end{aligned}$$

Or, written in matrix form

$$\begin{pmatrix} s & -i\Omega \\ -i\Omega & s \mp iA \end{pmatrix}\begin{pmatrix} \xi(s) \\ \chi(s) \end{pmatrix} = \begin{pmatrix} \xi(0) \\ \chi(0) \end{pmatrix}. \tag{3.22}$$

Multiplying by the inverse matrix

$$\left(\begin{array}{cc} s & -i\Omega \\ -i\Omega & s \mp iA \end{array} \right)^{-1} = \frac{1}{s\left(s \mp iA\right) + \Omega^2} \left(\begin{array}{cc} s \mp iA & i\Omega \\ i\Omega & s \end{array} \right). \tag{3.23}$$

We obtain a no coupled set of equations

$$\begin{aligned} \xi(s) &= \xi(0)\frac{s \mp iA}{s\left(s \mp iA\right) + \Omega^2} + \chi(0)\frac{i\Omega}{s\left(s \mp iA\right) + \Omega^2}, \\ \chi(s) &= \xi(0)\frac{i\Omega}{s\left(s \mp iA\right) + \Omega^2} + \chi(0)\frac{s}{s\left(s \mp iA\right) + \Omega^2}. \end{aligned}$$

It is necessary rewrite in a more convenient form so that their inverse Laplace Transform could be determinate immediately; we can complete the squares by using

$$\left(s \mp i\frac{1}{2}A\right)^2 = s^2 \mp isA - \frac{1}{4}A^2.$$

To obtain

$$\begin{aligned} \xi(s) &= \xi(0)\frac{s \mp iA}{\left(s \mp i\frac{1}{2}A\right)^2 + \left(\Omega^2 + \frac{1}{4}A^2\right)} + \chi(0)\frac{i\Omega}{\left(s \mp i\frac{1}{2}A\right)^2 + \left(\Omega^2 + \frac{1}{4}A^2\right)}, \\ &= \xi(0)\frac{s \mp i\frac{1}{2}A}{\left(s \mp i\frac{1}{2}A\right)^2 + \left(\Omega^2 + \frac{1}{4}A^2\right)} + \frac{i\Omega\chi(0) \mp \xi(0)i\frac{1}{2}A}{\left(s \mp i\frac{1}{2}A\right)^2 + \left(\Omega^2 + \frac{1}{4}A^2\right)}, \\ &= \xi(0)\frac{s \mp i\frac{1}{2}A}{\left(s \mp i\frac{1}{2}A\right)^2 + \left(\Omega^2 + \frac{1}{4}A^2\right)} \\ &\quad + \frac{i\Omega\chi(0) \mp \xi(0)i\frac{1}{2}A}{\sqrt{\Omega^2 + \frac{1}{4}A^2}} \frac{\sqrt{\Omega^2 + \frac{1}{4}A^2}}{\left(s \mp i\frac{1}{2}A\right)^2 + \left(\Omega^2 + \frac{1}{4}A^2\right)}. \end{aligned}$$

And similarly

$$\begin{aligned}
\chi(s) &= \xi(0)\frac{i\Omega}{\left(s\mp i\frac{1}{2}A\right)^2+\left(\Omega^2+\frac{1}{4}A^2\right)}+\chi(0)\frac{s}{\left(s\mp i\frac{1}{2}A\right)^2+\left(\Omega^2+\frac{1}{4}A^2\right)},\\
&= \frac{i\Omega\xi(0)}{\sqrt{\Omega^2+\frac{1}{4}A^2}}\frac{\sqrt{\Omega^2+\frac{1}{4}A^2}}{\left(s\mp i\frac{1}{2}A\right)^2+\left(\Omega^2+\frac{1}{4}A^2\right)}+\chi(0)\frac{s\mp i\frac{1}{2}A\pm i\frac{1}{2}A}{\left(s\mp i\frac{1}{2}A\right)^2+\left(\Omega^2+\frac{1}{4}A^2\right)},\\
&= \frac{i\Omega\xi(0)\pm i\chi(0)\frac{1}{2}A}{\sqrt{\Omega^2+\frac{1}{4}A^2}}\frac{\sqrt{\Omega^2+\frac{1}{4}A^2}}{\left(s\mp i\frac{1}{2}A\right)^2+\left(\Omega^2+\frac{1}{4}A^2\right)}\\
&\quad +\chi(0)\frac{s\mp i\frac{1}{2}A}{\left(s\mp i\frac{1}{2}A\right)^2+\left(\Omega^2+\frac{1}{4}A^2\right)}.
\end{aligned}$$

We have obtained a set of equations which have an immediate inverse Laplace transform

$$\begin{aligned}
\xi(\tau) &= \left[\xi(0)\cos\left(\sqrt{\Omega^2+\frac{1}{4}A^2t}\right)+\frac{i\Omega\chi(0)\mp\xi(0)i\frac{1}{2}A}{\sqrt{\Omega^2+\frac{1}{4}A^2}}\sin\left(\sqrt{\Omega^2+\frac{1}{4}A^2\tau}\right)\right]\\
&\quad\times\exp\left(\pm i\frac{1}{2}A\tau\right),\\
\chi(\tau) &= \left[\frac{\xi(0)i\Omega\pm\chi(0)\frac{1}{2}iA}{\sqrt{\Omega^2+\frac{1}{4}A^2}}\sin\left(\sqrt{\Omega^2+\frac{1}{4}A^2\tau}\right)+\chi(0)\cos\left(\sqrt{\Omega^2+\frac{1}{4}A^2\tau}\right)\right]\\
&\quad\times\exp\left(\pm i\frac{1}{2}A\tau\right).
\end{aligned}$$

Above equation is the most general possible, with accurate initial conditions we can find solutions when the system is initially separable or entangled. Now we will consider as initial condition the separable state with initial coefficients

$$\beta_3(0)=1,\ \beta_1(0)=\beta_2(0)=\beta_4(0)=0.$$

Which represents the QD-1 in the excited state and the QD-2 in the ground state. This means that

$$\begin{aligned}\xi_{\pm}(0) &= 0,\\ \chi_{\pm}(0) &= \pm 1.\end{aligned}$$

In this point we can choose different values of the electric field, in other words different values for B. An interesting case is when one of the QD is an isolated system and when the electric field amplitude of this QD is zero. This case allows us to investigate the effect of the coupling in a QD in absence of electric field, when $B = 0$.

$$\Omega_{\pm} = \pm\frac{1}{2}$$

And

$$\begin{aligned}\xi_{\pm}(\tau) &= \left[\frac{i}{\sqrt{1+A^2}}\sin\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\right]\exp\left(\pm i\frac{1}{2}A\tau\right),\\ \chi_{\pm}(\tau) &= \left[\frac{iA}{\sqrt{1+A^2}}\sin\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\pm\cos\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\right]\exp\left(\pm i\frac{1}{2}A\tau\right).\end{aligned}$$

Then, probabilityamplitude coefficients are

$$\beta_1(\tau) = \frac{i}{\sqrt{1+A^2}}\sin\left(\sqrt{1+A^2}\tau/2\right)\cos\left(A\tau/2\right), \tag{3.24}$$

$$\begin{aligned}\beta_2(\tau) &= i\frac{A}{\sqrt{1+A^2}}\sin\left(\sqrt{1+A^2}\tau/2\right)\cos\left(A\tau/2\right)\\ &\quad +i\cos\left(\sqrt{1+A^2}\tau/2\right)\sin\left(A\tau/2\right),\end{aligned} \tag{3.25}$$

$$\begin{aligned}\beta_3(\tau) &= -\frac{A}{\sqrt{1+A^2}}\sin\left(\sqrt{1+A^2}\tau/2\right)\sin\left(A\tau/2\right)\\ &\quad +\cos\left(\sqrt{1+A^2}\tau/2\right)\cos\left(A\tau/2\right),\end{aligned} \tag{3.26}$$

$$\beta_4(\tau) = -\frac{1}{\sqrt{1+A^2}}\sin\left(\sqrt{1+A^2}\tau/2\right)\sin\left(A\tau/2\right). \tag{3.27}$$

We can observe that $\beta_1(\tau)$ and $\beta_2(\tau)$ are imaginary complex numbers while $\beta_3(\tau)$ and $\beta_4(\tau)$ are real numbers. This fact will be important when calculated the expected value of the Pauli matrices.

3.5.1 Vector representation

As we have demonstrated in the last chapter the coefficients) are the amplitude probability in the rotating frame. The electric field frequencies are in resonance with the QD frequencies

$$\left|\tilde{\psi}(t)\right\rangle = \beta_1(\tau)\left|g(1), g(2)\right\rangle + \beta_2(\tau)\left|g(1), e(2)\right\rangle + \beta_3(\tau)\left|e(2), g(1)\right\rangle + \beta_4(\tau)\left|e(1), e(2)\right\rangle .$$

By taking the expectation values of the Pauli's matrices we can to define the real quantities

$$\begin{aligned} u_1(t) &= \left\langle\tilde{\psi}(\tau)\right|\sigma_1^x\left|\tilde{\psi}(\tau)\right\rangle, \\ v_1(t) &= \left\langle\tilde{\psi}(\tau)\right|\sigma_1^y\left|\tilde{\psi}(\tau)\right\rangle, \\ w_1(t) &= \left\langle\tilde{\psi}(\tau)\right|\sigma_1^z\left|\tilde{\psi}(\tau)\right\rangle. \end{aligned}$$

and

$$\begin{aligned} u_2(t) &= \left\langle\tilde{\psi}(\tau)\right|\sigma_2^x\left|\tilde{\psi}(\tau)\right\rangle, \\ v_2(t) &= \left\langle\tilde{\psi}(\tau)\right|\sigma_2^y\left|\tilde{\psi}(\tau)\right\rangle, \\ w_2(t) &= \left\langle\tilde{\psi}(\tau)\right|\sigma_2^z\left|\tilde{\psi}(\tau)\right\rangle. \end{aligned}$$

They are the components of the Bloch vector

$$
\begin{aligned}
r_1(\tau) &= u_1(\tau)\hat{x} + v_1(\tau)\hat{y} + w_1(\tau)\hat{z} \\
r_2(\tau) &= u_2(\tau)\hat{x} + v_2(\tau)\hat{y} + w_2(\tau)\hat{z}
\end{aligned}
$$

3.5.1.1 Expected values of the Pauli's matrices

With the wave function that we have found we take the expected value of the Pauli matrices

$\sigma^{\mathbf{x}}$ Pauli Matrices.

The expectation value of σ_1^x is

$$
\begin{aligned}
u_1(\tau) &= \left\langle \tilde{\psi} \right| \sigma_1^x \left| \tilde{\psi} \right\rangle . \\
&= \left\langle \tilde{\psi} \right| \sigma_1^x \left[\beta_1(\tau) \left| g(1), g(2) \right\rangle + \beta_2(\tau) \left| g(1), e(2) \right\rangle \right. \\
&\quad \left. + \beta_3(\tau) \left| e(1), g(2) \right\rangle + \beta_4(\tau) \left| e(1), e(2) \right\rangle \right] , \\
&= \left\langle \tilde{\psi} \right| \left[\beta_1(\tau) \left| e(1), g(2) \right\rangle + \beta_2(\tau) \left| e(1), e(2) \right\rangle \right. \\
&\quad \left. + \beta_3(\tau) \left| g(1), g(2) \right\rangle + \beta_4(\tau) \left| g(1), e(2) \right\rangle \right] , \\
&= \beta_3^*(\tau)\beta_1(\tau) + \beta_4^*(\tau)\beta_2(\tau) + \beta_1^*(\tau)\beta_3(\tau) + \beta_2^*(\tau)\beta_4(\tau) \\
&= 2\,\mathrm{Re}\left[\beta_3^*(\tau)\beta_1(\tau) + \beta_4^*(\tau)\beta_2(\tau) \right] .
\end{aligned}
$$

Because of the product of $\beta_3^*(\tau)\beta_1(\tau) + \beta_4^*(\tau)\beta_2(\tau)$ is imaginary follows

$$
u_1(\tau) = 0 \tag{3.28}
$$

And the expectation value of σ_2^x gives

$$
\begin{aligned}
u_2(\tau) &= \left\langle \tilde{\psi} \right| \sigma_2^x \left| \tilde{\psi} \right\rangle , \\
&= \left\langle \tilde{\psi} \right| \sigma_2^x \left[\beta_1(\tau) \left| g(1), g(2) \right\rangle + \beta_2(\tau) \left| g(1), e(2) \right\rangle \right. \\
&\quad \left. + \beta_3(\tau) \left| e(1), g(2) \right\rangle + \beta_4(\tau) \left| e(1), e(2) \right\rangle \right] , \\
&= \left\langle \tilde{\psi} \right| \left[\beta_1(\tau) \left| g(1), e(2) \right\rangle + \beta_2(\tau) \left| g(1), g(2) \right\rangle \right. \\
&\quad \left. + \beta_3(\tau) \left| e(1), e(2) \right\rangle + \beta_4(\tau) \left| e(1), g(2) \right\rangle \right] , \\
&= \beta_2^*(\tau)\beta_1(\tau) + \beta_1^*(\tau)\beta_2(\tau) + \beta_4^*(\tau)\beta_3(\tau) + \beta_3^*(\tau)\beta_4(\tau), \\
&= 2 \,\mathrm{Re} \left[\beta_1^*(\tau)\beta_2(\tau) + \beta_4^*(\tau)\beta_3(\tau) \right] .
\end{aligned}
$$

Then

$$
u_2(\tau) = \frac{A}{1+A^2} \sin^2 \left(\frac{1}{2} \sqrt{1+A^2} \tau \right) . \tag{3.29}
$$

There is a time evolution of $u_2(\tau)$ even in resonance due to the coupling.

$\sigma^{\mathbf{y}}$ Pauli Matrices.

The next operators for obtaining the expected value are σ_1^y

$$
\begin{aligned}
\langle \sigma_1^y \rangle &= \left\langle \tilde{\psi} \right| \sigma_1^y \left| \tilde{\psi} \right\rangle , \\
&= \left\langle \tilde{\psi} \right| \sigma_1^y \left[\beta_1(\tau) \left| g(1), g(2) \right\rangle + \beta_2(\tau) \left| g(1), e(2) \right\rangle \right. \\
&\quad \left. + \beta_3(\tau) \left| e(1), g(2) \right\rangle + \beta_4(\tau) \left| e(1), e(2) \right\rangle \right] , \\
&= \left\langle \tilde{\psi} \right| \left[-i\beta_1(\tau) \left| e(1), g(2) \right\rangle - i\beta_2(\tau) \left| e(1), e(2) \right\rangle \right. \\
&\quad \left. + i\beta_3(\tau) \left| g(1), g(2) \right\rangle + i\beta_4(\tau) \left| g(1), e(2) \right\rangle \right] , \\
&= i\beta_1^*(\tau)\beta_3(\tau) + i\beta_2^*(\tau)\beta_4(\tau) - i\beta_3^*(\tau)\beta_1(\tau) - i\beta_2(\tau)\beta_4^*(\tau), \\
&= 2 \,\mathrm{Re} \left[i\beta_1^*(\tau)\beta_3(\tau) + i\beta_2^*(\tau)\beta_4(\tau) \right] .
\end{aligned}
$$

When the coefficients are known:

$$
\begin{aligned}
v_1(t) &= 2\,\mathrm{Re}\left[i\beta_1^*(\tau)\beta_3(\tau) + i\beta_2^*(\tau)\beta_4(\tau)\right], \\
&= -4\frac{A}{1+A^2}\sin^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\cos\left(\frac{1}{2}A\tau\right)\sin\left(\frac{1}{2}A\tau\right) \\
&\quad +2\frac{1}{\sqrt{1+A^2}}\sin\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\cos\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\left[\cos^2\left(\frac{1}{2}A\tau\right) - \sin^2\left(\frac{1}{2}A\tau\right)\right], \\
&= -\frac{A}{1+A^2}\left[1-\cos\left(\sqrt{1+A^2}\tau\right)\right]\sin\left(A\tau\right) + \frac{1}{\sqrt{1+A^2}}\sin\left(\sqrt{1+A^2}\tau\right)\cos\left(A\tau\right).
\end{aligned}
$$

Is obtained

$$
\begin{aligned}
v_1(t) &= -\frac{A}{1+A^2}\left[1-\cos\left(\sqrt{1+A^2}\tau\right)\right]\sin\left(A\tau\right) \\
&\quad +\frac{1}{\sqrt{1+A^2}}\sin\left(\sqrt{1+A^2}\tau\right)\cos\left(A\tau\right).
\end{aligned} \tag{3.30}
$$

And for σ_2^y

$$
\begin{aligned}
v_2(\tau) &= \left\langle\tilde{\psi}\right|\sigma_2^y\left|\tilde{\psi}\right\rangle, \\
&= \left\langle\tilde{\psi}\right|\sigma_2^y\left[\beta_1(\tau)\left|g(1),g(2)\right\rangle + \beta_2(\tau)\left|g(1),e(2)\right\rangle\right. \\
&\quad \left. + \beta_3(\tau)\left|e(1),g(2)\right\rangle + \beta_4(\tau)\left|e(1),e(2)\right\rangle\right], \\
&= \left\langle\tilde{\psi}\right|\left[-i\beta_1(\tau)\left|g(1),e(2)\right\rangle + i\beta_2(\tau)\left|g(1),g(2)\right\rangle\right. \\
&\quad \left. - i\beta_3(\tau)\left|e(1),e(2)\right\rangle + i\beta_4(\tau)\left|e(1),g(2)\right\rangle\right], \\
&= i\beta_1^*(\tau)\beta_2(\tau) - i\beta_2^*(\tau)\beta_1(\tau) + i\beta_3^*(\tau)\beta_4(\tau) - i\beta_4^*(\tau)\beta_3(\tau), \\
&= 2\,\mathrm{Re}\left[i\beta_1^*(\tau)\beta_2(\tau) + i\beta_3^*(\tau)\beta_4(\tau)\right].
\end{aligned}
$$

The quantity $i\beta_1^*(\tau)\beta_2(\tau) + i\beta_3^*(\tau)\beta_4(\tau)$ is imaginary, then

$$
v_2(\tau) = 0 \tag{3.31}
$$

$\sigma^{\mathbf{z}}$ Pauli Matrices.

The single-population inversion for each system is given by the expected value of the following Pauli matrices:

$$
\begin{aligned}
w_1(\tau) &= \left\langle \tilde{\psi} \right| \sigma_1^z \left| \tilde{\psi} \right\rangle, \\
&= \left\langle \tilde{\psi} \right| \sigma_1^z \left[\beta_1(\tau) \left| g(1), g(2) \right\rangle + \beta_2(\tau) \left| g(1), e(2) \right\rangle \right. \\
&\quad \left. + \beta_3(\tau) \left| e(1), g(2) \right\rangle + \beta_4(\tau) \left| e(1), e(2) \right\rangle \right], \\
&= \left\langle \tilde{\psi} \right| \left[-\beta_1(\tau) \left| g(1), g(2) \right\rangle - \beta_2(\tau) \left| g(1), e(2) \right\rangle \right. \\
&\quad \left. + \beta_3(\tau) \left| e(1), g(2) \right\rangle + \beta_4(\tau) \left| e(1), e(2) \right\rangle \right], \\
&= -\left| \beta_1(\tau) \right|^2 - \left| \beta_2(\tau) \right|^2 + \left| \beta_3(\tau) \right|^2 + \left| \beta_4(\tau) \right|^2.
\end{aligned}
$$

And

$$
\begin{aligned}
w_2(\tau) &= \left\langle \tilde{\psi} \right| \sigma_2^z \left| \tilde{\psi} \right\rangle, \\
&= \left\langle \tilde{\psi} \right| \sigma_2^z \left[\beta_1(\tau) \left| g(1), g(2) \right\rangle + \beta_2(\tau) \left| g(1), e(2) \right\rangle \right. && (3.32) \\
&\quad \left. + \beta_3(\tau) \left| e(1), g(2) \right\rangle + \beta_4(\tau) \left| e(1), e(2) \right\rangle \right], \\
&= \left\langle \tilde{\psi} \right| \left[-\beta_1(\tau) \left| g(1), g(2) \right\rangle + \beta_2(\tau) \left| g(1), e(2) \right\rangle \right. && (3.33) \\
&\quad \left. - \beta_3(\tau) \left| e(1), g(2) \right\rangle + \beta_4(\tau) \left| e(1), e(2) \right\rangle \right], \\
&= -\left| \beta_1(\tau) \right|^2 + \left| \beta_2(\tau) \right|^2 - \left| \beta_3(\tau) \right|^2 + \left| \beta_4(\tau) \right|^2. && (3.34)
\end{aligned}
$$

In order to obtain analytical expressions for the inversion it is necessary to determinate the module of the coefficients:

$$
\begin{aligned}
|\beta_1(\tau)|^2 &= \frac{1}{1+A^2}\sin^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\cos^2\left(\frac{1}{2}A\tau\right), \\
|\beta_2(\tau)|^2 &= \frac{A^2}{1+A^2}\sin^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\cos^2\left(\frac{1}{2}A\tau\right)+\cos^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\sin^2\left(\frac{1}{2}A\tau\right) \\
&\quad +2\frac{A}{\sqrt{1+A^2}}\sin\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\cos\left(\frac{1}{2}A\tau\right)\cos\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\sin\left(\frac{1}{2}A\tau\right), \\
|\beta_3(\tau)|^2 &= \frac{A^2}{1+A^2}\sin^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\sin^2\left(\frac{1}{2}A\tau\right)+\cos^2\left(\frac{1}{2}A\tau\right)\cos^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right) \\
&\quad -2\frac{A}{\sqrt{1+A^2}}\sin\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\sin\left(\frac{1}{2}A\tau\right)\cos\left(\frac{1}{2}A\tau\right)\cos\left(\frac{1}{2}\sqrt{1+A^2}\tau\right), \\
|\beta_4(\tau)|^2 &= \frac{1}{1+A^2}\sin^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\sin^2\left(\frac{1}{2}A\tau\right).
\end{aligned}
$$

For the inversion of the system 1 we have

$$
\begin{aligned}
w_1(\tau) &= -|\beta_1(\tau)|^2-|\beta_2(\tau)|^2+|\beta_3(\tau)|^2+|\beta_4(\tau)|^2, \\
&= \cos\left(\sqrt{1+A^2}\tau\right)\cos(A\tau)-\frac{A}{\sqrt{1+A^2}}\sin\left(\sqrt{1+A^2}\tau\right)\sin(A\tau)
\end{aligned}
\tag{3.35}
$$

And the inversion of the system 2

$$
\begin{aligned}
w_2(\tau) &= -|\beta_1(\tau)|^2+|\beta_2(\tau)|^2-|\beta_3(\tau)|^2+|\beta_4(\tau)|^2, \\
&= \left[\frac{A^2-1}{A^2+1}\sin^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)-\cos^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right)\right] \\
&\quad +\frac{A}{\sqrt{1+A^2}}\sin\left(\sqrt{1+A^2}\tau\right)\sin(A\tau).
\end{aligned}
\tag{3.36}
$$

3.5.1.2 Resonance fluorescent spectrum

The expected value of the Pauli matrices σ_x and σ_y are related with the complex amplitude of the electric dipole through

$$
\begin{aligned}
d_1(\tau) &= u_1 + iv_1(\tau), \\
d_2(\tau) &= u_2 + iv_2(\tau).
\end{aligned}
$$

Explicitly

$$
\begin{aligned}
d_1(\tau) &= -\frac{A}{1+A^2}\left[1-\cos\left(\sqrt{1+A^2}\tau\right)\right]\sin\left(A\tau\right) \\
&\quad +\frac{1}{\sqrt{1+A^2}}\sin\left(\sqrt{1+A^2}\tau\right)\cos\left(A\tau\right), \\
d_2(\tau) &= \frac{A}{1+A^2}\sin^2\left(\frac{1}{2}\sqrt{1+A^2}\tau\right).
\end{aligned}
$$

If the interdo coupling is weak, then we can assume $A \ll 1$

$$
\begin{aligned}
d_1(\tau) &= -iA\sin\left(A\tau\right) + iA\cos\left(\tau\right)\sin\left(A\tau\right) + i\sin\left(\tau\right)\cos\left(A\tau\right), \\
d_2(\tau) &= A\left[\frac{1}{2}-\frac{1}{2}\cos\left(\tau\right)\right].
\end{aligned}
$$

The above equations are quite interesting. They can produce an approximation to the resonance fluorescence spectrum for each system. Let's notice that in absence of coupling $A = 0$, the expected result is reproduced: the system 1 has a spectrum studied in the last chapter while the QD-2 doesn't have oscillations.

$$
\begin{aligned}
d_1(\tau) &= -\frac{1}{2}A\exp\left(iA\tau\right) + \frac{1}{2}A\exp\left(-iA\tau\right) + \left(\frac{1}{4}A+\frac{1}{4}\right)\exp\left(i\tau\right)\exp\left(iA\tau\right) \\
&\quad - \left(\frac{1}{4}+\frac{1}{4}A\right)\exp\left(-i\tau\right)\exp\left(-iA\tau\right) + \left(\frac{1}{4}A-\frac{1}{4}\right)\exp\left(-i\tau\right)\exp\left(iA\tau\right) \\
&\quad - \left(\frac{1}{4}A-\frac{1}{4}\right)\exp\left(i\tau\right)\exp\left(-iA\tau\right), \\
d_2(\tau) &= \frac{1}{2}A - \frac{1}{4}A\exp\left(i\tau\right) - \frac{1}{4}A\exp\left(-i\tau\right).
\end{aligned}
$$

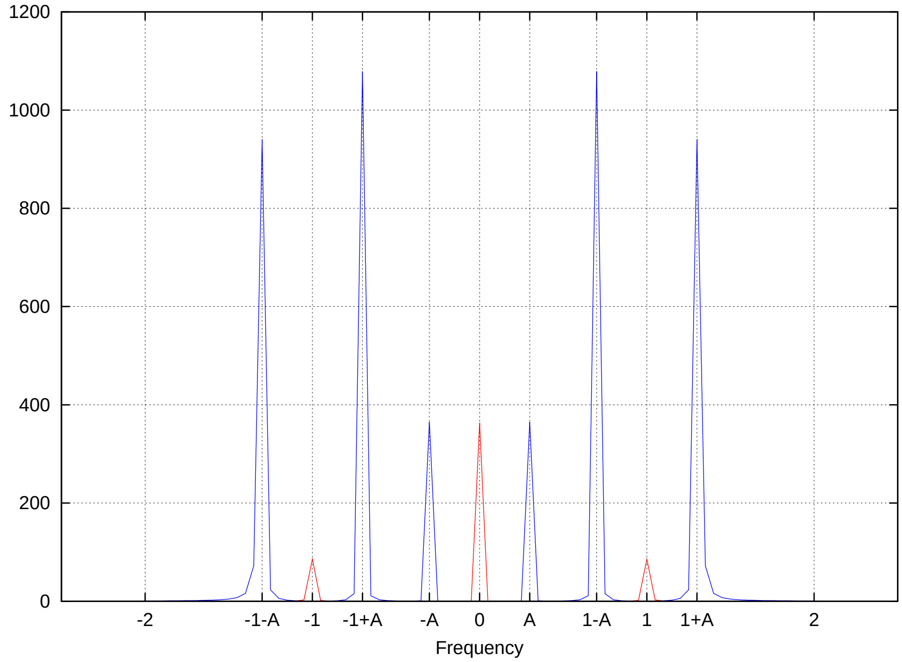


Figure 3.2: It is shown the resonance fluorescent spectrum of each system. Blue line corresponds to the non-isolated system while the red line represents to the isolated system. The isolated QD has oscillations due to the coupling and the non-isolated system has a different structure in comparison with the single QD spectrum.

Expressing the dipoles in this form, allow us to take the Fourier transform. Lets to start describing the system 2: it presents a three-peaked resonance fluorescent spectrum, the main peak is in the origin at the field excitation frequency of the system 1 while the secondary peaks are located at Rabi frequency of the system 1.By other hand, the first system is six-peaked distributed in pairs; each pair is shifted an amount A from the peaks of the isolated system.

3.5.1.3 Inversion

The expectation value of the σ_z Pauli matrices is called single population inversion. In the weak-coupling regime, where$A \ll 1$ we can neglect A^2 terms, equations) and) become

$$
\begin{aligned}
w_1(\tau) &= \cos(\tau)\cos(A\tau) - A\sin(\tau)\sin(A\tau)\,, \\
w_2(\tau) &= -\cos(A\tau) + A\sin(\tau)\sin(A\tau)\,.
\end{aligned}
$$

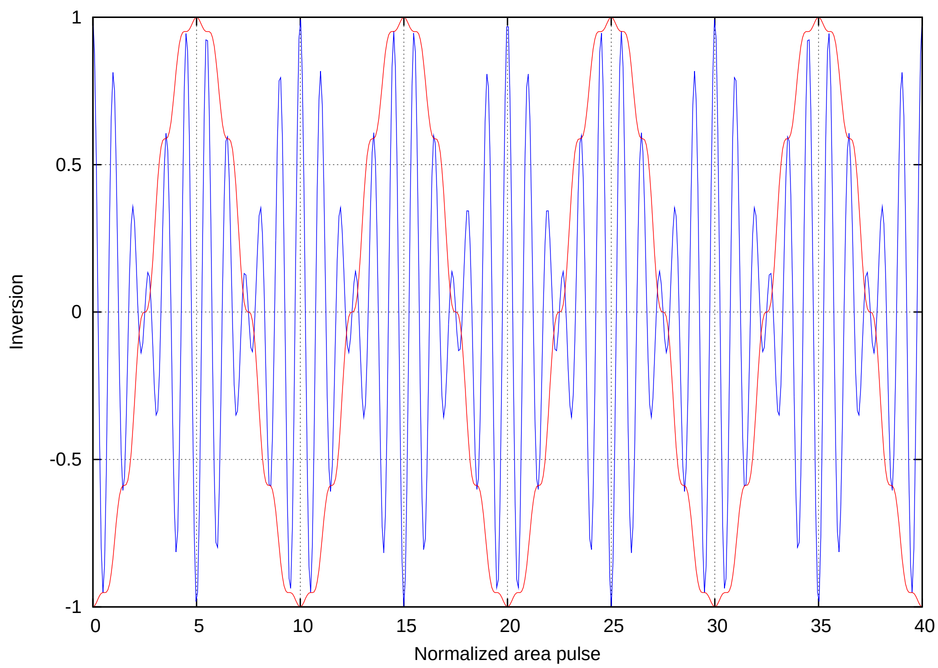


Figure 3.3: Red line indicates the inversion of the isolated system, blue line shows a modulation in the Rabi oscillations due to the coupling.

In above equations the terms contain to A as multiplicative factor are small, so the dominant part is contained in the other terms.

The dominant part of the system 2 shows Rabi oscillations for this system, the frequency of these oscillations is explicitly the strength of the coupling A. By other hand, the system 1 presents Rabi oscillations at Rabi frequency, remember $\tau = \Omega_1 t$, but its oscillations are modulated by the inversion of the isolated system.

3.5.2 Optcal Bloch's equations

We have already calculated the optical Bloch equations for the single quantum dot interacting with a classical electric field. Is it possible to obtain optical Bloch equations for the coupled system? To answer this question we will take the derivate of the expected value of the Pauli matrices. We have been studying resonance case and a null electric field for the isolated system, so we will work in this situation.

Taking the derivate of equations) we obtain the following set of equations

$$\begin{aligned}\dot{u}_1(\tau) &= \dot{\beta}_3^*(\tau)\beta_1(\tau) + \beta_3^*(\tau)\dot{\beta}_1(\tau) + \dot{\beta}_4^*(\tau)\beta_2(\tau) + \beta_4^*(\tau)\dot{\beta}_2(\tau) \\ &\quad +\dot{\beta}_1^*(\tau)\beta_3(\tau) + \beta_1^*(\tau)\dot{\beta}_3(\tau) + \dot{\beta}_2^*(\tau)\beta_4(\tau) + \beta_2^*(\tau)\dot{\beta}_4(\tau),\end{aligned} \tag{3.37}$$

$$\begin{aligned}\dot{v}_1(\tau) &= i\dot{\beta}_1^*(\tau)\beta_3(\tau) + i\beta_1^*(\tau)\dot{\beta}_3(\tau) + i\dot{\beta}_2^*(\tau)\beta_4(\tau) + i\beta_2^*(\tau)\dot{\beta}_4(\tau) \\ &\quad -i\dot{\beta}_3^*(\tau)\beta_1(\tau) - i\beta_3^*(\tau)\dot{\beta}_1(\tau) - i\dot{\beta}_2(\tau)\beta_4^*(\tau) - i\beta_2(\tau)\dot{\beta}_4^*(\tau),\end{aligned} \tag{3.38}$$

$$\begin{aligned}\dot{w}_1(\tau) &= -\dot{\beta}_1^*(\tau)\beta_1(\tau) - \dot{\beta}_1(\tau)\beta_1^*(\tau) - \dot{\beta}_2^*(\tau)\beta_2(\tau) - \dot{\beta}_2(\tau)\beta_2^*(\tau) \\ &\quad +\dot{\beta}_3^*(\tau)\beta_3(\tau) + \dot{\beta}_3(\tau)\beta_3^*(\tau) + \dot{\beta}_4^*(\tau)\beta_4(\tau) + \dot{\beta}_4(\tau)\beta_4^*(\tau).\end{aligned} \tag{3.39}$$

By choosing $B = 0$, the differential equation for the amplitude coefficients is:

$$\dot{\beta}_1(\tau) = i\frac{1}{2}\beta_3(\tau), \tag{3.40}$$

$$\dot{\beta}_2(\tau) = iA\beta_3(\tau) + i\frac{1}{2}\beta_4(\tau), \tag{3.41}$$

$$\dot{\beta}_3(\tau) = i\frac{1}{2}\beta_1(\tau) + iA\beta_2(\tau), \tag{3.42}$$

$$\dot{\beta}_4(\tau) = i\frac{1}{2}\beta_2(\tau). \tag{3.43}$$

With the equations) this set becomes for $u_1(\tau)$

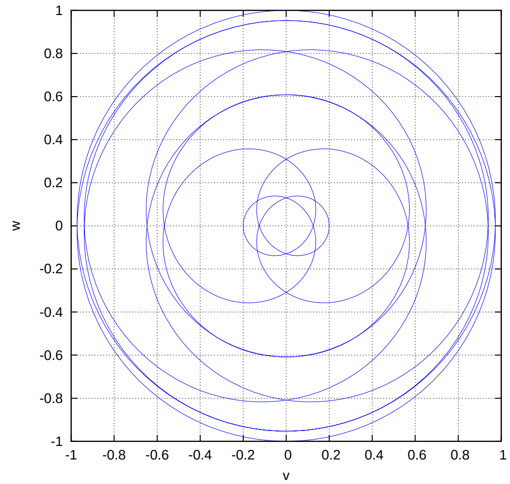
$$\begin{aligned}
\dot{u}_1(\tau) &= \left[-i\frac{1}{2}\beta_1^*(\tau) - iA\beta_2^*(\tau)\right]\beta_1(\tau) + \beta_3^*(\tau)\left[i\frac{1}{2}\beta_3(\tau)\right] + \left[-i\frac{1}{2}\beta_2^*(\tau)\right]\beta_2(\tau) \\
&\quad + \beta_4^*(\tau)\left[iA\beta_3(\tau) + i\frac{1}{2}\beta_4(\tau)\right] + \left[-i\frac{1}{2}\beta_3^*(\tau)\right]\beta_3(\tau) + \beta_1^*(\tau)\left[i\frac{1}{2}\beta_1(\tau) + iA\beta_2(\tau)\right] \\
&\quad + \left[-iA\beta_3^*(\tau) - i\frac{1}{2}\beta_4^*(\tau)\right]\beta_4(\tau) + \beta_2^*(\tau)\left[i\frac{1}{2}\beta_2(\tau)\right] \\
&= -iA\beta_2^*(\tau)\beta_1(\tau) + iA\beta_4^*(\tau)\beta_3(\tau) + iA\beta_1^*(\tau)\beta_2(\tau) - iA\beta_3^*(\tau)\beta_4(\tau) \\
&= 2A\,\mathrm{Re}\left[i\beta_4^*(\tau)\beta_3(\tau) + i\beta_1^*(\tau)\beta_2(\tau)\right].
\end{aligned}$$

For $v_1(\tau)$

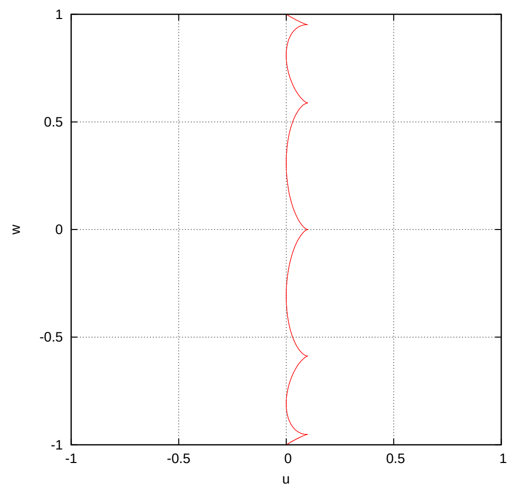
$$\begin{aligned}
\dot{v}_1(\tau) &= i\left[-i\frac{1}{2}\beta_3^*(\tau)\right]\beta_3(\tau) + i\beta_1^*(\tau)\left[i\frac{1}{2}\beta_1(\tau) + iA\beta_2(\tau)\right] + i\left[-iA\beta_3^*(\tau) - i\frac{1}{2}\beta_4^*(\tau)\right]\beta_4(\tau) \\
&\quad + i\beta_2^*(\tau)\left[i\frac{1}{2}\beta_2(\tau)\right] - i\left[-i\frac{1}{2}\beta_1^*(\tau) - iA\beta_2^*(\tau)\right]\beta_1(\tau) - i\beta_3^*(\tau)\left[i\frac{1}{2}\beta_3(\tau)\right] \\
&\quad - i\left[iA\beta_3(\tau) + i\frac{1}{2}\beta_4(\tau)\right]\beta_4^*(\tau) - i\beta_2(\tau)\left[-i\frac{1}{2}\beta_2^*(\tau)\right] \\
&= -\beta_1^*(\tau)\beta_1(\tau) - \beta_2(\tau)\beta_2^*(\tau) + \beta_3^*(\tau)\beta_3(\tau) + \beta_4^*(\tau)\beta_4(\tau) \\
&\quad - A\beta_1^*(\tau)\beta_2(\tau) + A\beta_3^*(\tau)\beta_4(\tau) - A\beta_2^*(\tau)\beta_1(\tau) + A\beta_3(\tau)\beta_4^*(\tau) \\
&= w_1(\tau) + 2A\,\mathrm{Re}\left[\beta_3^*(\tau)\beta_4(\tau) - \beta_2^*(\tau)\beta_1(\tau)\right].
\end{aligned}$$

And for the inversion

$$\begin{aligned}
\dot{w}_1(\tau) &= i\frac{1}{2}\beta_3^*(\tau)\beta_1(\tau) - i\frac{1}{2}\beta_3(\tau)\beta_1^*(\tau) + iA\beta_3^*(\tau)\beta_2(\tau) + i\frac{1}{2}\beta_4^*(\tau)\beta_2(\tau) - iA\beta_3(\tau)\beta_2^*(\tau) \\
&\quad - i\frac{1}{2}\beta_4(\tau)\beta_2^*(\tau) - i\frac{1}{2}\beta_1^*(\tau)\beta_3(\tau) - iA\beta_2^*(\tau)\beta_3(\tau) + i\frac{1}{2}\beta_1(\tau)\beta_3^*(\tau) + iA\beta_2(\tau)\beta_3^*(\tau) \\
&\quad - i\frac{1}{2}\beta_2^*(\tau)\beta_4(\tau) + i\frac{1}{2}\beta_2(\tau)\beta_4^*(\tau) \\
&= i\beta_3^*(\tau)\beta_1(\tau) - i\beta_3(\tau)\beta_1^*(\tau) + i\beta_4^*(\tau)\beta_2(\tau) - i\beta_4(\tau)\beta_2^*(\tau) \\
&\quad - iA\beta_2^*(\tau)\beta_3(\tau) + iA\beta_2(\tau)\beta_3^*(\tau) + iA\beta_3^*(\tau)\beta_2(\tau) - iA\beta_3(\tau)\beta_2^*(\tau) \\
&= -v_1(\tau) + 4A\,\mathrm{Re}\left[i\beta_3^*(\tau)\beta_2(\tau)\right].
\end{aligned}$$



(a) Bloch's vector of the non isolated system



(b) Bloch's vector of the isolated system

Figure 3.4: Blue line shows the behavior of the Bloch's vector; its dynamics is restricted to the $v - w$ plane and does not create a circumference in that plane. Red line shows the Bloch's vector of the isolated system; its time evolution is due to the coupling and is in the $u - w$ plane.

In this way we obtain the set of equations

$$\dot{u}_1(\tau) = 2A\,\mathrm{Re}\left[-i\beta_2^*(\tau)\beta_1(\tau) + i\beta_4^*(\tau)\beta_3(\tau)\right], \tag{3.44}$$

$$\dot{v}_1(\tau) = w_1(\tau) + 2A\,\mathrm{Re}\left[-\beta_1^*(\tau)\beta_2(\tau) + \beta_3^*(\tau)\beta_4(\tau)\right], \tag{3.45}$$

$$\dot{w}_1(\tau) = -v_1(\tau) + 4A\,\mathrm{Re}\left[i\beta_3^*(\tau)\beta_2(\tau)\right]. \tag{3.46}$$

Let's notice, they are the optical Bloch in resonance with additional terms due to the coupling which cannot be identified with any of the QDs variables. These equations cannot be written in terms of the components of the Bloch vector. They also depend on the normalized probability amplitude and they are different, depending of the initial conditions of the problem.

The derivate of the expected values of the Pauli matrices for the isolated system is

$$
\begin{aligned}
\dot{u}_2(\tau) &= \dot{\beta}_2^*(\tau)\beta_1(\tau) + \beta_2^*(\tau)\dot{\beta}_1(\tau) + \dot{\beta}_1^*(\tau)\beta_2(\tau) + \beta_1^*(\tau)\dot{\beta}_2(\tau) \\
&\quad + \dot{\beta}_4^*(\tau)\beta_3(\tau) + \beta_4^*(\tau)\dot{\beta}_3(\tau) + \dot{\beta}_3^*(\tau)\beta_4(\tau) + \beta_3^*(\tau)\dot{\beta}_4(\tau)
\end{aligned}
$$

$$
\begin{aligned}
\dot{v}_2(\tau) &= i\dot{\beta}_1^*(\tau)\beta_2(\tau) + i\beta_1^*(\tau)\dot{\beta}_2(\tau) - i\dot{\beta}_2^*(\tau)\beta_1(\tau) - i\beta_2^*(\tau)\dot{\beta}_1(\tau) \\
&\quad + i\dot{\beta}_3^*(\tau)\beta_4(\tau) + i\beta_3^*(\tau)\dot{\beta}_4(\tau) - i\dot{\beta}_4^*(\tau)\beta_3(\tau) - i\beta_4^*(\tau)\dot{\beta}_3(\tau)
\end{aligned}
$$

$$
\begin{aligned}
\dot{w}_2(\tau) &= -\dot{\beta}_1^*(\tau)\beta_1(\tau) - \dot{\beta}_1(\tau)\beta_1^*(\tau) + \dot{\beta}_2^*(\tau)\beta_2(\tau) + \dot{\beta}_2(\tau)\beta_2^*(\tau) \\
&\quad - \dot{\beta}_3^*(\tau)\beta_3(\tau) - \dot{\beta}_3(\tau)\beta_3^*(\tau) + \dot{\beta}_4^*(\tau)\beta_4(\tau) + \dot{\beta}_4(\tau)\beta_4^*(\tau).
\end{aligned}
$$

For $u_2(\tau)$

$$
\begin{aligned}
\dot{u}_2(\tau) &= \left[-iA\beta_3^*(\tau) - i\frac{1}{2}\beta_4^*(\tau)\right]\beta_1(\tau) + \beta_2^*(\tau)\left[i\frac{1}{2}\beta_3(\tau)\right] + \left[-i\frac{1}{2}\beta_3^*(\tau)\right]\beta_2(\tau) \\
&\quad + \beta_1^*(\tau)\left[iA\beta_3(\tau) + i\frac{1}{2}\beta_4(\tau)\right] + \left[-i\frac{1}{2}\beta_2^*(\tau)\right]\beta_3(\tau) \\
&\quad + \left[-i\frac{1}{2}\beta_1^*(\tau) - iA\beta_2^*(\tau)\right]\beta_4(\tau) + \beta_3^*(\tau)\left[i\frac{1}{2}\beta_2(\tau)\right] \\
&\quad = -iA\beta_3^*(\tau)\beta_1(\tau) + iA\beta_3(\tau)\beta_1^*(\tau) + iA\beta_2(\tau)\beta_4^*(\tau) - iA\beta_2^*(\tau)\beta_4(\tau)
\end{aligned}
$$

For $v_2(\tau)$

$$
\begin{aligned}
\dot{v}_2(\tau) &= i\left[-i\frac{1}{2}\beta_3^*(\tau)\right]\beta_2(\tau) + i\beta_1^*(\tau)\left[iA\beta_3(\tau) + i\frac{1}{2}\beta_4(\tau)\right] \\
&\quad -i\left[-iA\beta_3^*(\tau) - i\frac{1}{2}\beta_4^*(\tau)\right]\beta_1(\tau) - i\beta_2^*(\tau)\left[i\frac{1}{2}\beta_3(\tau)\right] \\
&\quad +i\left[-i\frac{1}{2}\beta_1^*(\tau) - iA\beta_2^*(\tau)\right]\beta_4(\tau) + i\beta_3^*(\tau)\left[i\frac{1}{2}\beta_2(\tau)\right] \\
&\quad -i\left[-i\frac{1}{2}\beta_2^*(\tau)\right]\beta_3(\tau) - i\beta_4^*(\tau)\left[i\frac{1}{2}\beta_1(\tau) + iA\beta_2(\tau)\right] \\
&= -A\beta_3(\tau)\beta_1^*(\tau) - A\beta_3^*(\tau)\beta_1(\tau) + A\beta_2^*(\tau)\beta_4(\tau) + A\beta_4^*(\tau)\beta_2(\tau)
\end{aligned}
$$

And for the inversion

$$
\begin{aligned}
\dot{w}_2(\tau) &= -\left[-i\frac{1}{2}\beta_3^*(\tau)\right]\beta_1(\tau) - \left[i\frac{1}{2}\beta_3(\tau)\right]\beta_1^*(\tau) + \left[-iA\beta_3^*(\tau) - i\frac{1}{2}\beta_4^*(\tau)\right]\beta_2(\tau) \\
&\quad +\left[iA\beta_3(\tau) + i\frac{1}{2}\beta_4(\tau)\right]\beta_2^*(\tau) - \left[-i\frac{1}{2}\beta_1^*(\tau) - iA\beta_2^*(\tau)\right]\beta_3(\tau) \\
&\quad -\left[i\frac{1}{2}\beta_1(\tau) + iA\beta_2(\tau)\right]\beta_3^*(\tau) + \left[-i\frac{1}{2}\beta_2^*(\tau)\right]\beta_4(\tau) + \left[i\frac{1}{2}\beta_2(\tau)\right]\beta_4^*(\tau) \\
&= -iA\beta_3^*(\tau)\beta_2(\tau) + iA\beta_3(\tau)\beta_2^*(\tau) + iA\beta_2^*(\tau)\beta_3(\tau) - iA\beta_2(\tau)\beta_3^*(\tau)
\end{aligned}
$$

And results into

$$
\begin{aligned}
\dot{u}_2(\tau) &= 2A\,\mathrm{Re}\left[i\beta_1^*(\tau)\beta_3(\tau) + i\beta_4^*(\tau)\beta_2(\tau)\right], \\
\dot{v}_2(\tau) &= 2A\,\mathrm{Re}\left[\beta_2^*(\tau)\beta_4(\tau) - \beta_3^*(\tau)\beta_1(\tau)\right], \\
\dot{w}_2(\tau) &= 4A\,\mathrm{Re}\left[i\beta_3(\tau)\beta_2^*(\tau)\right].
\end{aligned}
$$

We can see the time evolution of the isolated system is due to the coupling, when the coupling is null there is no time evolution.

Chapter 4

General Conclusions

We have studied the interaction between a pair of coupled quantum dots, where the coupling is because of the Foerster interaction. Additionally there are interactions with electric fields for each QD. The problem can be studied by using either Schrödinger's or Heisenberg's picture. The use of Heisenberg's picture give a set of six non linear coupled equations which solutions can be found only numerically. In this work we have used the Schrödinger's picture, it allow us to find analytical solutions to the problem of a pair of coupled quantum dots interacting with its own classical electric field.

In the chapter 2, we have developed the semi-classical study of a single quantum dot interacting with a classical electric field. We have used an amplitude probability method based on the Schrödinger's picture and we have shown the way to obtain the optical Bloch equations. Also we have proved that the atomic inversion react back to the initial state after a time such the area pulse is 2π.

The frequency of the exciting field needs to be resonant at:

$$\nu_1 = -\frac{E_0^{gg} - E_0^{eg}}{\hbar} = \varepsilon_1 - 2W$$

This frequency is smaller than the frequency necessary to produce a resonant effect in the two-level atom.

We have shown that in absense of electric field the Bloch's vector associated to the single quantum dot remains constant.

We also have developed a model to describe the coupling between a pair of quantum dots; this analysis has been based on the Schrödinger's picture because Heisenberg's picture gives non-linear set of equations whose analysis seems to be untreatable.

An ideal Foerster interaction between the QD requires that the frequencies of the QD be the same; this fact has been used and has helped to simplify the system of equations to solve but remains as a realistic possibility. Also we have worked under the resonance condition for simplicity. Under those considerations we have found analytical solutions for the slowly varying probability amplitude.

We have supposed, as a particular case, that one of the quantum dots is isolated and the electric field of its cavity is null. This allows us to point out the features in the isolated system due to the coupling and point out the differences in relation to the single quantum dot.

The first observation refers to the atomic inversion, the expected values of σ_1^z and σ_2^z. The dynamics of a single quantum does not indicate time evolution of the inversion; on other hand, we have obtained an analytical expression explicitly demonstrating a time evolution of the inversion of the isolated QD even in absence of electric field. When the coupling is weak, the oscillations of the inversion are fundamentally at the coupling frequency. An interesting fact is that the inversion of the non isolated QD shows oscillations at Rabi frequency, but those oscillations are modulated by the inversion of the isolated system. Therefore, its detection gives an evidence of the coupling and a measuring of its strength.

Also we give analytical expressions showing the resonance fluorescent spectrum. The single quantum dot in absence of electric field does not offer evidence of electric dipole oscillations. The isolated system shows oscillations that can be seen in the three-peaked RFS, it oscillates at exciting electric field and is modulated because of the coupling; notice the exciting field is a feature of the non isolated system. The RFS of the non isolated system is six-peaked distributed in pairs: each pair is shifted from the origin and from the Rabi frequency an amount related with the coupling strength. Again, a measure of the RFS of the non isolated system will give an evidence for the coupling and give a measuring of the strength of the coupling.

Also we calculate the equivalent to the optical Bloch's equations, that is, we found the differential equation for the derivate of the expected values of the Pauli matrices. For the non isolated system, it was possible to recover the Bloch's equation in resonance with additional terms due to the coupling. The isolated system shows a time evolution due exclusively to the coupling. It was no possible to express those additional quantities in terms of the components of the Bloch vector.

As a proof of physical and mathematical consistence, is necessary to mention that all the results demonstrated reduce to the predicted results for the single quantum dot in absence of coupling.

Appendices

Appendix A

Solving the differential equation of probability amplitude coefficients of a single quantum dot

In the study of the single quantum dot interacting with a classical electric field appears the system of differential equations

$$\dot{c}_g(t) = i\frac{\Omega_R(t)}{2}c_e(t) + i\frac{\Delta}{2}c_g(t)$$
$$\dot{c}_e(t) = i\frac{\Omega_R(t)}{2}c_g(t) - i\frac{\Delta}{2}c_e(t)$$

The application of the Laplace transform gives the matrix equation when is considered an electric field with constant amplitude, i.e, $\Omega_R(t) = \Omega_R$

$$A\begin{pmatrix} \tilde{c}_g(s) \\ \tilde{c}_e(s) \end{pmatrix} = \begin{pmatrix} c_g(0) \\ c_e(0) \end{pmatrix}$$

where the A matrix is

$$A = \begin{pmatrix} s - i\Delta/2 & -i\Omega_R/2 \\ -i\Omega_R/2 & s + i\Delta/2 \end{pmatrix}$$

To obtain the inverse matrix A^{-1} we have to calculate the determinant of A

$$\begin{aligned} \det(A) &= \left(s - i\frac{\Delta}{2}\right)\left(s + i\frac{\Delta}{2}\right) - \left(i\frac{\Omega_R}{2}\right)\left(i\frac{\Omega_R}{2}\right) \\ &= s^2 + \frac{\Delta^2}{4} + \frac{\Omega_R^2}{4} \end{aligned}$$

Then A^{-1}

$$A^{-1} = \left(s^2 + \frac{\Delta^2}{4} + \frac{\Omega_R^2}{2}\right)^{-1} \begin{pmatrix} s - i\Delta/2 & i\Omega_R/2 \\ i\Omega_R/2 & s + i\Delta/2 \end{pmatrix}$$

Multiplying on the left side by A^{-1} the equation)

$$\begin{aligned} \begin{pmatrix} \tilde{c}_g(s) \\ \tilde{c}_e(s) \end{pmatrix} &= \left(s^2 + \frac{\Delta^2}{4} + \frac{\Omega_R^2}{2}\right)^{-1} \begin{pmatrix} s - i\Delta/2 & i\Omega_R/2 \\ i\Omega_R/2 & s + i\Delta/2 \end{pmatrix} \begin{pmatrix} c_g(0) \\ c_e(0) \end{pmatrix} \\ &= \begin{pmatrix} s - i\Delta/2 & i\Omega_R/2 \\ i\Omega_R/2 & s + i\Delta/2 \end{pmatrix} \begin{bmatrix} c_g(0)\left(s^2 + \frac{\Delta^2}{4} + \frac{\Omega_R^2}{2}\right)^{-1} \\ c_e(0)\left(s^2 + \frac{\Delta^2}{4} + \frac{\Omega_R^2}{2}\right)^{-1} \end{bmatrix} \end{aligned}$$

In this form is obtained a pair of uncoupled algebraic equations

$$\begin{aligned} \tilde{c}_g(s) = c_g(0)\frac{s + i\frac{\Delta}{2}}{s^2 + \frac{\Delta^2}{4} + \frac{\Omega_R^2}{4}} + c_e(0)\frac{i\frac{\Omega_R}{2}}{s^2 + \frac{\Delta^2}{4} + \frac{\Omega_R^2}{4}} \\ \tilde{c}_e(s) = c_g(0)\frac{i\frac{\Omega_R}{2}}{s^2 + \frac{\Delta^2}{4} + \frac{\Omega_R^2}{4}} + c_e(0)\frac{s - i\frac{\Delta}{2}}{s^2 + \frac{\Delta^2}{4} + \frac{\Omega_R^2}{4}} \end{aligned}$$

They are written in a more suggestive form as

$$\tilde{c}_g(s) = c_g(0)\frac{s}{s^2+\frac{\Delta^2}{4}+\frac{\Omega_R^2}{4}} + \frac{ic_e(0)\frac{\Omega_R}{2}+ic_g(0)\frac{\Delta}{2}}{\sqrt{\frac{\Delta^2}{4}+\frac{\Omega_R^2}{4}}}\frac{\sqrt{\frac{\Delta^2}{4}+\frac{\Omega_R^2}{4}}}{s^2+\frac{\Delta^2}{4}+\frac{\Omega_R^2}{4}}$$

$$\tilde{c}_e(s) = \frac{ic_g(0)\frac{\Omega_R}{2}-ic_e(0)\frac{\Delta}{2}}{\sqrt{\frac{\Delta^2}{4}+\frac{\Omega_R^2}{4}}}\frac{\sqrt{\frac{\Delta^2}{4}+\frac{\Omega_R^2}{4}}}{s^2+\frac{\Delta^2}{4}+\frac{\Omega_R^2}{4}} + c_e(0)\frac{s}{s^2+\frac{\Delta^2}{4}+\frac{\Omega_R^2}{4}}$$

Written in this form, they are easily computed by calculating their inverse Laplace transform

$$c_g(t) = c_g(0)\cos\left(\frac{1}{2}\sqrt{\Delta^2+\Omega_R^2}t\right) + \frac{ic_e(0)\Omega_R+ic_g(0)\Delta}{\sqrt{\Delta^2+\Omega_R^2}}\sin\left(\frac{1}{2}\sqrt{\Delta^2+\Omega_R^2}t\right)$$

$$c_e(t) = \frac{ic_g(0)\Omega_R-ic_e(0)\Delta}{\sqrt{\Delta^2+\Omega_R^2}}\sin\left(\frac{1}{2}\sqrt{\Delta^2+\Omega_R^2}t\right) + c_e(0)\cos\left(\frac{1}{2}\sqrt{\Delta^2+\Omega_R^2}t\right)$$

The quantity $\sqrt{\Delta^2+\Omega_R^2}$ is the off resonance Rabi frequency and it is denoted by $\Omega = \sqrt{\Delta^2+\Omega_R^2}$

Finally, the required solution is

$$c_g(s) = c_g(0)\left[\cos\left(\frac{1}{2}\Omega t\right) + i\frac{\Delta}{\Omega}\sin\left(\frac{1}{2}\Omega t\right)\right] + ic_e(0)\frac{\Omega_R}{\Omega}\sin\left(\frac{1}{2}\Omega t\right)$$

$$c_e(s) = ic_g(0)\frac{\Omega_R}{\Omega}\sin\left(\frac{1}{2}\Omega t\right) + c_e(0)\left[\cos\left(\frac{1}{2}\Omega t\right) - i\frac{\Delta}{\Omega}\sin\left(\frac{1}{2}\Omega t\right)\right]$$

In exact resonance this quantity is denoted as $\Omega_0 = \Omega_R$.

List of Figures

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