Ann. Phys. (Berlin) **523**, No. 5, 402 – 407 (2011) / **DOI** 10.1002/andp.201000147

Many fields interaction: Beam splitters and waveguide arrays

R. Mar-Sarao, F. Soto-Eguibar, and **H. Moya-Cessa***

INAOE, Instituto Nacional de Astrofísica, Óptica y Electrónica, Apdo. Postal 51 y 216, 72000, Puebla, Pue., Mexico

Received 5 November 2010, revised 17 December 2010, accepted 2 February 2011 by B. Kramer
Published online 14 March 2011

Key words Master equations, nonclassical fields, evolution of coherent states.

We study the interaction of many fields. We obtain an effective Hamiltonian for this system by using a method recently introduced that produces a small rotation to the Hamiltonian that allows to neglect some terms in the rotated Hamiltonian. We show that coherent states remain coherent under the action of a quadratic Hamiltonian and by solving the eigenvalue and eigenvector problem for tridiagonal matrices we also show that a system of n interacting harmonic oscillators, initially in coherent states, remain coherent during the interaction.

1 Introduction

Major efforts have been directed towards the generation of nonclassical states of electromagnetic fields, in which certain observables exhibit less fluctuations (or noise) than in a coherent state, whose noise is referred to as the standard quantum limit (SQL). Nonclassical states that have attracted the greatest interest include (a) macroscopic quantum superpositions of quasiclassical coherent states with different mean phases or amplitudes, also called "Schrödinger cats" [1–5], (b) squeezed states [2, 6], whose fluctuations in one quadrature or the amplitude are reduced beyond the SQL, and (c) the particularly important limit of extreme squeezing, i.e. Fock or number states [7]. It is well known that such nonclassical states of the field are very sensitive to interference with an environment; for instance, although a coherent state subject to dissipation keeps its form during the dynamics, a superposition of coherent states (any field state may be written as a superposition of coherent states) will produce a mixed state that has lost all its nonclassical properties.

Because of the importance of this problem, it has been considered already by several authors, for instance Mista [8] discussed a Hamiltonian including third-order terms but preserving some special coherent states. Several Hamiltonians, with time-dependent and random frequencies, have been also considered in connection with preserving coherence [9–12]. Sánchez-Soto and Bernabeu [13] also considered a similar problem for generalized coherent states associated with arbitrary Lie groups.

Consider the master equation for a field in a lossy cavity at zero temperature

$$\frac{d\rho}{dt} = \gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \tag{1}$$

with a ($a^\dagger$) the annihilation (creation) operator for the cavity mode, γ the decay constant and ρ the density matrix. If the initial state of the cavity field is a coherent state $|\alpha\rangle$, then the dynamics shows that it will decay in time as $|\alpha e^{-\gamma t}\rangle$ (see for instance [14]). However, for no other states this occurs. One possible answer about why the coherent states preserve its form during decay is the fact that coherent states are eigenstates of the annihilation operator, but this argument does not hold for a dissipative two-photon process [15]

$$\frac{d\rho}{dt} = \gamma(2a^2\rho a^{\dagger 2} - a^{\dagger 2}a^2\rho - \rho a^{\dagger 2}a^2) \tag{2}$$

* Corresponding author E-mail: hmmc@inaoep.mx, Phone: +52 222 266 3100, Fax: +52 222 247 2940

even though coherent states are also eigenstates of the annihilation operator squared (so do even and odd coherent states [16]).

Both equations above are obtained using Born-Markov approximations [2,15]. In the case in which such approximations are not used, i.e. when the interaction between a harmonic oscillator and a set of harmonic oscillators (the environment) is considered, it is not clear how a coherent state decays. Here we will try to answer this question. First, we will consider the interaction between two fields, to later generalize the result to a field interacting with many. In particular, we will give expressions in terms of polynomials for eigenvectors of the matrices that diagonalize the Hamiltonian for the interaction of many fields.

2 Two fields interacting: beam splitters

Consider the Hamiltonian of two interacting fields (we set $\hbar = 1$)

$$H = \omega_a a^\dagger a + \omega_b b^\dagger b + \lambda(a^\dagger b + b^\dagger a). \tag{3}$$

This interaction occurs in beam splitters, however it may also be obtained by the interaction of two quantized fields with a two-level atom when the fields are far from resonance with the atom, in this case an effective Hamiltonian may be obtained, which has the form of the above Hamiltonian [17]. By transforming to the interaction picture, i.e. getting rid off the free Hamiltonians, we obtain

$$H_I = \Delta a^\dagger a + \lambda(a^\dagger b + b^\dagger a), \tag{4}$$

with $\Delta = \omega_a - \omega_b$, the detuning. It is useful to define normal-mode operators by [18]

$$A_1 = \delta a + \gamma b, \quad A_2 = \gamma a - \delta b, \tag{5}$$

with

$$\delta = \frac{2\lambda}{\sqrt{2\Omega(\Omega - \Delta)}}, \quad \gamma = \sqrt{\frac{\Omega - \Delta}{2\Omega}} \tag{6}$$

and $\Omega = \sqrt{\Delta^2 + 4\lambda^2}$ the Rabi frequency. The annihilation operators A_1 and A_2 are just like a and b, and obey the commutation relations

$$[A_1, A_1{}^\dagger] = [A_2, A_2^\dagger] = 1; \tag{7}$$

moreover, the normal-mode operators commute with each other

$$[A_1, A_2] = [A_1, A_2^\dagger] = 0. \tag{8}$$

In terms of these operators, the Hamiltonian (3) becomes

$$H_I = \mu_1 A_1^\dagger A_1 + \mu_2 A_2^\dagger A_2, \tag{9}$$

with $\mu_{1,2} = (\Delta \pm \Omega)/2$.

Up to here, we have translated the problem of solving Hamiltonian (3), into the problem of obtaining the initial states for the "bare" modes in the initial states for the normal modes. In order to have a way of transforming states from one basis to the other, we note that the vacuum states in both systems, $|0\rangle_a|0\rangle_b$ and $|0\rangle_{A_1}|0\rangle_{A_2}$, differ only by a phase [18]. First note that

$$A_1|0\rangle_a|0\rangle_b = 0, \tag{10}$$

and in a similar way, it may be seen the other normal-mode annihilation operator, A_2, has the same effect. We choose the phase so that

$$|0\rangle_a|0\rangle_b = |0\rangle_{A_1}|0\rangle_{A_2}. \tag{11}$$

If we consider coherent states as initial states for the interaction, we obtain the evolved wavefunction

$$\begin{aligned}|\psi(t)\rangle &= \mathrm{e}^{-it(\mu_1 A_1^\dagger A_1+\mu_2 A_2^\dagger A_2)} D_a(\alpha) D_b(\beta)|0\rangle_a|0\rangle_b,\\ &= \mathrm{e}^{-it(\mu_1 A_1^\dagger A_1+\mu_2 A_2^\dagger A_2)} D_a(\alpha) D_b(\beta)|0\rangle_{A_1}|0\rangle_{A_2}\end{aligned} \tag{12}$$

where the $D_c(\epsilon) = \exp(\epsilon c^\dagger - \epsilon^* c)$ is the Glauber displacement operator [19]. From (5), we can write the operators a and b in terms of the operator A_1 and A_2, and write (12) as

$$|\psi(t)\rangle = \mathrm{e}^{-it(\mu_1 A_1^\dagger A_1+\mu_2 A_2^\dagger A_2)} D_{A_1}(\alpha\delta+\beta\gamma) D_{A_2}(\alpha\gamma-\beta\delta)|0\rangle_{A_1}|0\rangle_{A_2}.$$

Passing the exponential in the above equation to the right, applying it to the vacuum states and using the following property

$$D_c(\epsilon_1)D_c(\epsilon_2) = D_c(\epsilon_1+\epsilon_1)\mathrm{e}^{\frac{1}{2}(\epsilon_1\epsilon_2^*-\epsilon_1^*\epsilon_2)}, \tag{13}$$

we obtain

$$\begin{aligned}|\psi(t)\rangle &= D_{A_1}([\alpha\delta+\beta\gamma]\mathrm{e}^{-i\mu_1 t}) D_{A_2}([\alpha\gamma-\beta\delta]\mathrm{e}^{-i\mu_2 t})|0\rangle_{A_1}|0\rangle_{A_2}\\ &= |[\alpha\delta+\beta\gamma]\mathrm{e}^{-i\mu_1 t}\rangle_{A_1}|[\alpha\gamma-\beta\delta]\mathrm{e}^{-i\mu_2 t}\rangle_{A_2}.\end{aligned} \tag{14}$$

Equation (14) shows that in the new basis, coherent states remain coherent during evolution.

By transforming back to the original basis, using again property (13), we obtain

$$|\psi(t)\rangle = |\delta[\alpha\delta+\beta\gamma]\mathrm{e}^{-i\mu_1 t}+\gamma[\alpha\gamma-\beta\delta]\mathrm{e}^{-i\mu_2 t}\rangle_a|\gamma[\alpha\delta+\beta\gamma]\mathrm{e}^{-i\mu_1 t}-\delta[\alpha\gamma-\beta\delta]\mathrm{e}^{-i\mu_2 t}\rangle_b,$$

i.e. coherent states remain coherent during evolution. This will be used next section as the building block for the interaction of many modes.

3 Generalization to n modes

Consider the Hamiltonian of the interaction of k fields

$$H = \sum_j^n \omega_j n_j + \sum_{j\neq i}^n \lambda_{ij}\left(a_i^\dagger a_j + a_i a_j^\dagger\right). \tag{15}$$

This Hamiltonian may be produced in waveguide arrays. From the Hamiltonian above, we can produce the following matrix

$$M = \begin{pmatrix} \omega_1 & \lambda_{21} & . & . & . & \lambda_{n1}\\ \lambda_{12} & \omega_2 & . & . & . & \lambda_{n2}\\ \lambda_{13} & \lambda_{23} & . & . & . & \lambda_{n3}\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ \lambda_{1n} & \lambda_{2n} & . & . & . & \omega_n \end{pmatrix}. \tag{16}$$

We can rewrite the Hamiltonian in the form (9), that is

$$H = \sum_m^n \mu_m A_m^\dagger A_m, \tag{17}$$

such that

$$\left[A_k, A_m^\dagger\right] = 0, \tag{18}$$

where we have defined the normal-mode operators A_k as

$$A_k = \sum_{i=1}^{n} r_{ki} a_i, \tag{19}$$

with r_{ki} a real number.

Equation (18) implies that

$$\left[A_k, A_m^\dagger\right] = \sum_{i,j=0}^{n} r_{ki} r_{mj} \left[a_i, a_j^\dagger\right] = \sum_{i}^{n} r_{ki} r_{mi} = 0. \tag{20}$$

By defining the vector

$$\vec{r}_k = (r_{k1}, r_{k2}, \ldots, r_{kn}), \tag{21}$$

Eq. (20) takes the form $\vec{r}_k \cdot \vec{r}_m = 0$, i.e. the vectors $\vec{r}_k$ are orthogonal; we will consider them also normalized, $\vec{r}_k \cdot \vec{r}_k = 1$. With these vectors we can form the matrix

$$R = \begin{pmatrix} r_{11} & r_{21} & . & . & . & r_{n1} \\ r_{12} & r_{22} & . & . & . & r_{n2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ r_{1n} & r_{2n} & . & . & . & r_{nn} \end{pmatrix}. \tag{22}$$

If we combine Eqs. (15), (17) and (19), we obtain the system of equations

$$\sum_m \mu_m r_{mi}^2 = \omega_i, \tag{23}$$

$$\sum_m \mu_m r_{mi} r_{mj} = \lambda_{ij}, \tag{24}$$

that may be re-expressed in the compact form

$$RDR^\dagger = M = \begin{pmatrix} \omega_1 & \lambda_{21} & . & . & . & \lambda_{n1} \\ \lambda_{12} & \omega_2 & . & . & . & \lambda_{n2} \\ \lambda_{13} & \lambda_{23} & . & . & . & \lambda_{n3} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \lambda_{1n} & \lambda_{2n} & . & . & . & \omega_n \end{pmatrix}, \tag{25}$$

with

$$D = \begin{pmatrix} \mu_1 & 0 & . & . & . & 0 \\ 0 & \mu_2 & . & . & . & 0 \\ 0 & 0 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \mu_n \end{pmatrix}, \tag{26}$$

i.e. D is a diagonal matrix whose elements are the eigenvalues of the matrix M, defined from the Hamiltonian. The matrix R is therefore M's eigenvectors matrix.

4 A particular interaction

Now we study a particular interaction, namely when $\lambda_{ij} = \lambda$ if $j = i + 1$ or $j = i - 1$ and it is zero otherwise. The frequencies ω_i are left arbitrary. The Hamiltonian governing this interaction then has an associated tridiagonal matrix of the form

$$M = \begin{pmatrix} \omega_1 & \lambda & 0 & . & . & 0 \\ \lambda & \omega_2 & \lambda & 0 & . & 0 \\ 0 & \lambda & \omega_3 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & \lambda \\ 0 & 0 & . & . & \lambda & \omega_k \end{pmatrix}. \tag{27}$$

We can use some properties of this matrix to find the eigenvectors; in particular, the characteristic polynomial for this matrix is given by the recurrence relation

$$F_0(\mu) = 1, \quad F_1(\mu) = \frac{\mu - \omega_1}{\lambda}, \quad F_n(\mu) = \frac{(\mu - \omega_n)}{\lambda} F_{n-1}(\mu) - F_{n-2}(\mu), \tag{28}$$

and the normalized eigenvectors are simply

$$\vec{r}_j = \frac{1}{\sqrt{N_j}} \begin{pmatrix} F_0(\mu_j) \\ F_1(\mu_j) \\ \vdots \\ \vdots \\ F_{n-1}(\mu_j) \end{pmatrix}, \tag{29}$$

with $N_j = \sum_{k=0}^{n-1} F_k^2(\mu_j)$, such that the matrix elements of the matrix R are given by

$$r_{ij} = \frac{F_{i-1}(\mu_j)}{\sqrt{N_j}}. \tag{30}$$

5 Coherent states as initial fields

The solution to the Schrödinger equation, subject to the Hamiltonian (15), with all the modes initially in coherent states, $|\psi(0)\rangle = |\alpha_1\rangle_1 |\alpha_2\rangle_2 \dots |\alpha_n\rangle_n$, is simply the direct product of coherent states

$$|\psi(t)\rangle = |\vec{r}_1 \cdot \vec{\beta}(t)\rangle_1 |\vec{r}_2 \cdot \vec{\beta}(t)\rangle_2 \dots |\vec{r}_n \cdot \vec{\beta}(t)\rangle_n \,, \tag{31}$$

with $\vec{\beta}(t) = (\vec{r}_1 \cdot \vec{\alpha} \mathrm{e}^{-i\mu_1 t}, \vec{r}_2 \cdot \vec{\alpha} \mathrm{e}^{-i\mu_2 t}, \dots, \vec{r}_n \cdot \vec{\alpha} \mathrm{e}^{-i\mu_n t})$ and the vector $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is composed by the coherent amplitudes of the initial wave function. Up to here, we have shown that the interaction of several modes, initially in coherent states, does not change the form of those states (remain coherent), but modifies their amplitudes. If we choose the interaction constants to be $\lambda_{1j} \neq 0$ for $j \neq 1$ and the rest as zero, we are dealing with the interaction between one field and $n - 1$ fields. If $n \to \infty$ and the amplitudes α_j are zero for $j > 1$, we deal with the interaction of one field with $n - 1$, one of them in a coherent state with amplitude α_1 and the rest in the vacuum. Therefore, the most likely situation we have is the coherent state decaying towards the vacuum while keeping its coherent form.

6 Conclusions

We have shown that coherent states remain coherent under the action of a quadratic Hamiltonian; this is an expected and known result. By solving the eigenvalue and eigenvector problem for tridiagonal matrices of the form (27), we have also shown that a system of n interacting harmonic oscillators, initially in coherent states, remain coherent during the interaction. In particular, if one considers one field (harmonic oscillator) interacting with many fields (harmonic oscillators), i.e. consider only $\lambda_{1j} \neq 0$ and $\lambda_{j1} \neq 0$ for $j > 2$, and all the others to be zero, we can model non-Markovian system-reservoir interaction. If we consider the system to be in a coherent state and all the others fields, that form the environment in a vacuum state (this is also in coherent states with zero amplitude), after evolution the amplitude of the coherent state will diminish, as one photon will go to another mode, keeping its coherent nature. If the number of modes that form the environment is very large, an event of the photon going back to the system is quite unlikely. Therefore, the next probable event is precisely the loss of another photon by the system, etc. until it arrives to a state close to the vacuum. In case the number of modes interacting with the system is infinite, then the vacuum would be the final state of the system. In other words, the total system perform the following transition

$$|\alpha\rangle_1 |0\rangle_2 \ldots |0\rangle_n \rightarrow |\delta_1\rangle_1 |\delta_2\rangle_2 \ldots |\delta_2\rangle_n, \tag{32}$$

where the coherent amplitudes, $\delta_k \rightarrow 0$, as $n \rightarrow \infty$.

In conclusion, we have given a complete algebraic solution to the problem of n interacting harmonic oscillators, without Born-Markov approximations.

References

[1] E. Schrödinger, Naturwissenschaften **23**, 807–812; 823–828; 844–849 (1935).
[2] D. F. Walls and G. J. Milburn, Quantum Optics, second edition (Springer, New York, 2008).
[3] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13–16 (1986).
[4] D. Vitali, M. Fortunato, P. Tombesi and F. De Martini, Fortschr. Phys. **48**, 437–446, (2000).
[5] M. A. Landau and C. R. Stroud Jr., Phys. Rev. A **77**, 062104 (2008).
[6] R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709–759 (1987).
[7] H. Moya-Cessa, P. L. Knight and A. Rosenhouse-Dantsker, Phys. Rev. A **50**, 1814 (1994).
[8] L. Mista, Phys. Lett. A **25**, 646 (1967).
[9] Y. Kano, Phys. Lett. A **56**, 7 (1976).
[10] B. Crosignani, B. D. di Porto, and S. Solimeno, Phys. Rev. **186**, 1342 (1969).
[11] P. Chaud, Nuovo Cimento B **50**, 17 (1979).
[12] D. A. Trifonov and V. N. Ivanov, Phys. Lett. A **64**, 269 (1977).
[13] L. L. Sánchez-Soto and E. Bernabeu, Nuovo Cimento B **100**, 619 (1987).
[14] H. Moya-Cessa, Phys. Rep. **432**, 1–41 (2008).
[15] L. Gilles, B. M. Garraway, and P. L. Knight, Phys. Rev. A **49**, 2785–2799 (1994).
[16] V. Buzek, H. Moya-Cessa, P. L. Knight, and S. J. D. Phoenix, Phys. Rev. A **45**, 8190–8203 (1992).
[17] F. O. Prado, N. G. de Almeida, M. H. Y. Moussa, and C. J. Villas-Bôas, Phys. Rev. A **73**, 043803 (2008).
[18] S. M. Dutra, P. L. Knight, and H. Moya-Cessa, Phys. Rev. A **48**, 3168–3173 (1993).
[19] R. J. Glauber, Phys. Rev. **131**, 2766–2788 (1963).