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Application of Rouche's theorem for MP filter design

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ABSTRACT

The method for the minimum-phase (MP) finite impulse response (FIR) filter design, based on Rouche's theorem from complex analysis is presented here. The filter is designed directly from a given specification. The method uses the cosine filters and the sharpening technique resulting in a multiplierless filter.

1. Introduction

There are some applications where high delay introduced by linear-phase FIR filters is not permitted like in communication data systems. If the application at hand does not require a linear-phase characteristic it is possible to obtain much lower delay by minimum-phase filter (MP) design which preserves desired magnitude response.

The minimum-phase requirement restricts the resulting filter to have all its zeros on or inside the unit circle.

There have been proposed different methods to obtain minimum-phase filters starting from a linear-phase filter and methods based on complex cepstrum [1–6]. The review of the methods can be found in [3].

Unlike the many known methods we propose here the design of minimum-phase filters which is multiplier-free. Besides the minimum-phase filter is designed directly from the given specification. Method is based on cascaded expanded cosine filters and sharpening method. The paper is organized as follows. Next section presents the cascaded expanded cosine filters followed by sharpening technique. Section 4 introduces minimum-phase filters based on the application of Rouche's theorem and is illustrated with one example.

2. Cosine-based linear-phase filters

The simplest low-pass finite impulse response (FIR) filter is the M-point moving-average (MA) filter, also known as the comb filter, with an impulse response

$$h_{comb}(n) = \begin{cases} 1/M, & \text{for } 0 \leqslant n \leqslant M-1, \\ 0, & \text{otherwise}. \end{cases}, \tag{1}$$

Its system function is given by

$$h_{comb}(z) = \frac{1}{M}(1 + z^{-1} + \cdots + z^{-(M-1)}) = \frac{1}{M}\sum_{k=0}^{M-1} z^{-k} = \frac{1}{M}\frac{1 - z^{-M}}{1 - z^{-1}}. \tag{2}$$

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The scaling factor $1/M$ is needed to provide a dc gain of 0 dB. This filter does not require any multiplications or coefficient storages.

The system function of the comb filter where M is a complete power of 2, 2^p, can be expressed as

$$H_{comb}(z) = (1 + z^{-1})(1 + z^{-2}) \cdots (1 + z^{-2^{p-1}})/M, \tag{3}$$

where M is an integer.

For $M = 2$ from (2) we have

$$H_2(z) = \frac{1}{2}(1 + z^{-1}). \tag{4}$$

The corresponding magnitude response is

$$|H_2(e^{j\omega})| = |\cos(\omega/2)|. \tag{5}$$

Because of this cosine form this filter is called a cosine filter.

The N-expanded filters are obtained by inserting $N - 1$ zeros between each sample of the impulse response. In z domain, each delay is consequently replaced by N delays

$$H_2(z^N) = \frac{1}{2}(1 + z^{-N}). \tag{6}$$

The magnitude response of this filter is

$$|H_2(e^{j\omega N})| = |\cos(N\omega/2)|. \tag{7}$$

The system function of the cascade of K cosine expanded filters is given by

$$H_{\cos}(z) = \prod_{k=1}^{K} H_2(z^k) = \prod_{k=1}^{K} \frac{1}{2}(1 + z^{-k}) \tag{8}$$

has all zeros on the unit circle i.e. it is a minimum-phase filter.

The corresponding magnitude response is then

$$|H_{\cos}(e^{j\omega})| = \left|\prod_{k=1}^{K} H_2(e^{j\omega k})\right| = \left|\prod_{k=1}^{K} \cos(k\omega/2)\right|. \tag{9}$$

The cascade of different expanded cosine filters results in one low-pass magnitude characteristic as shown in Fig. 1a, for $K = 5$. To improve the attenuation we can cascade L filters (8) as demonstrated in Fig. 1b for $L = 3$. Fig. 1c presents pole-zero plot to verify that the filter (8) is a minimum-phase filter.

However, the magnitude characteristic of filter (8) has an high passband droop and a low attenuation

To further improve the magnitude characteristic we use the sharpening technique briefly revised in the next section.

3. Sharpening technique

To improve the magnitude response characteristic of the filter (8) we propose to use the sharpening technique which can be used for simultaneous improvements of both the passband and stopband characteristics of a linear-phase FIR digital filter [7]. The technique uses the amplitude change function (ACF) which is a polynomial relationship of the form $H_{sh} = f(H)$ between the amplitudes of the overall and the prototype filters, H_{sh} and H, respectively.

We consider here simple sharpening polynomial [7]

$$H_{sh}(z) = 3H^2(z) - 2H^3(z). \tag{10}$$

We use the cascade of L filters (8) as the filter H in sharpening polynomial (10).

The resulted sharpened filter is a linear-phase filter $H_{LP}(z)$

$$H_{LP}(z) = H_{\cos}^{2L}(z)\{3z^{-(N-1)L/2} - 2H_{\cos}^{L}(z)\} = H_{\cos}^{2L}(z)B_{LP}(z), \tag{11}$$

where N is the length of the cascaded cosine filters $H_{\cos}^{L}(z)$, and

$$B_{LP}(z) = 3z^{-(N-1)L/2} - 2H_{\cos}^{L}(z). \tag{12}$$

As a result of the introduced delay $z^{-(N-1)L/2}$, the factor $B_{LP}(z)$ has a linear-phase. The magnitude characteristic of the sharpened filter from Fig. 1b is shown in Fig. 2.

In order to obtain a minimum-phase filter we propose to modify (12) as follows:

$$H_{\mathrm{MP}}(z) = H_{\cos}^{2L}(z)\{3 - 2H_{\cos}^{L}(z)\} = H_{\cos}^{2L}(z)B_{\mathrm{MP}}(z), \tag{13}$$

$$B_{\mathrm{MP}}(z) = 3 - 2H_{\cos}^{L}(z). \tag{14}$$

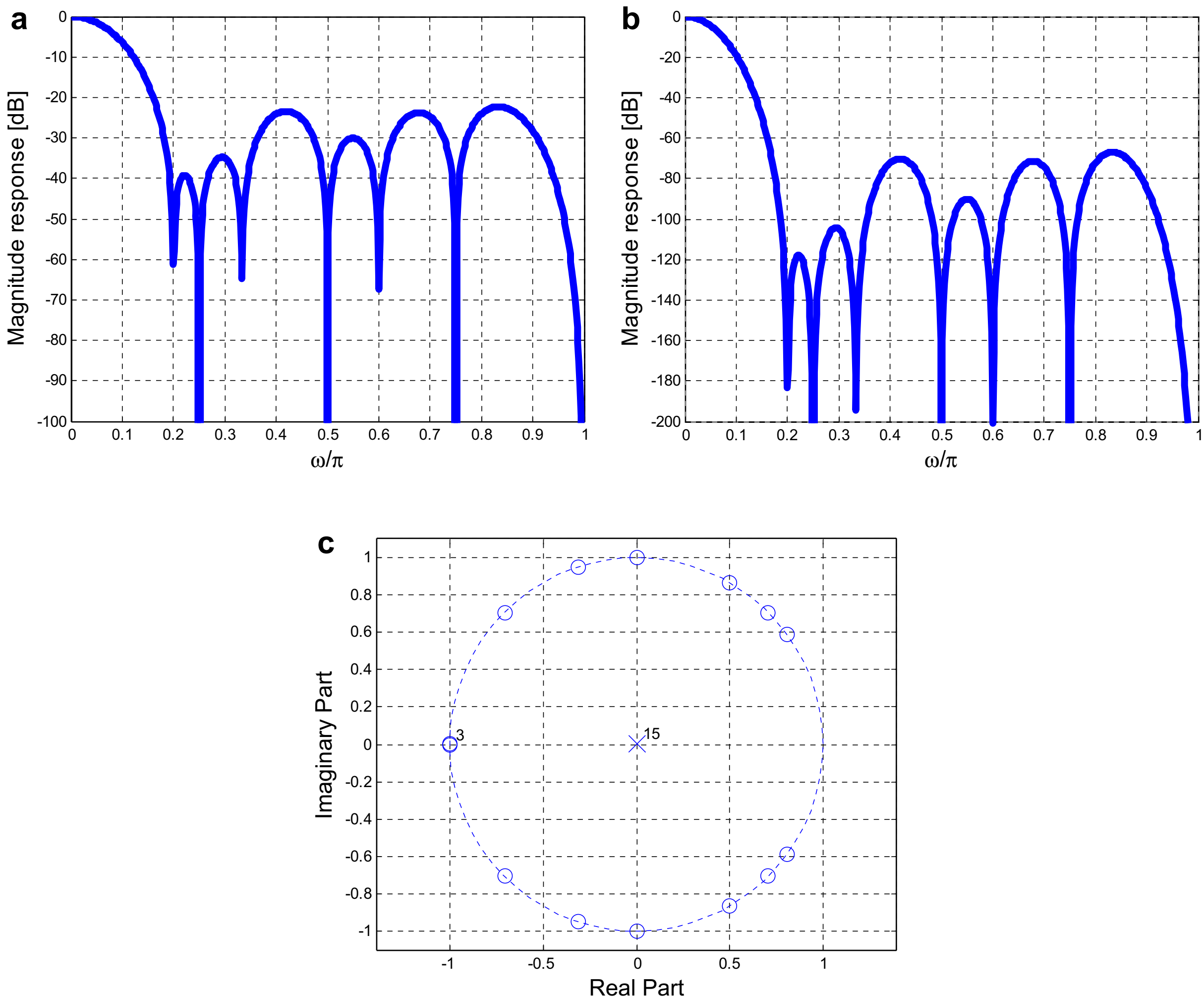

Fig. 1. Cascade of $K = 5$ cosine filters. (a) $L = 1$, (b) $L = 3$, (c) z-plane.

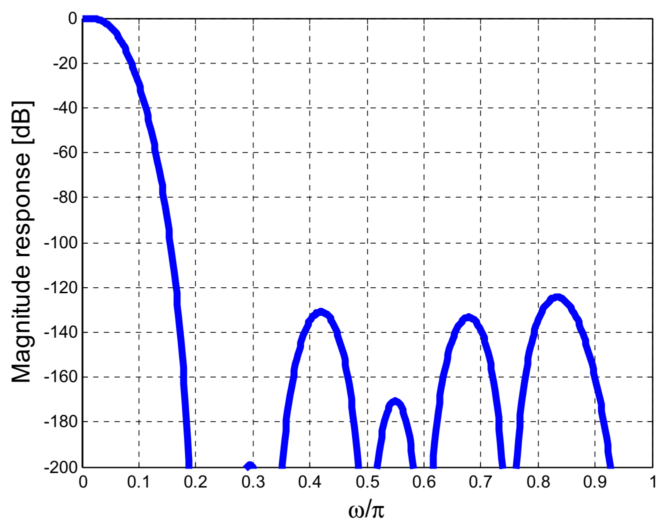

Fig. 2. Sharpened filter.

In the following we make use of the Rouche's theorem [8] to verify that the polynomial $B_{MP}(z)$ has all zeros inside the unit circle i.e. it is a minimum-phase filter.

4. Rouche's theorem

From (8) and (14) we have

$$B_{MP}(z) = 3 - 2\frac{1}{2^{KL}}(1+z^{-1})^L(1+z^{-2})^L \cdots (1+z^{-K})^L. \tag{15}$$

When $N-1$ can be expressed as a power-of-2 integer, Eq. (15) can be rewritten as

$$B_{MP}(z) = \frac{1}{2^{KL}}\left[3\cdot 2^{KL} - 2\left[\sum_{i=0}^{N-1} b_i z^{-i}\right]^L\right], \tag{16}$$

where [9]

$$N-1 = 1+2+3+\cdots+K, \tag{17}$$

which can be expressed as [9]

$$N = \frac{K(K+1)}{2} + 1. \tag{18}$$

The polynomial $\sum_{i=0}^{N-1} b_i z^{-i}$ has symmetrical integer coefficients b_i, and $b_i \geqslant 1$.
After some computation we arrive at

$$B_{MP}(z) = \frac{1}{2^{KL}}\left[3\cdot 2^{KL} z^{(N-1)L} - 2\cdot\left[\sum_{i=0}^{N-1} b_i z^{i}\right]^L\right]. \tag{19}$$

Let

$$B_{MP}(z) = \frac{1}{2^{KL}}P(z); \quad P(z) = f(z) + g(z), \tag{20}$$

where

$$f(z) = 3\cdot 2^{KL} z^{(N-1)L} \tag{21}$$

and

$$g(z) = -2\cdot(b_0 + b_1 z + \cdots + b_1 z^{N-2} + b_0 z^{N-1})^L. \tag{22}$$

The absolute values of $f(z)$ and $g(z)$ are

$$|f(z)| = 3\cdot 2^{KL}|z^{(N-1)L}|, \tag{23}$$

$$|g(z)| < 2\cdot|(1+z+\cdots+z^{N-1})^L| \leqslant 2\cdot(1+|z|+|z^2|+\cdots+|z^{N-1}|)^L. \tag{24}$$

On the unit circle $|z| = 1$, and it follows:

$$|f(z)|_{|z|=1} = 3\cdot 2^{KL}; \quad |g(z)|_{|z|=1} \leqslant 2\cdot N^L. \tag{25}$$

We write [9]

$$2^K = (1+1)^K = 1 + K + K(K-1)/2! + K(K-1)(K-2)/3! + \cdots + 1 > 1 + K + K(K-1)/2 = 1 + K(K+1)/2. \tag{26}$$

From (18) and (26) it follows:

$$2^K > N. \tag{27}$$

Consequently for z on the unit circle, it follows from (25) and (27) that

$$|f(z)| > |g(z)|. \tag{28}$$

The polynomial $f(z)$ has $(N-1)L$ zeros at the origin, i.e. inside the unit circle (see (21)). According to the Rouche's theorem, the polynomial $P(z)$ also has the same number of zeros inside the unit circle. Consequently, the polynomial $B_{MP}(z)$ is the minimum-phase filter, as the filter $H_{MP}(z)$ given in (13).

As an example consider cosine filters for $K = 4$ and $L = 2$

$$H^2_{\cos}(z) = \left[\frac{1}{2^4}(1+z^{-1})(1+z^{-2})(1+z^{-3})(1+z^{-4})\right]^2. \tag{29}$$

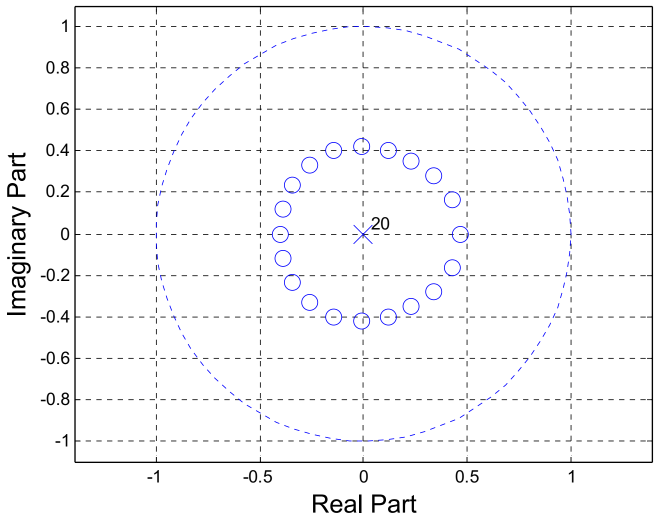

Fig. 3. Pole-zero plot for $B_{\mathrm{MP}}(z)$, $K = 4$, $L = 2$.

The minimum-phase factor $B_{\mathrm{MP}}(z)$ of the sharpening polynomial is

$$B_{\mathrm{MP}}(z) = 3 - 2\frac{1}{2^8}[(1+z^{-1})^2(1+z^{-2})(1+z^{-3})(1+z^{-3})]^2. \tag{30}$$

From (18), $N = 11$ and the number of zeros inside the unit circle is $(N-1)L = 20$ as shown in the z-plane plot in Fig. 3.

Example 1. As an illustration of the proposed method consider a MP filter with the following specification:

$$\omega_p = 0.002, \quad \omega_s = 0.15, \quad R_p = 0.0001\ \mathrm{dB}, \quad A_s = -100\ \mathrm{dB}, \tag{31}$$

where ω_p and ω_s are normalized passband and stopband frequencies, respectively, and R_p is the passband ripple, while A_s is the stopband attenuation.

We choose

$$K = \lceil 1/\omega_s \rceil = 7, \tag{32}$$

where $\lceil x \rceil$ is the ceiling of x (the smallest integer greater than or equal to x).

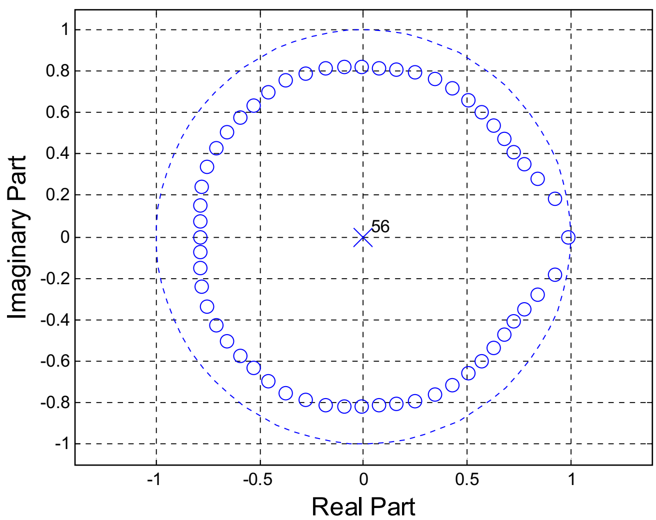

Fig. 4. Pole-zero plot for $B_{\mathrm{MP}}(z)$ in Example 1.

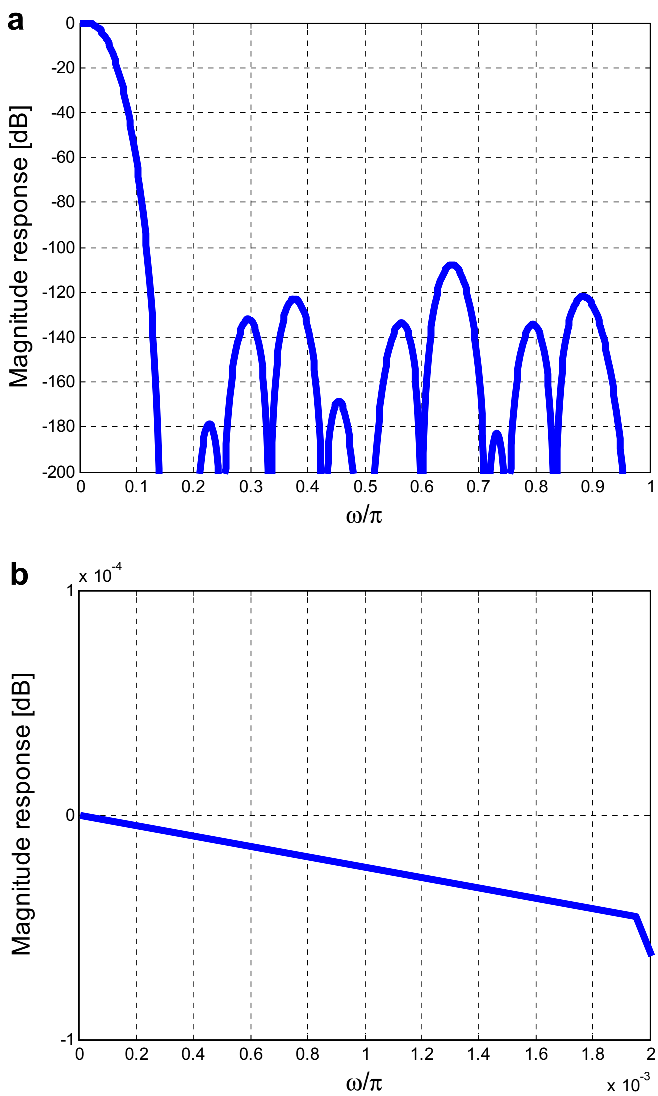

Fig. 5. Example 1. (a) Magnitude response of the designed MP filter and (b) passband zoom.

According to (3), the cascaded cosine filters for $K = 7$ and $L = 2$ can be written in the form

$$H^2_{\cos}(z) = \left[\frac{1}{2^7}\frac{1-z^{-8}}{1-z^{-1}}(1+z^{-3})(1-z^{-5})(1-z^{-6})(1-z^{-7})\right]^2. \tag{33}$$

Finally, the designed minimum-phase filter is

$$H_{\mathrm{MP}}(z) = H^4_{\cos}(z)\{3 - 2H^2_{\cos}(z)\} = H^4_{\cos}(z)B_{\mathrm{MP}}(z). \tag{34}$$

The polynomial $H^4_{\cos}(z)$ has all zeros at unit circle. As demonstrated in this paper the polynomial $B_{\mathrm{MP}}(z)$ has all zeros inside the unit circle as shown in Fig. 4.

The magnitude response and the passband zoom given in Fig. 5a and b, respectively, demonstrate that the specification is satisfied.

The interested reader may find details about the practical implementation of the filters containing polynomial terms of the form $(1 + z^{-n})$ and $(1 - z^{-n})$ in [10].

5. Concluding remarks

A method for a direct minimum-phase low-pass multiplier-free filter design has been presented. It can be effectively employed in the design of very narrow passband filters with small passband ripples and high stopband attenuation. The method is based on the cascade of expanded cosine filters and the sharpening technique. Rouche's theorem is crucial to transform the sharpened linear-phase cosine filters into a minimum-phase filters.

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