ON THE TIME-DEPENDENT SOLUTIONS OF THE SCHRÖDINGER EQUATION

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Abstract Recently, there has been some interest in finding exact solutions to the time-dependent Schrödinger equation, specifically in the case of a time-dependent linear potential, though surprisingly in all those works very cumbersome methods are used. In the present report we want to emphasize that there exists another method, quite general and simple, to solve such kind of problems. The method was proposed several years ago and it is based on the so called Wei–Norman theorem.

Recently, there has been some interest in the solutions of the Schrödinger equation for the time-dependent linear potential [1–6]. Most of the authors use the method of the Lewis–Riesenfeld invariant while Feng [6] used a a space time transformation method. In this work we want to emphasize that there is yet another method, simpler and straightforward, based on the Wei–Norman theorem [7]. A particular version of this method has been used by Rau *et al.* [8] to analyze the same problem and later on applied to the quantum Liouville–Bloch equation [9]. Curiously, in this publication [9], they do not give credit to the work of Wei and Norman although they do in the first one [8]. Our approach is quite different from that of Rau *et al.* since we avoid guessing the solution (ansatz) and, instead, a closed algebra is defined by adding some operators to the original problem. Although this would seem to complicate the problem, it happens to be just the opposite: the problem can be solved straightforward and, from the very beginning, the coefficients of the new operators are set equal to zero, thus leading to the solution we are looking for.

As an example of how this method works, let us consider the equation [8]

$$i\frac{\partial\psi}{\partial t} = \left\{-\frac{1}{2}\ \frac{\partial^2}{\partial x^2} + E_0 x \sin wt\right\}\psi, \tag{1}$$

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which can be written as

$$i\frac{\partial\psi}{\partial t} = \left\{\sum_{i=1}^{4} a_i H_i\right\}\psi \tag{2}$$

where $a_1 = 0,\ H_1 = 1;\ \ a_2 = E_0 \sin wt,\ H_2 = x;\ \ a_3 = 0,\ H_3 = \frac{\partial}{\partial x};\ \ a_4 = -\frac{1}{2},\ H_4 = \frac{\partial^2}{\partial x^2}$. It is very simple to show that $\pounds = \{H_1, H_2, H_3, H_4\}$ is a solvable Lie Algebra [7] since $\pounds'' = \{0\}$, so that the problem can be solved by quadratures.

Actually, solving the above equation is equivalent to do so for the evolution operator

$$i\frac{\partial U}{\partial t} = \dot{g}_1 H_1 U + \dot{g}_2 H_2 U + \dot{g}_3 U H_3 + \dot{g}_4 U H_4, \tag{3}$$

where $\Psi(x,t) = U(t)\Psi(x,0) = e^{g_1 H_1} e^{g_2 H_2} e^{g_3 H_3} e^{g_4 H_4}\varphi(x)$ and the upper dots denote differentiation with respect to t, and $\Psi(x,0) = \varphi(x)$ is the solution of the time independent Schrödinger equation.

Application of some well–known operator algebra techniques, leads to a set of four linear equations:

$$i\dot{g}_4 = -\frac{1}{2} \tag{4a}$$

$$i\dot{g}_2 = E_0 \sin wt \tag{4b}$$

$$\dot{g}_3 - 2\dot{g}_4 g_2 = 0 \tag{4c}$$

$$\dot{g}_1 - \dot{g}_3 g_2 + \dot{g}_4 g_2^2 = 0 \tag{4d}$$

which can be easily integrated to give the desired solution, that is, the one reported by Rau and Unnikrishnan [8]. The case with $E = E_0 \cos wt$ can be treated in a similar fashion, and also those analyzed in references [1–6]. The advantage of using properly the Wei–Norman method is that we can know in advance whether the problem is soluble or not, as it is shown to be the case here, because it corresponds to a solvable Lie algebra.

Another important feature of the Wei–Norman method is that the solution is global [10], i.e. it is valid in the whole domain of variable t, restricted only to time-dependent equations where solutions are of the exponential type.

Let us consider a more general equation, the one which is called the reduced velocity gauge or the Airy–Gordon–Volkov wave equation [11].

$$i\,\frac{\partial\psi}{\partial t} = \left\{-\frac{1}{2m}\,\frac{\partial^2}{\partial x^2} + i\,\frac{A_0 \cos wt}{m}\,\frac{\partial}{\partial x} + V - Fx\right\}\psi, \tag{5}$$

which can be transformed to an equivalent one for the evolution operator, as we did in the previous case:

$$\frac{dU}{dt} = \left\{\frac{i}{2m}\,\frac{\partial^2}{\partial x^2} + \frac{A_0 \cos wt}{m}\,\frac{\partial}{\partial x} + iFx - iV\right\} U(t) = H(t)U(t). \tag{6}$$

Introducing now the definitions

(7a) $$a_1(t) = \frac{i}{2m}$$

(7b) $$a_2(t) = \frac{A_0 \cos wt}{m}$$

(7c) $$a_3(t) = iF$$

(7d) $$a_4(t) = iV$$

(7e) $$H_1 = \frac{\partial^2}{\partial x^2}$$

(7f) $$H_2 = \frac{\partial}{\partial x}$$

(7g) $$H_3 = x$$

(7h) $$H_4 = I$$

This set of four operators forms a solvable Lie algebra, as we pointed out above, and the proposed Eq. (5) must have an elementary solution. In order to find it, we propose again:

(8) $$\Psi(x_1 t) = U(t)\Psi(x, 0) = e^{g_1 H_1} e^{g_2 H_2} e^{g_3 H_3} e^{g_4 H_4} \varphi(x)$$

Repeating the procedure outlined above we obtain a set of four *linear* differential equations:

(9a) $$\dot{g}_1 = a_1$$

(9b) $$\dot{g}_2 + 2\dot{g}_3 g_1 = a_2$$

(9c) $$\dot{g}_3 = a_3$$

(9d) $$\dot{g}_4 + \dot{g}_3 g_2 = a_4$$

which can be easily integrated thus obtaining:

(10a) $$g_1(t) = \frac{it}{2m}$$

(10b) $$g_2(t) = \frac{A_0 \sin wt}{mw} + \frac{F}{2m} t^2$$

(10c) $$g_3(t) = iFt$$

(10d) $$g_4(t) = iVt + i\frac{A_0 F \cos wt}{mw^2} - i\frac{F^2}{6m} t^3 - i\frac{A_0 F}{mw^2}$$

It is important to point out that this solution for the evolution operator $U(t)$ is not unique, since it depends on the order in which we arrange the elements of the corresponding Lie algebra, we can obtain a unique solution by using the BCH formula [12]:

$$\Psi(x,t) = e^{-iEt} e^{i\frac{FA_0 \cos wt}{mw^2}} A_i \left[-(2mF)^{1/3} \left(x + \frac{A_0 \sin wt}{mw} + \frac{E-V}{F} \right) \right] \tag{11}$$

where A_i is the Airy function.

There are several other similar cases, called length gauge, velocity gauge, Kramers–Henneberger frame, which are particular cases of Eq. (2) with appropriate coefficient $a_i^{\prime}s$. All of them are related to the same Lie algebra but with different Hamiltonians and their solution has been reported elsewhere [13].

We have shown in this work the power and elegance of Lie algebraic methods in the solution of differential equations, which, when used properly, lead easily to the desired solution. Application of this method to the case of the Caldirola–Kanai Hamiltonian is in progress and will be published elsewhere.

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