Holographic generation of a class of nondiffracting fields with optimum efficiency

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We discuss the accurate generation of complex optical fields using phase holograms that provide the optimum diffraction efficiency. In each considered case, the phase modulation of the employed hologram is identical to the phase of the desired optical field. We show that periodic and quasiperiodic nondiffracting optical fields, mathematically obtained through the superposition of multiple plane waves, can be generated with high fidelity using this approach.

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Generating an arbitrary optical field is an important task in physical optics. To achieve it, independent modulation of both the amplitude and the phase of the field are required. A convenient method to perform this task is provided by synthetic holography. Both the efficiency and the accuracy of synthetic holograms, however, depend on the hologram type and the field to be generated. Different types of synthetic holograms, for generation of arbitrary complex fields, are based on amplitude only modulation [1], real valued modulation [2], and phase-only modulation [3–9]. In the generation of a complex field it is possible to employ a synthetic phase hologram (SPH), whose phase modulation is identical to the phase of that field [10]. The performance of this hologram, which is referred to as kinoform of the complex field, is highly dependent on the nature of the optical field to be generated. We show that the periodic or quasiperiodic complex optical fields, equivalent to the superposition of multiple plane waves with symmetrically arranged propagation vectors, can be generated with high fidelity and the optimum efficiency using their kinoforms.

Let us denote the complex optical field to be generated as $f(x,y) = |f(x,y)|\exp[i\xi(x,y)]$, where $|f(x,y)|$ and $\xi(x,y)$ are the modulus and the phase, respectively. An optical element with complex transmittance $f(x,y)$, which is illuminated by a plane wave, generates the complex field $f(x,y)$ with efficiency

$$\eta_0 = A_\Omega^{-1}\iint_\Omega |f(x,y)|^2 \mathrm{d}x\mathrm{d}y, \tag{1}$$

where A_Ω is the area of the pupil, Ω, that limits the complex transmittance.

The transmittance of the SPH employed to generate the complex field is expressed as $h(x,y) = \exp[i\psi(x,y)]$. If the desired complex field $f(x,y)$ is generated on-axis, it can be related to the SPH by the identity

$$h(x,y) = \beta f(x,y) + e(x,y), \tag{2}$$

where β is a real positive constant, referred to as amplitude gain of the SPH, and $e(x,y)$ is the modulation error [7]. Multiplying Eq. (2) by $f^*(x,y)$, and integrating both sides of the resulting identity over the transmittance support, we obtain

$$\iint_\Omega \exp[i\psi(x,y)]f^*(x,y)\mathrm{d}x\mathrm{d}y = \beta\eta_0 A_\Omega + \iint_\Omega e(x,y)f^*(x,y)\mathrm{d}x\mathrm{d}y. \tag{3}$$

An interesting situation occurs when the Fourier spectra of the functions $e(x,y)$ and $f(x,y)$ have null overlapping. The Fourier spectra of functions $e(x,y)$ and $f(x,y)$ are denoted as $E(u,v)$ and $F(u,v)$, respectively, being (u,v) the spatial frequency coordinates, and the null overlapping of $E(u,v)$ and $F(u,v)$ is represented by the identity $E(u,v)F(u,v) = 0$. For further reference, this situation that allows the generation of the complex field $f(x,y)$, by applying a binary spatial filter to the Fourier spectrum of the hologram, is referred to as filtering condition. The second term at the right side of Eq. (3), which corresponds to the correlation between $e(x,y)$ and $f(x,y)$, evaluated at the origin, is null when the filtering condition is fulfilled. In this case, the SPH amplitude gain can be expressed as follows:

$$\beta = A_\Omega^{-1}\eta_0^{-1}\iint_\Omega |f(x,y)|\exp\{i[\psi(x,y) - \xi(x,y)]\}\mathrm{d}x\mathrm{d}y. \tag{4}$$

It is clear from Eq. (4) that the maximum possible amplitude gain is obtained when there is no difference between the SPH phase, $\psi(x,y)$, and the phase of the desired field, $\xi(x,y)$. Therefore, the upper bound for the amplitude gain is given by

$$\beta_L = \frac{\iint_\Omega |f(x,y)|\mathrm{d}x\mathrm{d}y}{\iint_\Omega |f(x,y)|^2\mathrm{d}x\mathrm{d}y}. \tag{5}$$

This limit amplitude gain β_L previously was derived by Wyrowski [11] with a different approach. The above discussion presents new and concise arguments to prove the result in Eq. (5), enhancing its meaning and relevance in the context of synthetic-phase holography. The field $f(x,y)$ can be recovered with the maximum amplitude gain, by performing spatial filtering on the Fourier spectrum of its kinoform, when this hologram fulfills the filtering condition. In this case, the efficiency also

gets its optimum value, which is obtained, considering Eqs. (1) and (2), as $\eta_L = \beta_L^2 \eta_0$. In the context of the above derivation, $f(x, y)$ corresponds to a complex transmittance, which is subject to the restriction $|f(x, y)| \leq 1$. Therefore, the optimum amplitude gain β_L is larger than or equal to 1, and the kinoform efficiency limit $\eta_L = \beta_L^2 \eta_0$ is larger than or equal to the efficiency (η_0) of the complex element with transmittance $f(x, y)$.

Next, we discuss a set of complex fields whose kinoforms fulfill the filtering condition. These fields are the periodic or quasiperiodic nondiffracting beams that result from the sum of multiple plane waves, subject to certain restrictions. The propagation vectors of the interfering waves have a common projection on the z axis, and the transverse projections of these vectors have azimuth angles uniformly distributed over the x-y plane. We assume that the nth plane wave has a phase shift $\theta_n = np\Delta\theta$, where $\Delta\theta = 2\pi/Q$, Q is the number of interfering waves and p is an integer number. The superposition of these Q waves at the plane $z = 0$ is given by

$$w(r, \theta) = C \sum_{n=0}^{Q-1} \exp(i\theta_n) \exp[i2\pi\rho_0 r \cos(\theta - n\Delta\theta)], \quad (6)$$

where r and θ are the cylindrical coordinates and $2\pi\rho_0$ is the modulus of the transverse components of the propagation vectors. The normalization constant C makes the maximum of $|w(r, \theta)|$ equal to 1. As a simple illustrative example, we consider the sum of two plane waves ($Q = 2$) assuming $p = 0$. In this case, the field at the plane $z = 0$ can be described, employing rectangular coordinates, by $f(x) = \cos(2\pi\rho_0 x)$. The kinoform of this function can be expressed by the Fourier series

$$h(x) = \sum_{m=1}^{\infty} c_m f(mx), \quad (7)$$

with coefficients $c_m = 4/(m\pi)$, for odd m, and $c_m = 0$, otherwise. Considering the form of function $f(x)$, it is easy to verify that its Fourier spectrum, formed by 2 Dirac deltas, is isolated from the Fourier spectra of the other terms in Eq. (7). Therefore, the kinoform of the complex field $f(x)$ fulfills the filtering condition. Moreover, the field $f(x)$ is generated from the kinoform with amplitude gain $\beta = c_1 = 4/\pi$. It is straightforward to show, employing Eq. (5), that the maximum amplitude gain attainable when this field is codified with a SPH is, as expected, $4/\pi$.

A more interesting example is the field obtained when the number of waves Q tends to infinity. In this case, the field in Eq. (6) corresponds to a nondiffracting Bessel beam $J_p(2\pi\rho_0 r) \exp(ip\theta)$, where J_p denotes the pth-order Bessel function of the first kind. It was proved in Arrizón *et al.* [9] that the kinoform of such a field fulfills the filtering condition.

To prove analytically the validity of the filtering condition in general, we first consider the periodicity of the exponential functions in Eq. (6), to establish the identity $w[r, \theta + n(2\pi/Q)] = \exp[inp(2\pi/Q)]w(r, \theta)$, for any integer number n. Using this result and the definition of the kinoform transmittance $k(r, \theta) = \exp[i\xi(r, \theta)]$, we also prove the relation

$$k[r, \theta + n(2\pi/Q)] = \exp[inp(2\pi/Q)]k(r, \theta). \quad (8)$$

Let us express the kinoform Fourier transform as $K(\rho, \varphi)$, where ρ and φ are the radial and angular polar coordinates, respectively, in the Fourier domain. Using the expression for the Fourier transform in polar coordinates and a variable change, it is shown that the kinoform Fourier spectrum also obeys the property in Eq. (8), i.e.,

$$K[\rho, \phi + n(2\pi/Q)] = \exp[inp(2\pi/Q)]K(\rho, \phi), \quad (9)$$

for an arbitrary integer number n. We can define the signal domain as the set of points (ρ, φ) where the Fourier spectrum of the field $w(r, \theta)$, denoted as $W(\rho, \varphi)$, is not null. It can be stated that the filtering condition is fulfilled if the Fourier spectrum of $K(\rho, \varphi)$ within the signal domain is proportional to $W(\rho, \varphi)$. Assuming that the field $w(r, \theta)$ is infinitely extended, its Fourier spectrum, $W(\rho, \varphi)$, will be formed by Q infinitely small spots, placed at the points of the circle of radius ρ_0, with angular coordinates $q(2\pi/Q)$, for $q = 0$ to $Q - 1$. The amplitudes of these spots have a common value and the qth spot is modulated by a phase shift $\exp[iqp(2\pi/Q)]$. Considering Eq. (9), it is obtained that the signal domain points $[\rho_0, q(2\pi/Q)]$, for $q = 0$ to $Q - 1$, in the kinoform Fourier spectrum have identical amplitudes among them and a phase distribution equal to that of the spots in the Fourier spectrum $W(\rho, \varphi)$. Therefore, the kinoform of $w(r, \theta)$ fulfills the filtering condition. This proof assumes that the signal spots are infinitely small, which is only possible for infinitely extended functions $w(r, \theta)$ and $k(r, \theta)$. We computed efficiencies η_0 and η_L versus Q for different values of p. In particular, the results for $p = 0$ (depicted in Fig. 1) clearly fulfill the relation $\eta_L > \eta_0$. Similar results are obtained for $p > 0$.

By means of numerical simulations it can be proved that the kinoforms with finite support fulfill the filtering condition with quite good approximation. To illustrate this fact, let us consider a couple of examples. For the first one, the field $w(r, \theta)$ is numerically computed for a number of waves $Q = 8$ and $p = 0$, assuming that the field is limited by a circular support of radius $R = 7.5\rho_0^{-1}$. Figure 2 shows the modulus and phase of the Fourier spectra of both the field $w(r, \theta)$ and its kinoform, computed in this case. Similar results, computed for the field $w(r, \theta)$ with parameters $Q = 5$ and $p = 1$, maintaining the same support, are depicted in Fig. 3.

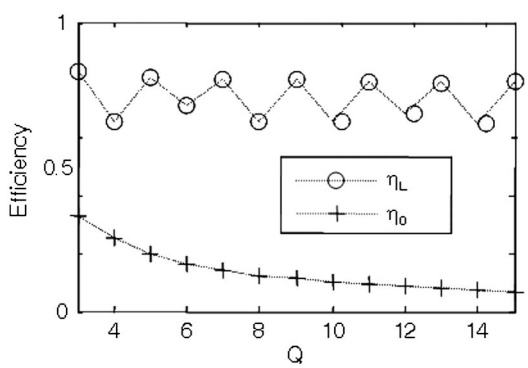

Fig. 1. Efficiencies η_0 and η_L versus the number Q of interfering waves for $p = 0$.

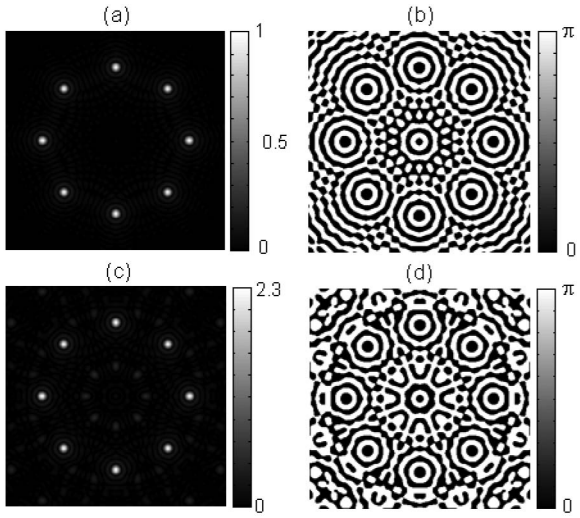

Fig. 2. (a) Modulus and (b) phase of the numerically computed Fourier spectrum of the field $w(r,\theta)$ with parameters for $Q = 8$ and $p = 0$. Similar results for the corresponding kinoform Fourier spectrum are shown in (c) and (d), respectively.

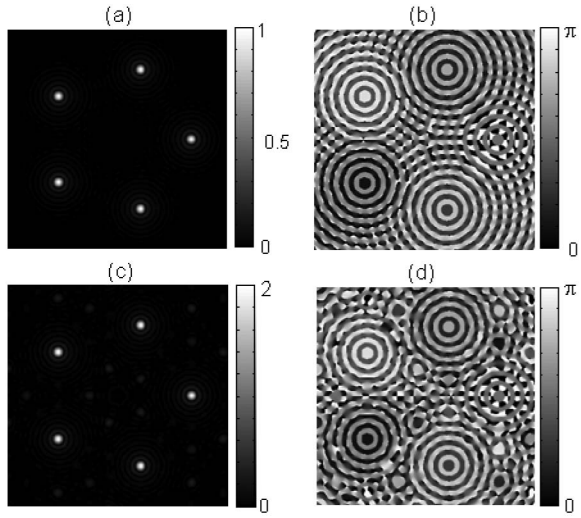

Fig. 3. (a) Modulus and (b) phase of the numerically computed Fourier spectrum of the field $w(r,\theta)$ with parameters for $Q = 5$ and $p = 1$. Similar results for the corresponding kinoform Fourier spectrum are shown in (c) and (d), respectively.

It is noted in each example that the kinoform Fourier spectrum shows a section (signal region) whose complex amplitude is quite similar to the Fourier spectrum of the field $w(r,\theta)$. Because the maximum amplitudes in the kinoforms spectra appear at the signal regions, the peak values in the gray level bars of Figs. 2(c) and 3(c), correspond to the amplitude gains, which are very approximated to the upper-bound limits obtained with Eq. (5). A consequence of the finite size of the kinoform SPH is that its Fourier spectrum is formed by extended spots, whose structure corresponds to the Fourier transform of the kinoform pupil (see the examples in Figs. 2 and 3). If the kinoform pupil is circular, the convenient spatial filter must be formed by finite-size circles, equal to the pupil's Airy disk. In this case, it is expected that the practical efficiency will roughly coincide with the ideal efficiency limit computed by the relation $\eta_L = \beta_L^2 \eta_0$. In general, if the size of the filter circles is further increased, unwanted energy will be transmitted through the filter, leading to an increase in the reconstructed signal error. A detailed discussion about this issue is out of the scope of the work presented herein.

In the context of conventional synthetic holography [1–8] the relation in Eq. (2), between the hologram $h(x,y)$ and the encoded complex field $f(x,y)$, is fulfilled with amplitude gains smaller than or equal to 1. The main result discussed establishes that any periodic or quasiperiodic nondiffracting field (PQNDF), which results from the superposition of an arbitrary number of plane waves, whose propagation vectors are symmetrically arranged around the optical axis, can be generated from its kinoform with the maximum possible amplitude gain, which is larger than 1. This is a new and significant result, in the context of synthetic phase holography, not only from a theoretical point of view but also because of its practical consequences. The generation of arbitrary complex fields with phase holograms has recently become common practice because of the wide availability of phase-only spatial light modulators. In particular, the possibility of generating a PQNDF with phase holograms that provide the maximum possible gain (and consequently the optimum efficiency) will affect several applications of such optical fields, e.g., fabrication of photonic crystals and quasicrystals, implementation of nonlinear optical guides, and particle manipulation with light. The kinoform hologram has been employed in other cases, i.e., in generation of Mathiew-Gauss beams [12]. For this and other cases that are different to the ones discussed here, it is still required to show whether the filtering condition is fulfilled, at least approximately.

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