Unifying distribution functions: some lesser known distributions

J. R. Moya-Cessa,[1] H. Moya-Cessa,[2,*] L. R. Berriel-Valdos,[2]
O. Aguilar-Loreto,[2] and P. Barberis-Blostein[2]

[1]Centro de Investigaciones en Optica, A.C., Apartado Postal 1-948, León, Gto., Mexico

[2]Instituto Nacional de Astrofísica, Optica, y Electronica, Coordinación de Optica, Apdo. Postal 51 y 216, 72000 Puebla, Pue., Mexico

*Corresponding author: hmmc@inaoep.mx

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We show that there is a way to unify distribution functions that describe simultaneously a classical signal in space and (spatial) frequency and position and momentum for a quantum system. Probably the most well known of them is the Wigner distribution function. We show how to unify functions of the Cohen class, Rihaczek's complex energy function, and Husimi and Glauber–Sudarshan distribution functions. We do this by showing how they may be obtained from ordered forms of creation and annihilation operators and by obtaining them in terms of expectation values in different eigenbases. © 2008 Optical Society of America

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1. Introduction

Distribution functions are widely used in optical physics (see [1] for a review) and in quantum mechanics, where they are usually called quasi-probability distribution functions [2–4]). Probably the best known quasi-probability distribution is the Wigner function [2,5], with applications in reconstruction of signals [6], image processing [7], and resolution [8] in the classical world and reconstruction of quantum states of different systems such as ions [9] or quantized fields [4,10–12] in the quantum world. In this contribution we would like to reintroduce a lesser known quasi-probability distribution function, namely the Kirkwood–Rihaczek function [1,13–16], show how it can be related to the Wigner function, and express it as an expectation value in some eigenbasis, just as the other quasi-probability functions may be also expressed.

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2. Best known quasi-probability distribution functions

A. Wigner function

We start by introducing the Wigner function, probably the best known. It may be written in two forms: series representation (see for instance [17]) and integral representation:

$$W(q,p) = \frac{1}{2\pi}\int \mathrm{d}u e^{iup}\langle q+\frac{u}{2}|\rho|q-\frac{u}{2}\rangle. \qquad (1)$$

For simplicity we use the Dirac notation here (see Appendix A). In Eq. (1), ρ is the so-called density matrix. In 1932, Wigner introduced this function $W(q,p)$, which is known now as his distribution function [2,5] and contains complete information about the state of the system ($\rho = |\psi\rangle\langle\psi|$).

It may also be written in terms of the (double) Fourier transform of the characteristic function:

$$W(\alpha) = \frac{1}{4\pi^2}\int \mathrm{d}^2\beta \exp(\alpha\beta^* - \alpha^*\beta)C(\beta), \qquad (2)$$

with $\alpha = (q + ip)/\sqrt{2}$ and $\beta = (u + iv)/\sqrt{2}$, being $\mathrm{d}^2\beta = \mathrm{d}u\mathrm{d}v$. $C(\beta)$ in terms of annihilation and creation operators,

$$a = \equiv \frac{\hat{q} + i\hat{p}}{\sqrt{2}} = \frac{q + \frac{\mathrm{d}}{\mathrm{d}q}}{\sqrt{2}} \qquad a^\dagger \equiv \frac{\hat{q} - i\hat{p}}{\sqrt{2}} = \frac{q - \frac{\mathrm{d}}{\mathrm{d}q}}{\sqrt{2}}, \quad (3)$$

is given by

$$C(\beta) = \mathrm{Tr}\{\rho \exp(\beta a^\dagger - \beta^* a)\}, \quad (4)$$

also known as the ambiguity function in classical optics [18,19].

B. Q-function

The Q or Husimi function [20]is expressed as the coherent state expectation value of the density operator,

$$Q(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta \exp(\alpha\beta^* - \alpha^*\beta)\mathrm{Tr}\{\rho \exp(-\beta^* a)\exp(\beta a^\dagger)\}, \quad (5)$$

and the alternative form is

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle. \quad (6)$$

C. Relating Q and Wigner functions in a differential form

It is possible to group the Wigner and the Husimi functions,

$$F(\alpha, s) = \frac{1}{\pi}\int \mathrm{d}^2\beta C(\beta, s)\exp(\alpha\beta^* - \alpha^*\beta), \quad (7)$$

where $C(\beta, s)$ is the characteristic function of order s,

$$C(\beta, s) = \mathrm{Tr}\{D(\beta)\rho\}\exp(s|\beta|^2/2), \quad (8)$$

with s a parameter that defines which function we want to obtain. For $s = 1$ it is obtained as the P function, for $s = 0$ the Wigner function, and for $s = -1$ the Q function.

The Q function is then

$$Q(\alpha) = \int \mathrm{d}^2\beta G(\beta)\exp(\alpha\beta^* - \alpha^*\beta), \quad (9)$$

and for $s = 0$ the Wigner function

$$W(\alpha) = \int \mathrm{d}^2\beta G(\beta)\exp(\alpha\beta^* - \alpha^*\beta)\exp(|\beta|^2/2), \quad (10)$$

where

$$G(\beta) = \frac{1}{\pi^2}\mathrm{Tr}\{D(\beta)\rho\}\exp(-|\beta|^2/2), \quad (11)$$

with $D(\beta) = e^{\beta a^\dagger - \beta^* a}$ the so-called Glauber displacement operator. The equation for $D(\beta)$ above may be written as an infinite (Taylor) series and inserted into Eq. (10) to obtain

$$W(\alpha) = \sum_{n=0}^{\infty}\frac{2^{-n}}{n!}\int \mathrm{d}^2\beta G(\beta)\exp(\alpha\beta^* - \alpha^*\beta)|\beta|^{2n}. \quad (12)$$

Considering the equality

$$\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\exp(\alpha\beta^* - \alpha^*\beta) = -|\beta|^2\exp(\alpha\beta^* - \alpha^*\beta), \quad (13)$$

we can insert Eq. (12) into

$$W(\alpha) = \sum_{n=0}^{\infty}\frac{2^{-n}}{n!}\left(-\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right)^n Q(\alpha), \quad (14)$$

or, finally [17]

$$W(\alpha) = \exp\left(-\frac{1}{2}\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right)Q(\alpha). \quad (15)$$

The analysis just done will help us to relate the Kirkwood–Rihaczek function with the Wigner function.

D. Glauber–Sudarshan

In order to complete the set of distribution functions most used in quantum mechanics, we introduce the well-known Glauber–Sudarshan P function [21,22]. First let us note that by using the coherent state eigenbasis, we can express the density matrix as the following double integral:

$$\rho = \frac{1}{\pi^2}\int\int \mathrm{d}^2\alpha \mathrm{d}^2\beta \langle\alpha|\rho|\beta\rangle|\alpha\rangle\langle\beta|. \quad (16)$$

This representation involves off-diagonal elements $\langle\alpha|\rho|\beta\rangle$and two integrations in phase space. The next diagonal representation was introduced independently by Glauber and Sudarshan [21,22],

$$\rho = \int \mathrm{d}^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|, \quad (17)$$

and involves only one integration. Using the equation above, we can write the Glauber–Sudarshan P function in the form

$$P(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta \exp(\alpha\beta^* - \alpha^*\beta)\mathrm{Tr}\{\rho \exp(\beta a^\dagger)\exp(-\beta^* a)\}, \quad (18)$$

or

$$P(\alpha) = F(\alpha, 1) = \frac{1}{\pi}\int \mathrm{d}^2\beta C(\beta, 1)\exp(\alpha\beta^* - \alpha^*\beta). \quad (19)$$

There are integral and differential relations between the three main quasi-probabilities used in quantum mechanics. In the next section we will show differential relations between them and a lesser known distribution function.

E. Cohen-class distribution functions

A function of the Cohen class is described by the general formula [1,23]

$$W_C(x,u)=\frac{1}{2\pi}\int\int\int \mathrm{d}y\mathrm{d}x'\mathrm{d}u'\phi\left(y+\frac{1}{2}x'\right)\phi^* \times\left(y-\frac{1}{2}x'\right)k(y,u,x',u')e^{-i(ux'-u'x+u'y)}, \tag{20}$$

and the choice of the kernel $k(y,u,x',u')$ selects one particular function of the Cohen class. The Wigner function, for instance, arises for $k(y,u,x',u')=1$, whereas the ambiguity function is obtained for $k(y,u,x',u')=2\pi\delta(y-x')\delta(u-u')$.

3. Lesser known distribution function: the Kirkwood–Rihaczek quasi-distribution function

Now we turn our attention to a lesser known distribution, the Kirkwood–Rihaczek function, which may be written using the notation above as [24]

$$K(\beta)=\int \mathrm{d}^2\alpha e^{\beta\alpha^*-\beta^*\alpha}e^{\frac{\alpha^2-\alpha^{*2}}{4}}C(\alpha). \tag{21}$$

This equation has been obtained from an equation similar to Eqs. (2), (5), and (18), i.e., taking the double Fourier transform

$$K(q,p)=\int \mathrm{d}u\mathrm{d}ve^{-iup}e^{ivq}\mathrm{Tr}\{\rho e^{iv\hat{q}}e^{iu\hat{p}}\} \tag{22}$$

and taking the trace as in Eq.(A18) in Appendix A.

We will now do an analysis similar to the one done in Subsection 2.D. We relate the Kirkwood–Rihaczek function to the Wigner function by using Eq. (21), via the following exponential of derivatives:

$$K(\beta)=e^{-\frac{1}{4}\frac{\partial^2}{\partial^2\beta}}e^{\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}}W(\beta). \tag{23}$$

We now use the nonintegral expression for the Wigner function [17]:

$$W(\beta)=\mathrm{Tr}[(-1)^{\hat{n}}D^\dagger(\beta)\rho D(\beta)], \tag{24}$$

with $\hat{n}=a^\dagger a$, the so-called number operator. We cast the above equation into the form

$$W(\beta)=\mathrm{Tr}[(-1)^{\hat{n}}\rho D(2\beta)], \tag{25}$$

where we have used the trace property $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$ and the identities $(-1)^{\hat{n}}D^\dagger(\beta)=D(\beta)(-1)^{\hat{n}}$.

Now we use the factorized form of the Glauber displacement operator [25], $D(2\beta)=e^{-2|\beta|^2}e^{2\beta a^\dagger}e^{-2\beta^* a}$, to obtain

$$W(\beta)=\mathrm{Tr}[(-1)^{\hat{n}}\rho e^{-2|\beta|^2}e^{2\beta a^\dagger}e^{-2\beta^* a}]. \tag{26}$$

Therefore, the Kirkwood–Rihaczek function may be written as

$$\begin{aligned}K(\beta,\beta^*)&=e^{-\frac{1}{4}\frac{\partial^2}{\partial^2\beta}}e^{\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}}W(\beta,\beta^*)\\&=\mathrm{Tr}[(-1)^{\hat{n}}\rho e^{-\frac{1}{4}\frac{\partial^2}{\partial^2\beta}}e^{\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}}D(2\beta)].\end{aligned} \tag{27}$$

The calculation of the exponential of derivatives of the Glauber operator will be tedious but straightforward. We will just write the main steps to obtain the final form; for instance, it is not difficult to show that

$$e^{\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}}D(2\beta)=e^{-\beta^2}e^{2\beta(a^\dagger+a-\beta^*)}e^{a^2}e^{-2\beta^* a}. \tag{28}$$

By using [26]

$$e^{-t^2+2tx}=\sum_{k=0}^{\infty}H_k(x)\frac{t^k}{k!}, \tag{29}$$

we can express the above equation as

$$e^{\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}}D(2\beta)=\sum_{k=0}^{\infty}H_k(a^\dagger+a-\beta^*)\frac{\beta^k}{k!}e^{a^2}e^{-2\beta^* a}, \tag{30}$$

with $H_k(x)$ as the Hermite polynomials. From the above equation, is easy to obtain

$$\frac{\partial^{2n}}{\partial\beta^{2n}}\sum_{k=0}^{\infty}H_k(x)\frac{\beta^k}{k!}=\sum_{k=0}^{\infty}H_{k+2n}(x)\frac{\beta^k}{k!}, \tag{31}$$

and therefore

$$e^{-\frac{1}{4}\frac{\partial^2}{\partial^2\beta}}e^{\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}}D(2\beta)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\frac{1}{4})^n}{n!}H_{k+2n}(a^\dagger+a-\beta^*)\frac{(\beta)^k}{k!}e^{a^2}e^{-2\beta^* a}. \tag{32}$$

Now we use the integral form of the Hermite polynomials [26],

$$H_p(x)=\frac{2^p}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}t(x+it)^p e^{-t^2}, \tag{33}$$

to obtain

$$K(\beta,\beta^*)=\frac{e^{-\beta^{*2}}e^{-2\beta\beta^*}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}te^{(-2\sqrt{2}x+2\beta^*+2\beta)it}e^{-2x^2}e^{2\sqrt{2}x(\beta^*+\beta)}\langle x|e^{a^2}e^{-2\beta^* a}(-1)^{\hat{n}}\rho|x\rangle \tag{34}$$

by using

$$\int_{-\infty}^{\infty} \mathrm{d}t e^{-ikt} = 2\pi\delta(k) \tag{35}$$

and taking $k = 2\sqrt{2}x - 2\beta^* - 2\beta$, we have

$$K(\beta,\beta^*) = 2\sqrt{\pi}e^{-\beta^{*2}}e^{-2\beta\beta^*}\int_{-\infty}^{\infty} \mathrm{d}x\delta \times \left(2\sqrt{2}x - 2\beta^* - 2\beta\right)e^{-2x^2}e^{2\sqrt{2}x(\beta^*+\beta)},$$

$$\times \langle x|e^{a^2}e^{-2\beta^* a}(-1)^{\hat{n}}\rho|x\rangle. \tag{36}$$

Making use of the identity $\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$ we finally obtain

$$K(\beta,\beta^*) = e^{-2\beta^{*2}}e^{-2\beta\beta^*} \sqrt{\frac{\pi}{2}}e^{2\left(\frac{\beta^*+\beta}{\sqrt{2}}\right)^2}\left\langle\frac{\beta^*+\beta}{\sqrt{2}}\middle|e^{(a-\beta^*)^2}(-1)^{\hat{n}}\rho\middle|\frac{\beta^*+\beta}{\sqrt{2}}\right\rangle, \tag{37}$$

or

$$K(\beta,\beta^*) = \sqrt{\frac{\pi}{2}}e^{\beta^2-\beta^{*2}}\langle X|D^\dagger(-\beta^*)e^{a^2}D(-\beta^*)(-1)^{\hat{n}}\rho|X\rangle, \tag{38}$$

with $X = \frac{\beta^*+\beta}{\sqrt{2}}$.

In this form, we have succeeded in obtaining the Kirkwood–Rihaczek function as an expectation value in terms of position eigenstates, just as we did the Q function in terms of coherent states [see Eq. (5)] and the Wigner and Glauber–Sudarshan functions in terms of number states [17].

4. Conclusions

We have shown that some distribution functions may be related through a method that allows the construction of some quasi-probability functions such as the Wigner, Glauber–Sudarshan, and Husimi functions [27]. This method consists of obtaining the distribution functions from a double Fourier transform of an averaged exponential operator. If we use the exponential operator in terms of creation and annihilation operators we construct the already mentioned distribution functions. Using this method, but leaving the exponential operator in terms of position and momentum and ordering (factorizing) the exponential in a convenient way, another lesser function used in classical optics, namely, Kirkwood–Rihaczek's distribution function, may be obtained. This function was recently introduced in quantum mechanics by Praxmeyer and Wódkiewicz [15,16] to have a phase representation of the hydrogen atom. The connection between Glauber–Sudarshan and Husimi functions and functions of the Cohen class has been given, i.e., the adequate kernels. Finally, the Kirkwood–Rihaczek function has been given in terms of an expectation value and in terms of position eigenstates, just as other distribution functions may also be given in terms of (sums of) expectation values. This may be of interest because it has been already exploited the fact that these forms allow reconstruction of quasi-probability distribution functions [11,12].

Appendix A

In Dirac notation, we denote functions "f" by means of "kets" $|f\rangle$. For instance an eigenfunction of the harmonic oscillator [26],

$$\psi_n(x) = \frac{\pi^{-1/4}}{\sqrt{2^n n!}}e^{-x^2/2}H_n(x), \tag{A1}$$

is represented by the ket $|n\rangle$, with $n = 0, 1, 2, \ldots$ In quantum mechanics, these states are called number or Fock states (see for instance [3]). Any function can be expanded in terms of eigenfunctions of the harmonic oscillator:

$$f(x) = \sum_{n=0}^{\infty} c_n\psi_n(x), \tag{A2}$$

where

$$c_n = \int_{-\infty}^{\infty} \mathrm{d}x f(x)\psi_n(x), \tag{A3}$$

and in the same way any ket may be expanded in terms of $|n\rangle$:

$$f\rangle = \sum_{n=0}^{\infty} c_n|n\rangle, \tag{A4}$$

where the orthonormalization relation

$$\langle m|n\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\psi_m^*(x)\psi_n(x) = \delta_{nm} \tag{A5}$$

has been used. The quantity $\langle m|$ is a so-called "bra."

The basis set of kets $|n\rangle$ is a discrete one. However, there are also continuous bases. We can form one continuous basis for example with the function $e^{ipq}/\sqrt{2\pi}$ and the corresponding ket $|p\rangle$. First note that

$$\langle p|p'\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x e^{-i(p-p')q} = \delta(p-p'), \tag{A6}$$

so that

$$e^{ip'q}/\sqrt{2\pi} = \int_{-\infty}^{\infty} \mathrm{d}p\delta(p-p')e^{ipq}/\sqrt{2\pi}, \tag{A7}$$

or, in bra-ket notation we have

$$|p'\rangle = \int_{-\infty}^{\infty} \mathrm{d}p \delta(p - p')|p\rangle = \int_{-\infty}^{\infty} \mathrm{d}p \langle p|p'\rangle |p\rangle, \quad \text{(A8)}$$

rearranging terms we have

$$|p'\rangle = \left(\int_{-\infty}^{\infty} \mathrm{d}p |p\rangle\langle p|\right)|p'\rangle = \mathbf{1}|p'\rangle, \quad \text{(A9)}$$

i.e., we have the completeness relation

$$\int_{-\infty}^{\infty} \mathrm{d}p |p\rangle\langle p| = \mathbf{1}. \quad \text{(A10)}$$

Finally note that the function $e^{ipq}/\sqrt{2\pi}$ is an eigenfunction of the operator $-i\frac{d}{dq}$ with eigenvalue p. For position, an "eigenket" of $\hat{q}$ is

$$\hat{q}|q\rangle = q|q\rangle \quad \text{(A11)}$$

and an "eigenbra"

$$\langle q'|q' = \langle q'|\hat{q}. \quad \text{(A12)}$$

We therefore find

$$\langle q'|q\rangle(q' - q) = 0, \quad \text{(A13)}$$

which has as a solution [3]

$$\langle q'|q\rangle = \delta(q' - q). \quad \text{(A14)}$$

We then can express the completeness relation

$$\mathbf{1} = \int_{-\infty}^{\infty} \mathrm{d}q |q\rangle\langle q|, \quad \text{(A15)}$$

such that

$$|\Psi\rangle = \mathbf{1}|\Psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}q |q\rangle\langle q||\Psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}q \Psi(q)|q\rangle, \quad \text{(A16)}$$

where $\Psi(q) = \langle q||\Psi\rangle = \langle q|\Psi\rangle$. The density matrix ρ is defined simply as the ket-bra operator $\rho = |\Psi\rangle\langle\Psi|$.

The completeness relation serve us among other things to calculate averages, for instance

$$\begin{aligned}\langle\Psi|\hat{A}|\Psi\rangle &= \langle\Psi|\hat{A}\mathbf{1}|\Psi\rangle = \langle\Psi|\hat{A}\int_{-\infty}^{\infty} \mathrm{d}q |q\rangle\langle q||\Psi\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}q \langle\Psi|\hat{A}|q\rangle\langle q|\Psi\rangle, \quad \text{(A17)}\end{aligned}$$

or finally,

$$\langle\Psi|\hat{A}|\Psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}q \langle q|\Psi\rangle\langle\Psi|\hat{A}|q\rangle. \quad \text{(A18)}$$

Note that in the above equation we are simply adding "diagonal" elements, i.e., we have the trace of the operator $|\Psi\rangle\langle\Psi|\hat{A}$. As the trace is independent of the basis, we can have it in terms of the discrete basis $|n\rangle$:

$$\langle\Psi|\hat{A}|\Psi\rangle = \sum_{n=0}^{\infty}\langle n|\Psi\rangle\langle\Psi|\hat{A}|n\rangle \equiv \mathrm{Tr}\{\rho\hat{A}\}. \quad \text{(A19)}$$

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