

# Optical realization of quantum-mechanical invariants

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We study the problem of paraxial propagation in two grade-index media by using invariant techniques that allow a continuous solution of the problem. By using the well-known fact that this problem is analogous to the time-dependent harmonic oscillator in quantum mechanics, known methods there may be imported producing on the one hand a solution to the propagation problem and on the other hand a realization of a quantum-mechanical invariant. © 2009 Optical Society of America

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It is well known that there exists a correspondence between phenomena in quantum mechanics and paraxial wave optics [1,2]. This allows us to interchange the use of the extensively studied methods of solution for the wave equation in quantum mechanics as well as to look for optical phenomena analogous to those in quantum-mechanical systems.

In the past decade the rapid evolution of the field of photonics has motivated a renewed interest in materials with a gradient index owing to their demonstrated and potential applications in optical communications, medical optics, micro-optics, and optical computing, to mention a few [3,4]. This yields the necessity of having the methods to obtain the solutions that describe the physical behavior of light propagating in such materials. If the mathematical methods used come from quantum mechanics, this also allows us to find a classical system that might emulate a quantum system, and deeper insights of these systems can be gained.

Periodic piecewise media are common in the study of pulse propagation in optical dispersion management [5]. Here we will assume a more complex medium that will be composed of two sections, each having a transverse parabolic gradient index but two different strength parameters. In this study the properties of the Lewis–Ermakov invariant [6], are employed to investigate the solutions of the Schrödinger wave equation for the composed grade-index (GRIN) medium. This system can be engineered using modern material techniques [3]. Although the term *diffraction management* has been used mainly for waveguide arrays [7], strictly speaking the proposed system, which also modifies the diffraction characteristics of an input beam, also delivers diffraction management.

There exist classical optical systems related to the waveguiding of optical beams that can be described by the paraxial wave equation, which has an analogous form to the Schrödinger equation, namely,

$$2ik_0 \frac{\partial E}{\partial z} = \nabla_{\perp}^2 E + k^2(x, y, z)E, \quad (1)$$

where  $k_0 = \pi n_0 / \lambda$  is the wavenumber for the propagating light beam of wavelength  $\lambda$  and  $n_0$  is the homogeneous refractive index. The function  $k^2(x, y, z)$  describes the inhomogeneity of the medium responsible for the waveguiding of the optical field  $E$ . A medium like this is, for example, a GRIN fiber, a GRIN lens, or a laser active medium [4,8,9]. In Fig. 1, we show two GRIN media with the quadratic profile in contact. We study here how propagation takes place in such a system when it is illuminated, at the plane  $z=0$ , by an arbitrary optical field (boundary condition). The important issue here is to establish the appropriate conditions in order to solve this equation. The obvious choice would be to consider the incident field amplitude and its derivative at a given plane. However, such a proposal is inadequate, since at any plane where the incident wave exists there is also a contribution from the reflected wave, which is so far unknown. This assertion is true even far away from the region, where the refractive index varies considerably, since we are dealing with infinite wave trains.

An alternative is to establish the conditions for the refracted wave, which is simply the transmitted wave. The required assumption is that far from the interface region the transmitted wave is constant and that in this region there is no reflected wave.

The function  $k^2(x, y, z)$  may be written as



Fig. 1. (Color online) Arbitrary field created at  $z=0$  and propagated through a first GRIN medium,  $z \leq L$  (left side), and a second GRIN medium,  $z > L$  (right side).

$$k^2(x,y,z) = \left\{ \begin{array}{l} k_1^2 = \beta_1^2 + \frac{\nu_1^2}{2}x^2 + \frac{\mu_1^2}{2}y^2, \quad 0 < z < L \\ k_2^2 = \beta_2^2 + \frac{\nu_2^2}{2}x^2 + \frac{\mu_2^2}{2}y^2 \quad z \geq L \end{array} \right\}, \quad (2)$$

where  $\mu_i(z), \nu_i(z), \beta_i(z)$ , with  $i=1,2$ , are piecewise continuous real valued functions. Renormalizing the  $z$  coordinate in Eq. (1) with respect to  $k_0$ , we may write

$$i \frac{\partial E}{\partial z} = \left( -\frac{p_x^2 + \nu^2(z)x^2}{2} - \frac{p_y^2 + \mu^2(z)y^2}{2} + \beta(z) \right) E, \quad (3)$$

where we have defined a canonical momentum  $p_q = -i\partial/\partial q$ , with  $q=x,y$  and

$$\alpha(z) = \left\{ \begin{array}{l} \alpha_1 \quad z < L \\ \alpha_2 \quad z \geq L \end{array} \right\}, \quad (4)$$

with  $\alpha(z) = \mu(z), \nu(z), \beta(z)$ . The functions  $\alpha(z)$  represent then simple step functions and are plotted in Fig. 2. With a transformation of the form

$$E = e^{-i\int^z \beta(z') dz'} \epsilon, \quad (5)$$

we get rid of the function  $\beta(z)$  in Eq. (3), obtaining

$$i \frac{\partial \epsilon}{\partial z} = \left( -\frac{p_x^2 + \nu^2(z)x^2}{2} - \frac{p_y^2 + \mu^2(z)y^2}{2} \right) \epsilon. \quad (6)$$

Equation (6) may represent a pair of ( $z$ -dependent) quantum harmonic oscillators [9,10].

It is well known that the Hamiltonian of the time-dependent harmonic oscillator [6],

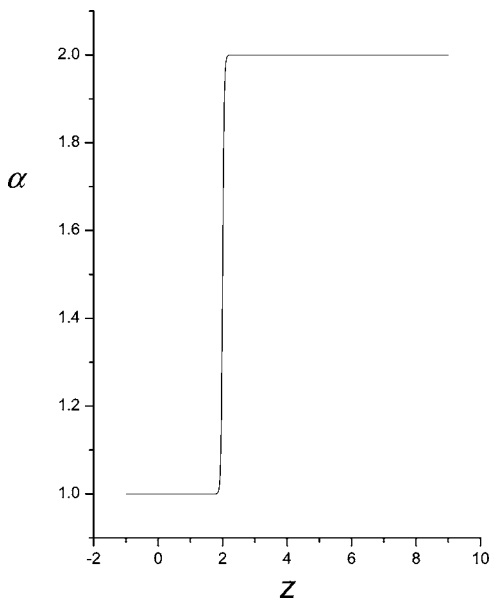


Fig. 2. Example of a step function for the functions  $\nu(z), \mu(z)$ , or  $\beta(z)$ .

$$\hat{H}(t) = \frac{1}{2}(\hat{p}_q^2 + \Omega^2(t)\hat{q}^2), \quad (7)$$

has the so-called Lewis–Ermakov invariant

$$\hat{I} = \frac{1}{2} \left[ \left( \frac{\hat{q}}{\rho_q} \right)^2 + (\rho_q \hat{p} - \dot{\rho}_q \hat{q})^2 \right], \quad (8)$$

with  $\rho_q$  an auxiliary function that obeys the Ermakov equation

$$\ddot{\rho}_q + \Omega^2(t)\rho_q = \rho_q^{-3}. \quad (9)$$

The solution for this equation for a step function is plotted ( $t \rightarrow z$ ) in Fig. 3. Making use of some properties of the invariant, namely, that it may be transformed into an operator that does not depend explicitly on time by means of a Bogoliubov (squeezing) and a displacement transformation [10,11], one can solve the Schrödinger equation for the time-dependent Hamiltonian in a straightforward manner.

Now, we look at the field propagation through the two GRIN media in contact. Consider the transformation

$$T_q = S_q D_q, \quad q = x,y, \quad (10)$$

where

$$S_q = e^{i(qp_q + p_q q) \frac{\ln \rho_q}{2}}, \quad (11)$$

$$D_q = \exp \left\{ -i \frac{q^2}{2\rho_q} \frac{d\rho_q}{dz} \right\}. \quad (12)$$

By doing  $\epsilon = T_x T_y \psi$  we can readily solve Eq. (6) subject to the boundary condition  $\epsilon(z=0) = \epsilon_1(x)\epsilon_2(y)$  as

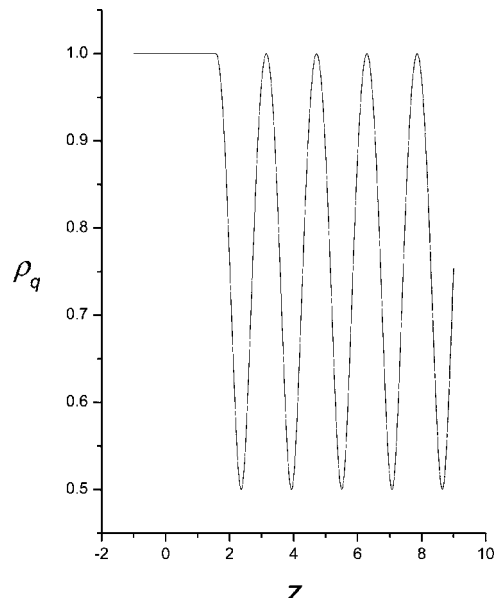


Fig. 3. Solution of the Ermakov equation for the function of Fig. 2.

$$\begin{aligned} \epsilon(x,y,z) = & T_x e^{i\int^z dz' / \rho_x^2(z') (N_x + 1/2)} \\ & \times \epsilon_1(x) T_y e^{i\int^z dz' / \rho_y^2(z') (N_y + 1/2)} \epsilon_2(y), \end{aligned} \quad (13)$$

with  $N_q = a_q^\dagger a_q$  the so-called number operator, with  $a_q = q + ip_q / \sqrt{2}$  and  $a_q^\dagger = q - ip_q / \sqrt{2}$ , the so-called annihilation and creation operators, respectively. The functions  $\epsilon_j(q)$ ,  $j=1,2$  may always be developed in Hermite functions,

$$u_n(q) = \frac{1}{2^n n!} H_n(q) e^{-q^2/2}, \quad (14)$$

that is

$$\epsilon_j(q) = \sum_{n=0}^{\infty} C_n^{(j)} u_n(q), \quad (15)$$

making it relatively easy to apply the solution in Eq. (13) to them, because the functions  $u_n(q)$  are eigenfunctions of the number operator,  $N_q u_n(q) = n u_n(q)$ .

In conclusion, we have presented an operational method to solve a problem of propagation that uses an invariant method introduced by Lewis [6]. We would like to stress that this approach is continuous, in the sense that it does not require one solution before the interface between the GRIN pieces and another one after it, with the subsequent matching of these solutions. It is worth noting that Lewis–Ermakov methods are also used in trap ions, where we have systems with time-dependent frequency [12].

The method presented here is not restricted to step functions of  $z$  but to arbitrary functions. The purpose

of this work was twofold: to show how to use quantum-optics methods to solve classical optics propagation problems and to create a classical system to emulate a quantum one, showing the potential for studying quantum optics with classical systems. In this context we rephrase Feynman [13], pointing out that there is always the hope that this point of view will inspire more and better optical simulations of quantum systems.

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