On the Relation between the Number of Scrolls and the Lyapunov Exponents in PWL-functions-based η-Scroll Chaotic Oscillators

Article *in* International Journal of Nonlinear Sciences and Numerical Simulation · November 2010
DOI: 10.1515/IJNSNS.2010.11.11.903

5 authors, including:

Esteban Tlelo-Cuautle
Instituto Nacional de Astrofísica, Óptica y Electrónica (INAOE)
275 PUBLICATIONS **2,347** CITATIONS
SEE PROFILE

J.M. Munoz-Pacheco
Benemérita Universidad Autónoma de Puebla
83 PUBLICATIONS **615** CITATIONS
SEE PROFILE

Carlos Sanchez-Lopez
Autonomous University of Tlaxcala, Mexico
107 PUBLICATIONS **1,266** CITATIONS
SEE PROFILE

©Freund Publishing House Ltd., International Journal of Nonlinear Sciences & Numerical Simulation 11(11): 903-910, 2010

On the Relation between the Number of Scrolls and the Lyapunov Exponents in PWL-functions-based n-Scroll Chaotic Oscillators

R. Trejo-Guerra[1], E. Tlelo-Cuautle[1], J.M. Muñoz-Pacheco[2], C. Sánchez-López[3], C. Cruz-Hernández[4]

[1] *INAOE, Department of Electronics, Luis Enrique Erro No. 1, Tonantzintla, Puebla. 72840, MEXICO*
[2] *UPP, Department of Electronics and Telecomunications, Puebla, 72640, MEXICO*
[3] *UAT, Department of Electronics Engineering, Apizaco, Tlaxcala, 70300 MEXICO*
[4] *CICESE, Department of Electronics and Telecommunications, Km. 107, Carretera Tijuana-Ensenada, 22860 Ensenada, Baja California, MEXICO; ccruz@cicese.mx*

Abstract

A challenge in the physical design of n-scroll attractors is to generate a large number of scrolls. However, an open question is: Does the large number of scrolls determine a better chaotic behavior? We present a partial answer to this question by computing the Lyapunov exponents for piecewise-linear (PWL) functions based on n-scroll third order chaotic oscillators, namely: Zhong modified Chua's circuit, Yu modified Chua's circuit, saturated function series based chaotic oscillator, and generalized Chua's circuit with saw-tooth function. Three different integration methods (forward Euler, fourth order Runge Kutta and *Matlab*© ODE 45) are used in order to show that the positive Lyapunov exponent is not significantly incremented in relation with the number of scrolls.

Keywords: Lyapunov exponent, Multiscroll oscillators, scroll number, chaos

1. Introduction*

Chaotic oscillators have been investigated to generate n-scroll attractors [1]. Some of them can be modeled by piecewise-linear (PWL) approach, so that the nonlinear problem can be transformed to a linear one. Four known third order PWL-function-based n-scroll attractors are: Zhong modified Chua's circuit [2], Yu modified Chua's circuit [3], saturated function series based n-scroll chaotic oscillator [4], and generalized Chua's circuit with saw-tooth function [5]. These oscillators have already been realized by using electronic devices [1]-[5], and the challenge has been oriented to generate a large number of scrolls. Also, as an application, they have been already proposed in the secure communications field [6,7]. However, an interesting question not considered by the research community is: Does the large number of scrolls determine a better chaotic behavior? The answer to this question may involve different perspectives for future research. Here, we present a partial answer focused on the calculation of the Lyapunov exponents in PWL-function-based n-scroll attractors.

Since control of chaotic systems is still subject of research [8]-[10], some approaches to compute the Lyapunov exponents have been introduced in [11]-[15]. In chaotic oscillators, there is at least one positive Lyapunov exponent, showing not only that the dynamical

* **Corresponding Authors:**
[1]rodolfo13@inaoep.mx, e.tlelo@ieee.org; rodotg13@gmail.com; +52-222-2470517; [2]jpacheco@uppuebla.edu.mx; [3]carlsan@ieee.org; [4]ccruz@cicese.mx

system is unpredictable [13], but also they are used to determine the asymptotical stability [14]. Furthermore, from all the Lyapunov exponents, the bigger exponent determines the predictability of dynamical systems and almost often is considered as a good metric to guarantee chaotic behavior. For instance, the Lyapunov exponents have been calculated in the double scroll chaotic oscillator [16], Colpitts oscillator [17], modified Lorenz system [18], hysteretic system [19], MOSFET-based electronic circuit inducing chaos [20], Jerk function based circuit [21], and inductorless Chua's circuit [22].

In this paper we will show some relations between the number of scrolls and the value of the positive Lyapunov exponent in PWL-function-based attractors by numerical computations. That way, the descriptions of four PWL-function-based n-scroll chaotic oscillators are given in Section 2. In Section 3, we review an approach to compute Lyapunov exponents. In Section 4 we show the relation between the number of scrolls and the value of the Lyapunov exponents for the four PWL-function-based chaotic oscillators. Finally, the conclusions are given in Section 5.

2. PWL-function-based multiscroll chaotic oscillators

The chaotic oscillators described herein are third order systems with PWL-based functions. Some characteristic issues are the mathematical tractability, since partial solutions can be obtained; and the easy change in the attractor shape, due to the system dependence on the PWL-function.

a. Zhong modified Chua's circuit

Chua's circuit has been realized with different kinds of electronics devices [1], a modification on the nonlinear function is given by (1), and leads us to generate up to 10-scroll attractors [2]. The values of the coefficients are given in section 4, in order to compute the Lyapunov exponents.

$$\begin{aligned} \dot{x}_1 &= \alpha\left(x_2 - x_1 - g(x_1)\right), \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -\beta x_2, \\ g(x_1) &= m_{2n-1}x_1 + \frac{1}{2}\sum_{i-q}^{2n-1}(m_{i-1} - m_i)\left(|x_1 + b_i| - |x_1 - b_i|\right) \end{aligned} \tag{1}$$

b. Yu modified Chua's circuit

Based on (1), a later modification of Chua's parameters system [3] allows to calculate breakpoints thus leading to generate n-scroll attractors in an iteratively manner. The used expression (2) requires only the first break-point of the PWL function and calculates the rest. Two slopes are subsequently repeated. The values of the coefficients are given in Section 4 to compute the Lyapunov exponents.

$$b_{i+1} = \frac{2\sum_{j}^{i}(m_j - m_{j-1})b_j}{1 + m_i} - b_i \tag{2}$$

c. Saturated function series based n-scroll chaotic oscillator

Another interesting generator of n-scroll attractors is the one based on saturated function series. It has been realized with some electronic devices [4],[23], and it can be extended to generate multidimensional n-scroll attractors [24]. This oscillator is described by (3), and the values of the coefficients are given in section 4 to compute the Lyapunov exponents.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -ax_1 - ax_2 - ax_3 + af(x_1; k, h, p, q)$$

$$f(x;k,h,p,q) = \begin{cases} (2q+1)k, & \text{if} \quad x > qh+1 \\ k(x-ih)+2ik, & \text{if} \quad x - ih \le 1, \quad -p \le i \le q \\ (2i+1)k, & \text{if} \quad ih+1 < x < (i+1)h-1 \\ & \qquad -p \le i \le q-1 \\ -(2p+1)k, & \text{if} \quad x < -ph-1. \end{cases} \tag{3}$$

d. Generalized Chua's circuit with saw-tooth function

Chua's circuit can be generalized in different ways: by a nonlinear function with positive and negative slopes [1], by adding more break-points to Chua's diode [2],[3], by a saw-tooth function [5] or other non PWL approaches. For the last case of concern, the generalized Chua's circuit is described by (4),

where $f_1(x_1)$ is given by (5) to generate even scrolls, and by (6) to generate odd scrolls [5]. The values of the coefficients are given in Section 4 to compute the Lyapunov exponents.

$$\dot{x}_1 = \alpha(x_2 - f_1(x_1)),$$
$$\dot{x}_2 = x_1 - x_2 + x_3,$$
$$\dot{x}_3 = -\beta x_2, \tag{4}$$

$$f_1(x_1) = \xi\left\{x_1 - A_1\left[-\operatorname{sgn}(x_1) + \sum_{i=0}^{N-1}\left(\operatorname{sgn}(x_1 + 2iA_1) + \operatorname{sgn}(x_1 - 2iA_1)\right)\right]\right\} \tag{5}$$

$$f_1(x_1) = \xi\left\{x_1 - A_1\left[\sum_{i=0}^{N-1}\left(\operatorname{sgn}(x_1 + (2i+1)A_1) + \operatorname{sgn}(x_1 - (2i+1)A_1)\right)\right]\right\} \tag{6}$$

3. Calculation of Lyapunov exponents

The Lyapunov exponents can be computed by applying the methods given in [11]-[15]. They provide information on the average exponential rate of divergence or convergence of trajectories which are very close in phase space. The number of Lyapunov exponents equals the number of state variables in a dynamical system, and if at least one is positive, this is an indication of chaotic behavior. In this section we introduce the used method to calculate the Lyapunov exponents in third-order PWL-function-based n-scrolls attractors. To guarantee the validity of the results, the integration process is performed by applying three different numerical integration methods, namely: forward Euler (FE), fourth order Runge Kutta (RK4) and Matlab ODE 45. The method is summarized as follows [25]:

1. Initial conditions of the system and the variational system are set to $\boldsymbol{X_0}$ and $\mathbf{I}_{\mathbf{nxn}}$, respectively.
2. The systems are integrated by several steps until the orthonormalization period *TO* is reached. The integration of the variational system **Y**=*[y₁, y₂, y₃]* depends on the specific Jacobian that the original system ***X*** is using in the current step.

3. The variational system is orthonormalized by using the standard Gram-Schmidt method [26], the logarithm of the norm of each Lyapunov vector contained in **Y** is obtained and accumulated in time.
4. The next integration is carried out by using the new orthonormalized vectors as initial conditions. This process is repeated until the full integration period T is reached.
5. The Lyapunov exponents are obtained by using:

$$\lambda_i :\approx \frac{1}{T}\sum_{j=TO}^{T} \ln \|\mathbf{y}_i\| .$$

Time step selection was made by using the minimum absolute value of all the eigenvalues of the system λ_{min} [27], and Ψ was chosen well above the sample theorem as 50.

$$t_{step} = \frac{1}{\lambda_{min}\Psi} .$$

We choose to give an orthogonalization period TO of about 95 t_{step}, which is a third of the period corresponding to the maximum central frequency of the chaotic system. The integration was carried in about 500,000 steps, thus having a full period T of 10,000s for the saturated series system, 3200s for the Yu Chua's system and 4,000s for the rest. Initial conditions were $X_0 = [0.1, 0, 0]$ in all cases, where the ergodicity property implies that λ_i is unique as $T \to \infty$.

4. Relation between the number of scrolls and the Lyapunov exponent

The calculation of the Lyapunov exponents for the Zhong modified Chua's circuit [2] described by (1), is performed by setting: $\alpha=10$, $[m_0...m_9]=[-4.416, -0.276, -3.036, -0.276, -3.036, -0.276, -3.036, -0.276, -3.036, -0.276]$, $[b_1...b_9]=[0.1, 1.1, 1.55, 3.2, 3.85, 5.84, 6.6, 8.7, 9.45]$ to generate even scrolls, and $[b_2...b_9]=[0.8, 1.4, 3.2, 3.9, 5.8, 6.4, 8.3, 9.2]$ to generate odd scrolls. However, some different values of constant β from 15 to 10.3 were required to really generate the n-scroll attractors with the chosen integration step. The selection of n depends on the number of scrolls, $2n$ for even numbers or $2n-1$ for odd number. These parameters allowed generating up to ten scrolls. The results of the positive Lyapunov exponent are shown in Table 1, the maximum second exponent was $\lambda_2 \approx 0.008$ for the FE process, $\lambda_2 \approx 0.003$ for the RK4 process and $\lambda_2 \approx 0.002$ for the ODE45 process. The negative exponent $\lambda_3 \in [-7.994$ to $-6.895]$ show a decrease in magnitude as λ_1 increases.

Lyapunov exponents where calculated for the same system [3] under the parameters $\alpha=9.47$, $\beta=14$, 13, 10 for 2, 3-10 and 11-15 scroll attractors, respectively; $m_0=m_2=...=m_{14}$, $m_1=m_3=...=m_{13}$ with $m_0 = -1.41$, $m_1 = -0.528$ for odd n-scrolls, and $m_0 = -0.528$, $m_1 = -1.41$ for even n-scrolls; thus having $[b_1...b_{14}] = [0.2, 0.665, 1.07, 1.53, 1.93, 2.4, 2.8, 3.26, 3.66, 4.13, 4.53, 4.99, 5.39, 5.86]$ for odd n-scrolls and $[b_1...b_{13}] = [0.2, 0.544, 0.944, 1.29, 1.69, 2.03, 2.43, 2.78, 3.18, 3.52, 3.92, 4.26, 4.66]$ for even n-scrolls. The results of the positive Lyapunov exponent are shown in Table 1 the maximum magnitude of the second exponent was $\lambda_2 \approx 0.005$ for the FE process, $\lambda_2 \approx 0.008$ for the RK4 process, and $\lambda_2 \approx 0.001$ for the ODE45 process. The negative exponent $\lambda_3 \in [-4.91$ to $-3.99]$ show an irregular decreasing magnitude as λ_1 increases.

For the saturated function series based chaotic oscillator described by (2), the calculation is performed by setting: $a=0.7$, $k=10$, $h=2$; p and q are adjusted to generate several scrolls. The results of the positive Lyapunov exponent are shown in Table 1. The maximum second exponent was $\lambda_2 \approx 0.01$ for the FE process, $\lambda_2 \approx 0.0008$ for the RK4 process as well as the ODE45 process. The negative exponent $\lambda_3 \in [-0.86$ to $-0.78]$ show an increasing magnitude as λ_1 increases.

For the generalized Chua's circuit with saw-tooth function described by (3), the calculation is performed by setting: $\alpha=10$, $\beta=16$, $\xi=0.25$, $A_1=1$. It is worthy to mention that for this oscillator, the Jacobian is always the same for any number of scrolls ($2N$ or $2N+1$), so that the Lyapunov exponent is

Table 1
Calculation of the positive Lyapunov exponent

Scrolls	Zhong Modified Chua			Yu Modified Chua		
	FE	*RK4*	*Matlab ODE45*	*FE*	*RK4*	*Matlab ODE45*
2	0.40884	0.23720	0.24144	0.43009	0.41390	0.41119
3	0.49882	0.36999	0.37958	0.46237	0.40714	0.41378
4	0.50620	0.41298	0.41808	0.47426	0.42061	0.40292
5	0.52163	0.41002	0.41051	0.48414	0.39354	0.40152
6	0.55911	0.44154	0.43760	0.41400	0.38767	0.40267
7	0.57303	0.44652	0.45613	0.45197	0.39254	0.38952
8	0.63320	0.49336	0.51609	0.48472	0.41582	0.42209
9	0.64162	0.53176	0.51772	0.47069	0.39259	0.40155
10	0.68270	0.52967	0.54934	0.48958	0.65408	0.66798
11				0.70011	0.66261	0.66900
12				0.68698	0.65842	0.68653
13				0.69638	0.66850	0.66162
14				0.68121	0.66582	0.67615
15				0.70895	0.67435	0.68519

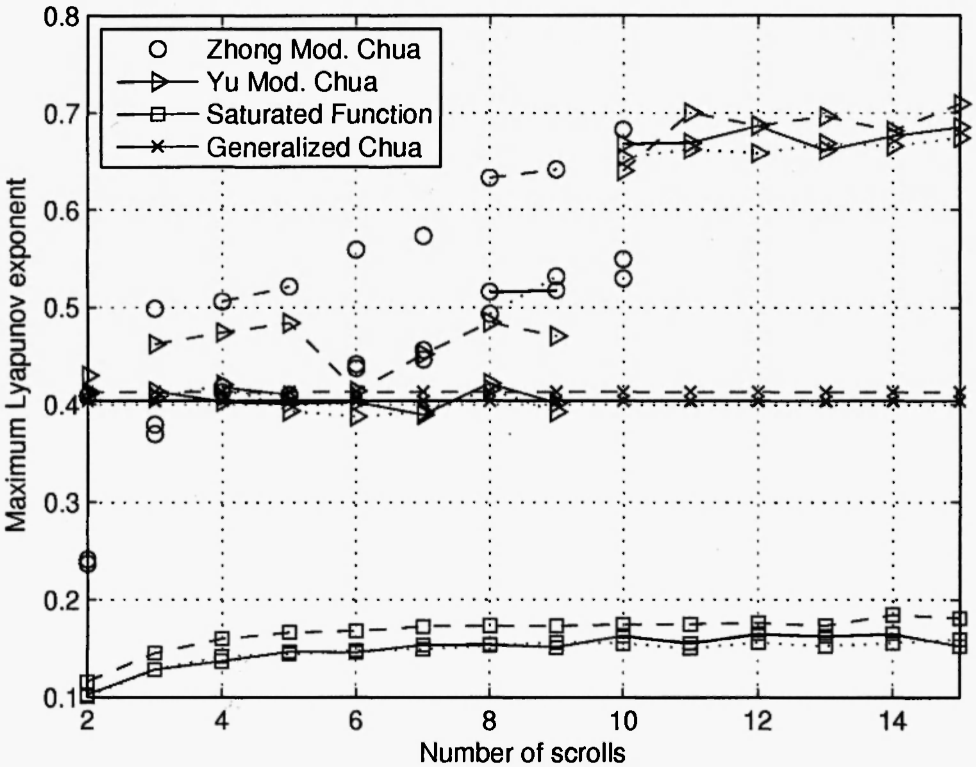

Fig. 1: Comparison of the maximum Lyapunov exponent vs number of scrolls of the four chaotic oscillators by applying ODE45 integration (solid line), RK4 integration (dotted line) and FE integration (dashed line). Note that this is a discrete-valued plot.

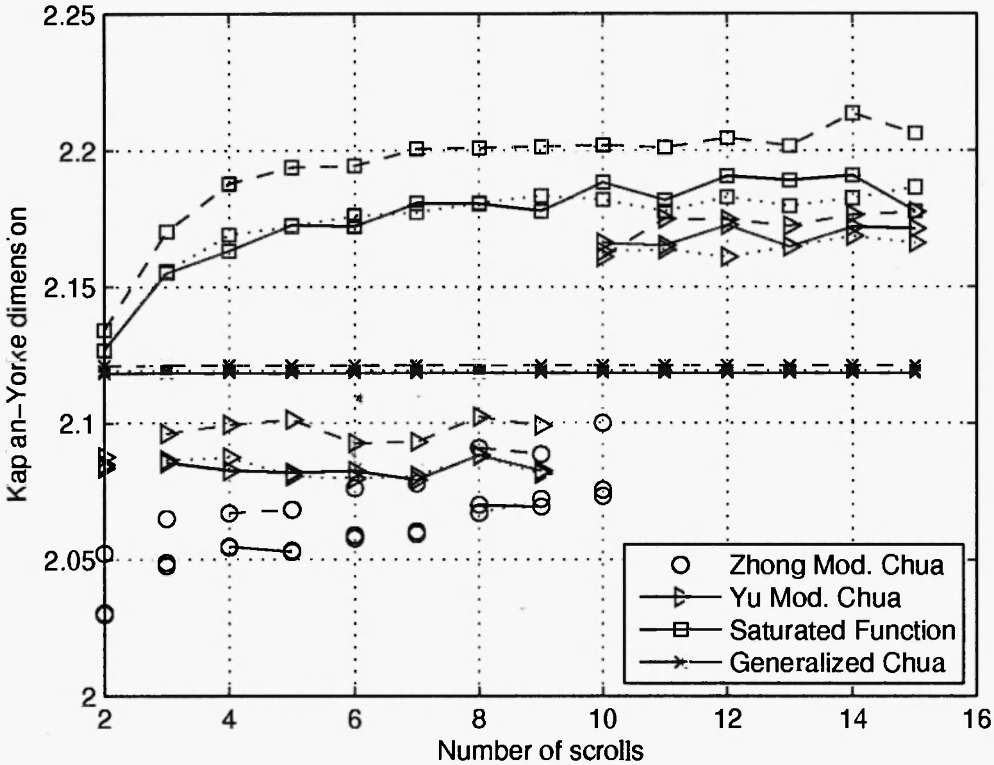

Fig. 2: Kaplan-Yorke dimension of the attractors obtained in ODE45 integration (solid line), RK4 integration (dotted line) and FE integration (dashed line). Note that this is a discrete-valued plot.

invariable (somehow as [21]). Here, we have introduced a short extra slope to give continuity to the nonlinear function. The result of the positive Lyapunov positive exponent is $\lambda_1 \approx 0.41394$, 0.40363, 0.40415 for FE, RK4 and ODE45, respectively. The other exponents were $\lambda_2 \approx 0.001$ and $\lambda_3 \approx -3.41$.

The results are summarized in Figure 1, where the obtained exponents have been joined by a line to remark the use of identical parameter set and to show each integration process. Discontinuities recall for the different values of parameter β which was adjusted to generate the attractors at the given integration step. The Kaplan-Yorke dimension (Lyapunov dimension) was also calculated and is shown in Figure 2 according to:

$$d_L = j + \frac{\lambda_1 + \lambda_2 + \ldots + \lambda_j}{|\lambda_{j+1}|}.$$

Where all Lyapunov exponents are ordered in decreasing order and j is the index of the smallest nonnegative exponent. The bigger dimension of the saturated function series system indicates less contraction over the trajectories.

5. Conclusion

The behavior of the positive Lyapunov exponent of four third order PWL-function-based n-scroll attractors has been computed by using the ODE45 package in MATLAB and by applying FE and RK4 algorithms. Convergences show that the same precision is achieved with more simulation time, since the orthogonalization is the same; i. e. the convergence of the computed values depended on the integration algorithm.

From the obtained results, we conclude that the relation between the number of scrolls in third order dynamical systems, and the Lyapunov exponent (λ_1) does not introduce an important trade-off, because the variation of the values is very small, especially if compared with the variation produced by a change in the system parameters as reveals the discontinuous lines of the graphics. In this manner, from the electronic design point of view, it could be comparable to generate few scrolls since the generation of a large number of scrolls leads us to large circuits, while the Lyapunov exponents are quite similar. A complete parameter space analysis is always required.

Among the four simulated systems, it is clear that the most regular is the generalized Chua because of its constant Jacobian, each scroll evolve in the same manner generating new initial conditions for the next one. The sum of all its Lyapunov exponents is approximately constant, since $\sum_i \lambda_i \approx \nabla \cdot f$ [13] where $\dot{x} = f(x)$, some inaccuracies are present due to the extra slope used in the computations. The saturated function series system exhibited the same property for all the simulations, this time, with great accuracy. In contrast, both of the modified Chua's systems seem to be less dissipative as the positive Lyapunov exponent grows.

Finally we remark that chaotic behavior is quite complex to analyze, since local Lyapunov exponents have probe to be very different to the mean exponent as reveals [28]; thus, working in the attractor shape can still bring new useful results.

Acknowledgment

We acknowledge the support of CONACyT-MEXICO under the projects 48396-Y and P-50051-Y, for the scholarship 204229, and for the sabbatical stay of the second author at University of California Riverside during 2009-2010.

References

[1] J. Lü and G. Cheng, "Generating multiscroll chaotic attractors: Theories, Methods and Applications," *International Journal of Bifurcation and Chaos*, vol. 16, no. 4, pp. 775-858, 2006.

[2] G. Zhong, K.F. Man, and G. Chen, "A Systematic Approach to Generating n-Scroll Attractors", *International Journal of Bifurcation and Chaos*, vol. 12, no. 12, pp. 2907-2915, 2002.

[3] S. M. Yu, S. S. Qiu, and Q. H. Lin, "New results of study on generating multiple-scroll chaotic attractors," *Science in China Series* F 46, pp. 104–115, 2003.

[4] J. Lü, G. Chen, X. Yu, and H. Leung, "Design and Analysis of Multiscroll Chaotic Attractors From Saturated Function Series", *IEEE Transactions on Circuits and Systems*, vol 51, no. 12, pp. 2476-2490, 2004.

[5] S. Yu, "Generation of nxm-Scroll Attractors under a Chua-Circuit Framework", *International Journal of Bifurcations and Chaos*, vol. 17, no. 11, pp. 3951-3964, 2007.

[6] R. Trejo-Guerra, E. Tlelo-Cuautle, C. Cruz-Hernández, and C. Sánchez-López, "Chaotic communication system using Chua's oscillators realized with CCII+s," *International Journal of Bifurcations and Chaos*, vol. 19, no. 12, 2009.

[7] L. Gámez-Guzmán, C. Cruz-Hernández, R. M. López-Gutiérrez, and E. E. García-Guerrero, "Synchronization of Chua's Circuits with Multi-Scroll Attractors: Application to Communication", Communications in Nonlinear Science and Numerical Simulation vol. 14, no. 6, pp. 2765-2775, 2009.

[8] J. H. Park, "Adaptive synchronization of a unified chaotic system with an uncertain parameter", *International Journal of Nonlinear Sciences and Numerical Simulation,* vol. 3, no. 2, pp. 201-206, 2005.

[9] C. L. Kuo, "Design of an Adaptive Fuzzy Sliding-Mode Controller for Chaos Syn-

chronization", *International Journal of Nonlinear Sciences and Numerical Simulation,* vol. 8, no. 4, pp. 631-636, 2007.

[10]T. H. S. Li, M. Y. Hsiao, J. Z. Lee, and S. H. Tsai, "Adaptive synchronization of a unified chaotic system with an uncertain parameter", *International Journal of Nonlinear Sciences and Numerical Simulation,* vol. 10, no. 2, pp. 171-180, 2009.

[11]A. Stefanski, and T. Kapitaniak, "Using chaos synchronization to estimate the largest lyapunov exponent of nonsmooth systems," *Discrete Dynamics in Nature and Society*, vol. 4, no. 3, pp. 207-215, 2000.

[12]Y. Rongfei, H. Anpin, Z. Sheng, and Z. Junwei, "Maximum Lyapunov exponent analysis for compressors," Acta Aeronautica et Astronautica Sinica, 30(4) pp. 622-624, 2009.

[13]K. Shin, and J. K. Hammond, "The instantaneous Lyapunov exponent and its application to chaotic dynamical systems," *Journal of sound and vibration*, vol. 218, no. 3, pp. 389-403, 1998.

[14]S. Rugonyi, and K. J. Bathe, "An evaluation of the Lyapunov characteristic exponent of chaotic continuous systems," *International Journal for Numerical Methods in Engineering*, vol. 56, no. 1, pp. 145-163, 2003.

[15]C. Cruz-Hernández, and A. A. Martynyuk, Advances in chaotic dynamics and applications; *Series: Stability, Oscillations and Optimization of Systems*, Cambridge Scientific Publishers, vol. 4, 2009.

[16]T. Matsumoto, L. O. Chua, and M. Komuro, "The double scroll," *IEEE Transactions on Circuits and Systems*, vol. 32, no. 8, pp. 797-818, 1985.

[17]M. P. Kennedy, "Chaos in the Colpitts oscillator," *IEEE Transactions on Circuits and Systems I*, vol. 41, no. 11, pp. 771-774, 1994.

[18]L. Liu, Y. C. Su, C. X. Liu, and T. Liu, "A modified Lorenz System", *International Journal of Nonlinear Sciences and Numerical Simulation,* vol. 7, no. 2, pp. 187-190, 2006.

[19]L. G. Machado, D.C. Lagoudas, and M. A. Savi, "Lyapunov exponents estimation for hysteretic systems," *International Journal of Solids and Structures*, vol. 46, no. 6, pp. 1269-1286, 2009.

[20]S. Gopal, and Y. C. Lai, "Inducing Chaos in MOSFET-Based Electronic Circuits," *Circuits, Systems and Signal Processing*, vol. 28, no. 4, pp. 535-545, 2009.

[21]Li Chun-Biao and Wang De-Chun, "An attractor with invariable Lyapunov exponent spectrum and its Jerk circuit implementation," *Acta Physica Sinica*, vol. 58, no, 2, pp. 764-70, 2009.

[22]R. M. Rubinger, A. W. M. Nascimento, L. F. Mello, C. P. L. Rubinger, N. Manzanares Filho, and H. A. Albuquerque, "Inductorless Chua's Circuit: Experimental Time Series Analysis," *Mathematical Problems in Engineering*, vol. 2007, 2007.

[23]C. Sánchez-López, R. Trejo-Guerra, J. M. Munoz-Pacheco, and E. Tlelo-Cuautle, "N-scroll chaotic attractors from saturated functions employing CCII+s," *Nonlinear Dynamics*, 2010. DOI 10.1007/s11071-009-9652-3.

[24]J.M. Muñoz-Pacheco, and E. Tlelo-Cuautle, "Automatic synthesis of 2D-n-scrolls chaotic systems by behavioral modeling," *Journal of Applied Research and Technology*, vol. 7, no. 1, pp. 5-14, 2009.

[25]T. S. Parker, and L.O. Chua, *Practical Numerical Algorithms for Chaotic Systems*, Springer-Verlag, NY, 1989.

[26]G. H. Golub, and C. V. Loan, *Matrix Computations*, 3rd ed. The Johns Hopkins University Press, 1996.

[27]E. Tlelo-Cuautle, J.M. Munoz-Pacheco, and J. Martínez-Carballido, "Frequency-scaling simulation of Chua's circuit by automatic determination and control of step-size", *Applied Mathematics and Computation*, vol. 194, no. 2, pp. 486-491, 2007.

[28]H. D. I. Abarbanel, R. Brown, and M. B. Kennel, "Variation of Lyapunov Exponents on a Strange Attractor", Journal of Nonlinear Science, vol. 1, no. 2, pp. 175-199, 1991.