A Second-Order Lowpass Parameter-Varying Filter Based on the Interconnection of First-Order Stages

Miguel Ángel Gutiérrez de Anda, *Member, IEEE*, and Isabelo Meza Dector, *Student Member, IEEE*

***Abstract*—In recent years, a new class of filters known as parameter-varying filters has been proposed. These filters are characterized by having a transient response of reduced duration while preserving their frequency response properties. The reduction of the transient response of these filters is a consequence of the automatic adjustment of their parameters. These filters may be used in signal processing applications in which the influence of the transient behavior of the filter is seen as an unwanted component of the filter output. In this article, a second-order lowpass parameter-varying filter based on the cascade of first-order parameter-varying stages is presented. The stability properties of the proposed filter, and particularly its bounded-input, bounded-output stability, are verified using a linear time-varying model which accurately represents its behavior when it is subject to the variation of its parameters. The parameters of the filter are varied according to a predefined control strategy. Simulation results confirm the filter capabilities to shorten the duration of its transient behavior when its parameters are varied through the action of a nonlinear control loop.**

***Index Terms*—Linear time-varying systems, lowpass filters, parameter-varying filters, transient response reduction.**

I. Introduction

In recent times, a new class of continuous-time filters known as parameter varying filters has been proposed in [1]–[7]. These filters are subject to temporary changes in the value of some of their parameters to reduce the duration of their transient response without altering its frequency domain behavior. These changes are induced as a response to a sudden variation in the power content of the input signal in a given frequency band. This frequency band is related to the passband of the filter in question (lowpass, passband or notch filter).

Parameter-varying filters are used in applications in which it is required to speed up the acquisition of signals with unwanted components in the frequency domain. In such applications, the transient response introduced by the same filter is seen as an unwanted component of the filter output response. These filters have been already applied in the compensation of the dynamic behavior of load cells [6]–[10]. Moreover, it has been suggested in [5] that parameter-varying filters may be used in the acquisition of brainstem auditory evoked potentials.

Manuscript received November 02, 2010; revised December 21, 2010; accepted December 31, 2010. Date of publication February 10, 2011; date of current version July 27, 2011. This work was supported in part by Mexican CONACyT under grant contract 84819-Y and in part by the Foundation for Polish Science (Fundacja na rzecz Nauki Polskiej – Kasa im. Józefa Mianowskiego). This paper was recommended by Associate Editor S. Pavan.

The authors are with the Electronics Department, National Institute for Astrophysics, Optics and Electronics (INAOE), 72840 Tonantzintla, Mexico (e-mail:mdeanda@inaoep.mx; dector2@yahoo.com.mx).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org

Digital Object Identifier 10.1109/TCSI.2011.2107230

In [1], a continuous-time mathematical model for a lowpass parameter-varying filter was proposed. In principle, the dynamic behavior of this type of filter may be understood by means of the linear time-varying (LTV) differential equation

$$y''(t) + 2\zeta(t)\omega_n(t)y'(t) + \omega_n^2(t)y(t) = \omega_n^2(t)u(t). \qquad (1)$$

In this differential equation, $u(t)$ and $y(t)$, represent, respectively, the input and the output of the filter, whereas $\zeta(t)$ and $\omega_n(t)$ are functions of the time variable t. Functions $\zeta(t)$ and $\omega_n(t)$ are used to model the parameters which control, respectively, the damping factor and the undamped frequency of oscillations of the filter's transient response with $u(t) = 0$ for a set of initial conditions different from zero. The transient behavior of the filter represented by (1) may be reduced in duration if functions $\omega_n(t)$ and $\zeta(t)$ are increased momentarily in value whenever the input signal shows an abrupt change in its amplitude.

In [2], it was established that functions $\zeta(t)$ and $\omega_n(t)$ must satisfy the following conditions in order to ensure that the homogeneous solutions of (1) are asymptotically stable

$$\zeta(t) > 0 \qquad (2a)$$

$$\omega_n(t) > 0 \qquad (2b)$$

$$|\omega_n'(t)| < 2\omega_n(t)\zeta(t). \qquad (2c)$$

Unfortunately, these conditions do not guarantee that the response of the filter represented by (1) to a bounded input may be bounded as well. In [1], [5]–[7] it was heuristically found that functions $\zeta(t)$ and $\omega_n(t)$ in (1) must vary exponentially in time. In [11], an analytic solution of (1) is presented considering such variations in the parameter functions $\zeta(t)$ and $\omega_n(t)$. However, this work does not establish conditions for the boundedness of the response of a parameter-varying filter described by (1) to a bounded input. Despite all the theoretical work done in the field of parameter-varying filters based on continuous-time LTV models, there are few analog implementations of these systems. In [5]–[7], discrete-time models are considered for (1) for their usage in the aforementioned applications. Very recently, analog circuits implementing first and second-order lowpass parameter varying filters have been proposed in [12] and [13].

The aim of this work is to present a second-order continuous-time lowpass parameter-varying filter whose stability behavior is determined by a less restrictive set of conditions compared to those given in expression (2). Moreover, thanks to these stability conditions, the filter shows a bounded response to a bounded input and allows its implementation as an analog system. The dynamic model associated to the new filter is different from the model formulated in (1). The transient behavior of the proposed parameter-varying filter is reduced in duration by means of a temporary increase of its bandwidth. This is attained through the temporary increase in value of only one of the describing parameters associated to the filter. In order to achieve an additional improvement of its transient response, its overshoot may be also reduced by temporarily varying the value of a second describing parameter. Both parameters may be varied independently to control the bandwidth of the filter and its overshoot.

This article is organized as follows. In Section II, preliminary considerations on the theory behind the operation of parameter-varying filters will be made. Given that the dynamics of parameter-varying filters may be modeled by means of LTV systems, some concepts on the stability of these systems will be introduced as well. For the reader not acquainted with the theory of LTV systems, [14]–[18] may give a good introduction to the basic concepts. The general structure of the proposed filter will be presented in Section III. In this section, a complete analysis of its stability properties will be carried out. The results of this analysis will be used in the formulation of conditions which guarantee that the transient behavior of the parameter-varying filter subject to a temporary change in its parameters has a shorter duration compared to the response of the same filter not subject to variations in the values of its parameters. The scheme used to induce variations in the values of the describing parameters of the filter as a consequence of the sudden change of the amplitude of the input signal is presented in Section IV. In Section V, design guidelines will be established for the synthesis of a parameter-varying filter and its associated parameter control scheme based on the models developed in Sections III and IV. In the same section, a design example will be presented. Finally, some concluding remarks are made in Section VI.

II. Preliminary Considerations

The dynamic behavior of parameter-varying filters may be better understood if the following state-space model is used to represent them:

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)u(t) \tag{3a}$$

$$y(t) = \mathbf{C}(t)\mathbf{x}(t) + D(t)u(t). \tag{3b}$$

In this model, $u(t)$ and $y(t)$ stand, respectively, for the input and the output of the filter. The vector $\mathbf{x}(t)$ contains n state variables which define the dynamic behavior of the filter, whereas matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$ and $\mathbf{C}(t)$ represent, respectively, the state, input and output matrices. The scalar factor $D(t)$ is a coupling factor between the input and the output.

In general, a parameter-varying filter must satisfy two important conditions for its successful implementation as an electronic circuit:

- The filter must have a stable behavior when $u(t)$ is zero. In other words, the voltages and currents of the electronic circuitry of the filter should return to their quiescent values,
- The filter should generate a bounded response $y(t)$ to a bounded input signal $u(t)$.

In the case of linear time-invariant (LTI) filters, these criteria are automatically fulfilled when their poles are located on the left half of the complex plane. However, in the case of LTV systems, in order to guarantee that the system in question will show a bounded response to a bounded input, Lemmas 1 and 2 have to be considered. Before stating these lemmas, the following definition is presented:

Definition: The response of the homogeneous LTV system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{4}$$

is exponentially asymptotically stable if, for any initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, *the following relation is satisfied for* $t \geq t_m \geq t_o$*:*

$$\|\mathbf{x}(t)\| < Me^{-k(t-t_m)} \tag{5}$$

where M and k are positive constants. In other words, the response of the system must decay exponentially before reaching its equilibrium point $\mathbf{x}(t) = \mathbf{0}$*[19].*

Lemma 1: Consider a LTV system whose state variables are defined by (3a). *The response of the state variables to a bounded input signal will be also bounded if the following conditions are met:*

- *The entries in matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are bounded,*
- *The homogeneous response of* (3a) *shows exponential asymptotic stability [20].*

Proof: See [20]. ∎

Lemma 2: The response of system (3) *to a bounded input will be also bounded if the following conditions are met:*

- *The entries of matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$ and the scalar factor $D(t)$ in* (3) *must be bounded for all t,*
- *The response of the homogeneous system* (4) *shows exponential asymptotic stability.*

Casted in other terms, system (3) *will show bounded-input bounded-output (BIBO) stability if the conditions previously stated are satisfied.*

Proof: If the aforementioned conditions are satisfied, the response of the state variables $\mathbf{x}(t)$ to a bounded input $u(t)$ will be bounded according to Lemma 1. Given that $\mathbf{x}(t)$ and $u(t)$ are bounded, the output $y(t)$ defined by (3b) will be bounded since the entries of matrix $\mathbf{C}(t)$ and the coupling factor $D(t)$ are bounded as well. ∎

In the case of LTI systems, the conditions for having BIBO stability as introduced in Lemma 2 are naturally enforced provided that their poles are located on the left half of the complex plane. However, for the LTV case, it is very difficult to demonstrate that system (4) will show exponential asymptotic stability for any initial condition. For most of the cases, numerical methods must be used to determine the stability properties of the homogeneous solutions of system (4). In [21], a method based on the determination of the magnitude of the modes associated to a set of linearly independent solutions of (4) is applied in the analysis of the stability properties of a lowpass parameter-varying filter.

The time-varying coefficients of (3) are used to model the temporary changes which are induced in the value of one or more describing parameters of a given filter to improve its transient response. Moreover, the describing parameters of the filter are varied such that the conditions previously mentioned for ensuring BIBO stability are met. In order to guarantee that the frequency domain characteristics of a parameter-varying filter are not unnecessarily altered, the values of its describing parameters are only varied when transient behavior is expected to occur. A

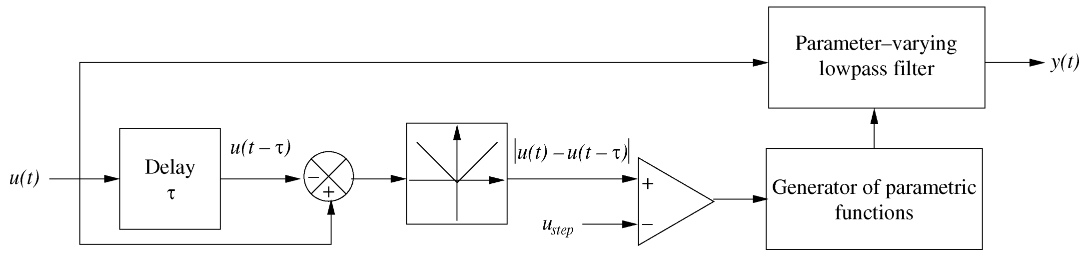

Fig. 1. Block diagram of an amplitude-based parameter control scheme for a parameter-varying lowpass filter.

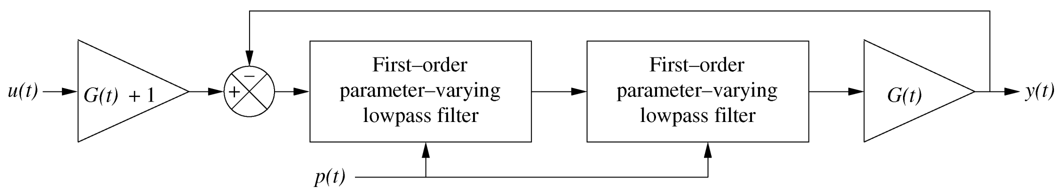

Fig. 2. Block diagram of the proposed second-order parameter-varying lowpass filter.

simple way to detect the possible occurrence of transient behavior at the output of a lowpass parameter-varying filter, for instance, is through the detection of sudden variations in amplitude of the input signal [21]. These variations are linked to a change in the instantaneous power of the DC component of the input signal. In Fig. 1, a control scheme based on this concept is presented. In this scheme, the input signal $u(t)$ is delayed by a delay constant τ. This constant is chosen to enable the detection of sudden changes in its amplitude. If the absolute difference $|u(t) - u(t-\tau)|$ exceeds a given threshold value u_{step}, the block responsible for the variation of the filter parameters is activated.

Finally, due to the introduction of the control scheme shown in Fig. 1, a lowpass parameter-varying filter is a nonlinear system since the property of homogeneity (which is a characteristic of any linear system) does not hold anymore [14]. However, the stability properties of the filter will be still determined by the LTV model formulated in (3) since the generator of the parametric functions used to improve its transient behavior is controlled by a static nonlinear circuit.

III. General Structure of the Proposed Lowpass Filter and Its Properties

A block representation of the proposed second-order parameter-varying filter is presented in Fig. 2. This filter is formed by the cascade connection of two identical first-order lowpass parameter-varying filters. In this scheme, a negative feedback loop is used to place the poles of the filter in a given location on the complex plane. Each of the first-order parameter-varying filters present in the block representation of Fig. 2 is modeled by means of the LTV differential equation

$$y'(t) + p(t)y(t) = p(t)u(t). \tag{6}$$

In this expression, $u(t)$ and $y(t)$ are, respectively, the input and output variables of the first-order parameter-varying filter. Function $p(t)$ may be interpreted as a function which controls its 3 dB cutoff frequency. The following lemma states under which condition the output of the filter described by (6) is bounded when a bounded input signal is applied.

Lemma 3: The response of the filter modeled by (6) *to a bounded input signal will be also bounded if function $p(t)$ is bounded and positive for all t.*

Proof: The time-domain response of this filter is given by the following expression: [22]

$$y(t) = Ce^{-\int p(t)dt} + e^{-\int p(t)dt}\int e^{\int p(t)dt} p(t)u(t)dt \tag{7}$$

where C is a constant whose value depends on the initial condition imposed on (6). In this expression, the term $Ce^{-\int p(t)dt}$ is the homogeneous part of the response of the filter modeled by (6), whereas the next term is the nonhomogeneous part of the response. From an examination of expression (7), and taking into account the statement presented in Lemma 2, it can be concluded that in order to guarantee the bounded response of the first-order parameter-varying filter, function $p(t)$ must be bounded and positive for all t. ■

If the homogeneous response of the parameter-varying filter represented by (6) is associated to the actual transient response of the filter, a temporary increase of the filter parameter $p(t)$ will induce a reduction of the duration of its transient response. This assertion will be later used to formulate a strategy of variation of the parameters for the second-order filter proposed in this article.

The behavior of the system resulting from the cascade connection of two first-order parameter varying filters with a negative feedback loop as depicted in Fig. 2 is given by the following system of LTV differential equations:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = p(t)\begin{bmatrix} -1 & -G(t) \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} p(t)(G(t)+1) \\ 0 \end{bmatrix} u(t) \tag{8a}$$

$$y(t) = x_2(t). \tag{8b}$$

In this set of equations, it is assumed that functions $p(t)$ and $G(t)$ are bounded and positive for all t. This assumption is required in order to establish conditions for the boundedness of the response of the proposed filter to a bounded input. Functions $p(t)$ and $G(t)$ may be interpreted as functions which control the position of the real and imaginary parts of the complex-conjugate poles associated to the second-order filter when it behaves as a LTI system. If functions $p(t)$ and $G(t)$ assume positive constant values which will be denoted respectively as p_c and G_c, the complex conjugate poles of the filter p_1 and p_2 will be given by

$$p_1 = -p_c + p_c\sqrt{G_c}j \tag{9a}$$
$$p_2 = -p_c - p_c\sqrt{G_c}j. \tag{9b}$$

From this equation, some rules may be inferred on the mechanism of variation that $p(t)$ and $G(t)$ should experience in order to reduce the transient response of the second-order lowpass parameter-varying filter. If function $p(t)$ is temporarily increased, this will induce a temporary increase in the filter bandwidth. Therefore, its transient behavior will be improved. If function $G(t)$ is temporarily reduced in magnitude such that it always adopt positive values, the overshoot generated by the filter will be reduced in amplitude. This reduction will be more noticeable if the imaginary part of the complex conjugate poles of the original LTI filter is much larger in magnitude than the real part of its poles. These variations must occur within finite time intervals since they must be *temporary*.

Assuming that parameters $p(t)$ and $G(t)$ vary in time as indicated below

$$p(t) = \begin{cases} p_c + p_i(t) & 0 \le t \le t_c \\ p_c & t > t_c \end{cases} \tag{10a}$$
$$G(t) = \begin{cases} G_c + G_d(t) & 0 \le t \le t_c \\ G_c & t > t_c \end{cases} \tag{10b}$$

subject to the following constraints:

- Function $G_d(t)$ is defined as follows:

$$G_d(t) = -G_x e^{-t/t_d} \tag{11}$$

- Function $p_i(t)$ is a function of t which adopts positive values and is bounded in value in the interval $0 \le t \le t_c$,
- Constants p_c, G_c, G_x and t_d present in expressions (10) and (11) are real and greater than zero,

the homogeneous response of the filter described by (8) starting at $t = 0$ will be shorter in duration compared to the response starting at $t = 0$ of the same filter when $p(t) = p_c$ and $G(t) = G_c$. In both cases, the initial conditions considered at $t = 0$ for the determination of the response of the filter described by (8) with different choices for functions $p(t)$ and $G(t)$ must be the same. In order to prove this result, the following lemmas are presented.

Lemma 4: The response of the dynamical system modeled by the differential equation

$$\mathbf{x}'(t) = p(t)\mathbf{f}(\mathbf{x}(t)) \tag{12}$$

subject to the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ *is given by [23]*

$$\mathbf{x}(t) = \mathbf{x_o}\left(\int_0^t p(t)dt\right) \tag{13}$$

where $\mathbf{x_o}(t)$ *is a solution of the dynamical system modeled by the differential equation*

$$\mathbf{x_o}'(t) = \mathbf{f}(\mathbf{x_o}(t)). \tag{14}$$

Proof: Expression (13) may be differentiated with respect to t. The resulting expression will be equal to the right side of (12)[23]. ■

Lemma 5: For the following set of initial conditions at $t = 0$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{1_c} \\ x_{2_c} \end{bmatrix} \tag{15}$$

the homogeneous response $y(t)$ *of the system described by* (8) *is equal to*

$$y(t) = \begin{cases} e^{-a(t)}\left[C_1 I_\nu(b_1(t)) + C_2 I_{-\nu}(b_1(t))\right] & 0 \le t \le t_c \\ e^{-p_c t}\left[D_1 b_2(t) + D_2 b_3(t)\right] & t > t_c \end{cases} \tag{16}$$

provided that

- *Functions* $p(t)$ *and* $G(t)$ *are given as in expression* (10),
- *Function* $G_d(t)$ *is given as in expression* (11),
- *Constants* G_c *and* G_x *are real and greater than zero.*

In expression (16), C_1 *and* C_2 *are constants which depend on the set of initial conditions (15), whereas* D_1 *and* D_2 *are constants which ensure the continuity of the system response at* $t = t_c$. *Function* $I_\nu(x)$ *represents the modified Bessel function of the first kind of order* ν. *Functions* $a(t)$, $b_1(t)$, $b_2(t)$ *and* $b_3(t)$ *as well as parameter* ν *are defined as follows:*

$$a(t) = p_c t + \int_0^t p_i(t)dt \tag{17a}$$
$$b_1(t) = 2t_d\sqrt{G_x}e^{-a(t)/(2t_d)} \tag{17b}$$
$$b_2(t) = e^{jp_c\sqrt{G_c}t} \tag{17c}$$
$$b_3(t) = e^{-jp_c\sqrt{G_c}t} \tag{17d}$$
$$\nu = -2t_d\sqrt{G_c}j. \tag{17e}$$

Proof: If $u(t)$ is equal to zero and $p(t) = 1$[1], a second-order scalar differential equation involving $x_2(t)$ may be formulated as follows:

$$x_2''(t) + 2x_2'(t) + [1 + G(t)]x_2(t) = 0. \tag{18}$$

This equation was obtained through the manipulation of (8a) in order to get rid of $x_1(t)$ and its derivative.

In order to solve this equation in the time interval $0 \le t \le t_c$, the following substitution will be considered: [22]

$$w = e^{-t/t_d} \tag{19}$$

to transform (18) into a simpler one. After the change of independent variables in (18) via the previous relation, the following equation is obtained:

$$w^2x''(w) + w(1 - 2t_d)x'(w) + t_d^2(1 + G_c - G_x w)x(w) = 0. \tag{20}$$

[1]The value of function $p(t)$ has units of radians per second. However, these units are omitted in this proof for the sake of simplicity.

This equation has to be simplified by means of a second change of variables. For this purpose, the transformation

$$x(w) = w^m v(w) \tag{21}$$

will be used [22]. In this expression, coefficient m stands for any of the solutions of the following equation:

$$m^2 - 2t_d m + t_d^2(G_c + 1) = 0. \tag{22}$$

If $m = t_d(1 + \sqrt{G_c}j)$, (20) may be cast out as follows:

$$wv''(w) + (1 - 2\sqrt{G_c}j)v'(w) - t_d^2 G_x v(w) = 0. \tag{23}$$

This equation has the following solution [22]:

$$v(w) = w^{-t_d\sqrt{G_c}j} Z_\nu(2jt_d\sqrt{G_x w}) \tag{24}$$

where $Z_\nu(\eta)$ is an arbitrary solution of the Bessel equation

$$\eta^2 z''(\eta) + \eta z'(\eta) + (\eta^2 + 4t_d^2)z(\eta) = 0. \tag{25}$$

In expression (24), parameter ν is defined as in (17e). Given that ν is, in general, a quantity which is not necessarily integer, $Z_\nu(\eta)$ may be written as follows: [24]

$$Z_\nu(\eta) = C_1 J_\nu(\eta) + C_2 J_{-\nu}(\eta) \tag{26}$$

where $J_\nu(\eta)$ is the Bessel function of the first kind. Using this expression, the solution of (23) is given by

$$v(w) = w^{-t_d\sqrt{G_c}j}\left[C_1 I_\nu\left(2t_d\sqrt{G_x w}\right) + C_2 I_{-\nu}\left(2t_d\sqrt{G_x w}\right)\right]. \tag{27}$$

In this expression, $J_\nu(\eta)$ has been replaced by the modified Bessel function of the first kind $I_\nu(\eta)$ since the original argument of $Z_\nu(\eta)$ in expression (24) is imaginary [24].

If the variable transformations given in expressions (19) and (21) are applied to expression (27), the solution of (18) is given by the following expression:

$$x_2(t) = e^{-t}\left[C_1 I_\nu\left(2t_d\sqrt{G_x}e^{-t/(2t_d)}\right) + C_2 I_{-\nu}\left(2t_d\sqrt{G_x}e^{-t/(2t_d)}\right)\right]. \tag{28}$$

If $p(t) = 1$ in (8), $x_1(t)$ may be written as follows:

$$x_1(t) = x_2'(t) + x_2(t). \tag{29}$$

The solution given in expression (28) for (18) may be used together with the previous relation in order to obtain a valid solution for $x_1(t)$ when $p(t) = 1$ and $u(t) = 0$. After some algebra, it turns out that $x_1(t)$ is given by

$$\begin{aligned} x_1(t) = -\frac{1}{2}\sqrt{G_x}e^{-t(1+1/(2t_d))}\cdot \Big(&C_1\left[I_{\nu+1}\left(2t_d\sqrt{G_x}e^{-t/(2t_d)}\right) + I_{\nu-1}\left(2t_d\sqrt{G_x}e^{-t/(2t_d)}\right)\right] \\ &+ C_2\left[I_{-\nu+1}\left(2t_d\sqrt{G_x}e^{-t/(2t_d)}\right) + I_{-\nu-1}\left(2t_d\sqrt{G_x}e^{-t/(2t_d)}\right)\right]\Big). \end{aligned} \tag{30}$$

The solutions already calculated for $x_1(t)$ and $x_2(t)$ with $p(t) = 1$ will be used to formulate a solution for (8a) with $u(t) = 0$. For this purpose, Lemma 4 will be considered. After the direct application of this Lemma to expressions (28) and (30), the following expression may be formulated for $x_1(t)$ and $x_2(t)$ which is valid in the time interval $0 \le t \le t_c$:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-a(t)} \begin{bmatrix} c_1(t) & c_2(t) \\ I_\nu(b_1(t)) & I_{-\nu}(b_1(t)) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{31}$$

where

$$c_1(t) = c_3(t)[I_{\nu+1}(b_1(t)) + I_{\nu-1}(b_1(t))] \tag{32a}$$

$$c_2(t) = c_3(t)[I_{-\nu+1}(b_1(t)) + I_{-\nu-1}(b_1(t))] \tag{32b}$$

$$c_3(t) = -\frac{1}{2}\sqrt{G_x}e^{-a(t)/(2t_d)}. \tag{32c}$$

In expression (31), constants C_1 and C_2 are chosen such that the initial conditions given in expression (15) are satisfied. For this purpose, expressions (15) and (31) are equated at $t = 0$.

For $t > t_c$, $x_1(t)$ and $x_2(t)$ may be written as follows:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-p_c t} \begin{bmatrix} \sqrt{G_c}jb_2(t) & -\sqrt{G_c}jb_3(t) \\ b_2(t) & b_3(t) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}. \tag{33}$$

In (31)–(33), functions $b_1(t)$, $b_2(t)$ and $b_3(t)$ are defined as indicated in (17). Moreover, it should be noted that the response of (18) is continuous for $t \ge 0$. For that reason, constants D_1 and D_2 in (33) must be computed such that $x_1(t)$ and $x_2(t)$ in expressions (31) and (33) will have the same value at $t = t_c$.

Finally, given that $y(t) = x_2(t)$, expression (16) is obtained. This concludes the demonstration. ■

Lemma 6: For the set of initial conditions (15)*, the homogeneous response $y(t)$ of the system described by* (8) *shows exponential asymptotic stability provided that*

- *Functions $p(t)$ and $G(t)$ are given as in expression* (10)*,*
- *Function $G_d(t)$ is given as in expression* (11)*,*
- *Function $p_i(t)$ is a function of t which adopts positive values in the interval $0 \le t \le t_c$,*
- *Constants p_c, G_x and t_d are greater than zero.*

Proof: It should be noticed that function $b_1(t)$ given in (17b) will decay from $2t_d\sqrt{G_x}$ at $t = 0$ and will tend to zero for large values of t. This function serves as an argument η to function $I_\nu(\eta)$. A series expansion for this function around $\eta = 0$ is given as follows: ([25, eq. 9.6.10]

$$I_\nu(\eta) = \left(\frac{1}{2}\eta\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}\eta^2\right)^k}{k!\Gamma(\nu + k + 1)}. \tag{34}$$

The expansion given in (34) deserves a more detailed analysis. If parameter ν is imaginary, the term $((1/2)\eta)^\nu$ will be a complex quantity. However, its magnitude is always equal to one. If α and δ are real quantities, it can be demonstrated that the term $\alpha^{j\delta}$ is given by the following expression:

$$\alpha^{j\delta} = \cos\left(\frac{1}{2}\delta\ln\left(\alpha^2\right)\right) + j\sin\left(\frac{1}{2}\delta\ln\left(\alpha^2\right)\right). \tag{35}$$

Taking into account this argument, it can be inferred that the sum on the right side of (34) will determine the magnitude of

$I_\nu(\eta)$. Function $\Gamma(\eta)$ is analytic as long as $\mathrm{Re}(\eta) > 0$. Moreover, it has no zeros in the aforementioned region of the complex plane. From the series expansion presented in expression (34), it can be seen that $I_\nu(\eta)$ will tend to zero as long as η is real and approaches zero from the right. Therefore, the magnitude of the solution given in expression (16) will be bounded by the term $e^{-a(t)}$ as t approaches t_c. Given that function $a(t)$ increases monotonically and is positive in the interval $0 \le t \le t_c$, the term $e^{-a(t)}$ will decrease in value as t approaches t_c. As $t \to \infty$, the magnitude of the solution will be bounded by function $e^{-p_c t}$. Given that $p_c > 0$, this function will also tend to zero for large t. Therefore, it can be concluded that the homogeneous response $y(t)$ associated to (8) will show exponential asymptotic stability. ■

Theorem 1: The response of the filter described by (8) *to a bounded input will be bounded as well provided that*
- *Functions $p(t)$ and $G(t)$ are given as in expression* (10),
- *Function $G_d(t)$ is given as in expression* (11),
- *Constants p_c, G_c, G_x and t_d present in expressions* (10) *and* (11) *are real and greater than zero,*
- *Function $p_i(t)$ is a function of t which adopts positive values and is bounded in value in the interval $0 \le t \le t_c$.*

Proof: All the functions present in the system matrix of (8) are bounded. According to the statement presented in Lemma 2, this condition is required to ensure that the homogeneous response of system (8) shows exponential asymptotic stability. As demonstrated in the previous lemma, $x_2(t)$ is bounded and tends exponentially to zero for large t. Therefore, it is necessary to prove that $x_1(t)$ is also bounded and tends exponentially to zero for large t. The solutions indicated for $x_1(t)$ in expressions (31) and (33) are similar to the solution given for $x_2(t)$ in (16). Following the same reasoning presented in the proof of Lemma 6, it can be concluded that the homogeneous solution of $x_1(t)$ in (8) will show exponential asymptotic stability as well. Therefore, all the conditions expressed in Lemma 2 are met and it can be concluded that the response to a bounded input of the filter represented by (8) subject to the constraints established in the main statement of this theorem will be bounded as well. ■

The previous theorem guarantees that the response of the filter to a bounded input will be also bounded provided that $p(t)$ and $G(t)$ satisfy a set of conditions which in principle are not so restrictive compared to the conditions for asymptotic stability described in expression (2) for the parameter-varying filter modeled by (1). However, a word of caution should be issued here. As established by Theorem 1, constant G_x present in expression (11) must be real and greater than zero. However, this constant should not be greater than G_c for practical reasons. In Fig. 3, the response of the filter modeled by (8) to a unit step is shown with different values of G_x. In this example, $p(t) = 1\ \mathrm{rad/s}$ and $G(t) = 100 - G_x e^{-t}$. The response of this filter is compared to the response of (8) with $p(t) = 1\ \mathrm{rad/s}$ and $G(t) = 100$. When G_x is equal to 90, the response of the parameter-varying filter has a smaller overshoot compared to the overshoot associated to the response of the LTI filter. However, if G_x is equal to 120, it can be seen that the response of the parameter-varying filter remains indeed bounded and its overshoot is still smaller than the overshoot generated by the LTI filter. However, a small undershoot is generated as well. For this reason, it was established

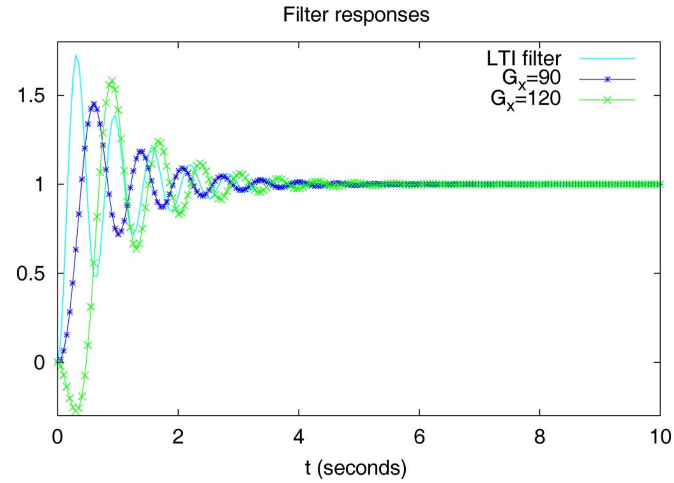

Fig. 3. Response of the filter modeled by (8) with $p(t) = 1\ \mathrm{rad/s}$ and $G_c = 100$ to a unit step for different values of G_x. The response of this system is compared to the response of a reference LTI system.

at the beginning of this section that $G(t)$ should remain positive for all t. This control rule for $G(t)$ has another justification: if $G(t)$ is positive for all t, the negative feedback loop around the cascade of the first-order stages will be retained.

From the simulation results presented in Fig. 3, it may be concluded that parameter G_x has a direct influence in the magnitude of the overshoot generated by the filter. The arguments leading to the next theorem offer a better understanding of the role of function $p_i(t)$ in expression (10) in the control of the transient behavior of the parameter-varying filter. This theorem establishes conditions to guarantee that the transient behavior of the parameter-varying filter will be shorter in duration compared to the behavior of a filter with constant parameters with the same pole location.

Theorem 2: Assuming the following constraints:
- *Functions $p(t)$ and $G(t)$ are given as in expression* (10),
- *Function $G_d(t)$ is given as in expression* (11),
- *Constants p_c, G_c, G_x and t_d present in expressions* (10) *and* (11) *are real and greater than zero,*
- *Function $p_i(t)$ is a function of t which adopts positive values and is bounded in value in the interval $0 \le t \le t_c$,*

the homogeneous response of the filter described by (8) *for the set of initial conditions* (15) *will be shorter in duration compared to the homogeneous response of the same filter determined for the same set of initial conditions with functions $p(t) = p_c$ and $G(t) = G_c$.*

Proof: According to the proof of Theorem 1, if the parameter-varying filter modeled by (8) satisfies the constraints established in the main statement of this theorem, its state variables will show exponential asymptotic stability. If functions $p(t)$ and $G(t)$ assume constant values, a LTI filter will be obtained. Its poles, which are given by expression (9), will be located on the left half of the complex plane. Therefore, its state variables will show exponential asymptotic stability as well. In both cases, the state variables of the systems considered here with $u(t) = 0$ will tend towards the equilibrium state $\mathbf{x}(t) = \mathbf{0}$ as $t \to \infty$. In order to show that the homogeneous response of the proposed parameter-varying filter tends to zero faster than the response of a filter with constant parameters, a geometric argument will be used. The solutions found for $x_1(t)$ and $x_2(t)$ in Lemma 5 for the parameter-varying filter with functions $p(t)$, $G(t)$ and $G_d(t)$ as defined in expressions (10) and (11) may be used to define a basis for the space of solutions of (8). This will be achieved by finding the transition matrix $\mathbf{\Phi}_1(t,0)$ associated to the homogeneous solution of (8).

In order to formulate the aforementioned matrix, it is necessary to calculate a fundamental matrix $\mathbf{X}_1(t)$ for system (8). This fundamental matrix may be readily identified from the general solution given for $x_1(t)$ and $x_2(t)$ in expressions (31) and (33) and it is given by

$$\mathbf{X}_1(t) = \begin{cases} \mathbf{X}_{1,1}(t) & 0 \le t \le t_c \\ \mathbf{X}_{1,2}(t) & t > t_c \end{cases} \tag{36}$$

where

$$\mathbf{X}_{1,1}(t) = e^{-a(t)} \begin{bmatrix} c_1(t) & c_2(t) \\ I_\nu(b_1(t)) & I_{-\nu}(b_1(t)) \end{bmatrix} \tag{37a}$$

$$\mathbf{X}_{1,2}(t) = e^{-p_c t} \begin{bmatrix} \sqrt{G_c} j b_2(t) & -\sqrt{G_c} j b_3(t) \\ b_2(t) & b_3(t) \end{bmatrix} \mathbf{D}. \tag{37b}$$

In the last expression, $b_1(t)$, $b_2(t)$, $b_3(t)$, $c_1(t)$ and $c_2(t)$ are defined in expressions (17) and (32). Matrix $\mathbf{D}$ is a nonsingular matrix required to ensure the continuity of $\mathbf{X}_1(t)$ at $t = t_c$. This matrix may be computed such that $\mathbf{X}_{1,1}(t_c) = \mathbf{X}_{1,2}(t_c)$.

The transition matrix $\mathbf{\Phi}_1(t,0)$ may be now calculated as follows:

$$\mathbf{\Phi}_1(t,0) = \mathbf{X}_1(t)\mathbf{X}_1^{-1}(0) \tag{38}$$

where $\mathbf{X}_1^{-1}(0)$ is the inverse of $\mathbf{X}_1(t)$ evaluated at $t = 0$. The columns of the transition matrix define a very convenient basis for the space of solutions of (8). The solution of the LTV system given in (8) subject to the set of initial conditions (15) may be written as follows:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{\Phi}_1(t,0) \begin{bmatrix} x_{1_c} \\ x_{2_c} \end{bmatrix}. \tag{39}$$

Geometrically speaking, the columns of the transition matrix constitute a set of linearly independent vectors which form a basis capable of spanning the complete space of solutions of (8). The magnitude and the direction of these vectors will change over time. Moreover, these vectors will tend to the origin of the phase plane formed by the state variables $x_1(t)$ and $x_2(t)$. At $t = 0$, the columns of $\mathbf{\Phi}_1(t,0)$ at $t = 0$ are equal to $[1\ 0]^T$ and $[0\ 1]^T$. As indicated by expression (39), the initial conditions given in expression (15) will serve only as scaling factors of the columns of $\mathbf{\Phi}_1(t,0)$. The sum of these scaled columns will define a particular homogeneous solution for (8).

In the state plane, the column vectors of the transition matrix may be thought of representing the adjacent sides of a parallelogram with one of its vertexes located at the origin. At $t = 0$, this parallelogram will be a square with area equal to 1. If the set of initial conditions (15) is considered, the scaled columns of the transition matrix will now form at $t = 0$ a rectangle whose area will be given by $x_{1_c} x_{2_c}$. These scaled columns will be also linearly independent to each other.

As time advances, the shape and the area of the parallelogram formed by the scaled column vectors of the transition matrix associated to an arbitrary homogeneous solution of (8) will change. Moreover, the general form of the homogeneous solutions of (8) will tend to zero with increasing t. Therefore, the area of the parallelogram will also tend to zero as well. A measure of the area of the parallelogram may be calculated by means of the determinant of the transition matrix. The determinant of the transition matrix $\mathbf{\Phi}(t,0)$ of an arbitrary linear system of the form (4) may be computed according to the following formula [17]:

$$\det \mathbf{\Phi}(t,0) = e^{\int_0^t \mathrm{tr}\ \mathbf{A}(t) dt} \tag{40}$$

where $\mathrm{tr}\ \mathbf{A}(t)$ stands for the trace of the system matrix $\mathbf{A}(t)$. It should be noticed that the determinant of $\mathbf{\Phi}(t,0)$ will be always different from zero for all t. This is a consequence of the linear independence of the columns of $\mathbf{\Phi}(t,0)$. The determinant of the transition matrix $\mathbf{\Phi}(t,0)$ may be interpreted as a joint measure of the amplitude of the state variables of the filter. As it will be seen soon, this quantity may be used as an indicator of the overall evolution of the dynamic behavior of the parameter-varying filter.

Using relation (40), the oriented area $A_1(t)$ of the parallelogram associated to a given homogeneous solution of (8) with function $p(t)$ defined in expression (10) and subject to the set of initial conditions (15) may be written as follows:

$$\begin{aligned} A_1(t) &= x_{1_c} x_{2_c} e^{-2\int p(t)dt} \\ &= \begin{cases} x_{1_c} x_{2_c} e^{-2a(t)} & 0 \le t \le t_c \\ A_x e^{-2p_c(t-t_c)} & t > t_c. \end{cases} \end{aligned} \tag{41}$$

In this expression, constant A_x is given by

$$A_x = x_{1_c} x_{2_c} e^{-2a(t_c)}. \tag{42}$$

It should be noticed that function $A_1(t)$ is continuous at $t = t_c$. This is a consequence of the continuity of the fundamental matrix given in (36) for the determination of $\det \mathbf{\Phi}_1(t,0)$.

If $p(t) = p_c$ and $G(t) = G_c$, the filter represented by (8) will be LTI and its poles will define how fast its transient response will decay. For this system, it is relatively simple to determine the transition matrix $\mathbf{\Phi}_2(t,0)$ which will characterize its solution space. This matrix is given below:

$$\mathbf{\Phi}_2(t,0) = d_1(t) \begin{bmatrix} d_2(t) + 1 & j\sqrt{G_c}(d_2(t) - 1) \\ -\frac{j(d_2(t)-1)}{\sqrt{G_c}} & d_2(t) + 1 \end{bmatrix} \tag{43}$$

where

$$d_1(t) = \frac{e^{-p_c t(1+j\sqrt{G_c})}}{2} \tag{44a}$$

$$d_2(t) = e^{2jp_c\sqrt{G_c}t}. \tag{44b}$$

It turns out that the oriented area $A_2(t)$ of the parallelogram associated to the homogeneous solution of (8) with constant parameters and subject to the set of initial conditions (15) is given by the following expression:

$$A_2(t) = x_{1_c} x_{2_c} e^{-2p_c t}. \tag{45}$$

If the arguments of the exponential functions in $A_1(t)$ and $A_2(t)$ are compared for $0 < t \leq t_c$, the argument of the exponential function in $A_1(t)$ is bigger in absolute value than the argument of the exponential function in $A_2(t)$. Therefore, the area of the parallelogram defined by the columns of the fundamental matrix associated to (8) with function $p(t)$ as indicated in expression (10) decreases faster in the time interval $0 \leq t \leq t_c$ compared to the area of the parallelogram defined by the columns of the fundamental matrix of the LTI filter.

After $t > t_c$, the area of the parallelogram of the LTV system will decrease with the same speed as the area of the parallelogram of the system with constant parameters. However, given that the area is also a continuous function of t, the area of the parallelogram defined by the columns of the transition matrix of the LTV system will be also smaller after $t > t_c$ compared to the area of the parallelogram defined by the columns of the transition matrix of the LTI system. In fact, the following relation may be established for the ratio $R(t)$ of the two areas at $t = t_c$:

$$R(t_c) = \frac{A_1(t_c)}{A_2(t_c)} = A_x e^{2p_c t_c} = e^{-2\int_0^{t_c} p_i(t)dt}. \qquad (46)$$

In the previous expression, $p_i(t)$ is positive in the range $0 \leq t \leq t_c$. Therefore, the quantity given in expression (46) will be smaller than 1.

Given that all the solutions involved in the formulation of the fundamental matrices of the systems considered here have a decreasing character for large values of t, the progressive reduction of the area of the parallelogram defined by the columns of the transition matrix of the LTV system indicates that its homogeneous behavior must be of shorter duration compared to the homogeneous behavior of the LTI system. This completes the proof. ■

Theorem 2 may be conveniently explained by graphical means. In order to do this, the homogeneous response of a LTI filter which will serve as a reference point will be compared to the homogeneous response of the proposed parameter-varying filter. If functions $p(t)$ and $G(t)$ adopt constant values in (8), a LTI system will be obtained. For demonstrative purposes, it will be assumed that $p(t) = 1\ \mathrm{rad/s}$ and $G(t) = 100$. It should be noticed that with this choice for $p(t)$ and $G(t)$, the position of the poles of the filter (which are given by (9)) will not lead to a useful lowpass approximation (Butterworth, Chebyshev or Bessel, for instance). However, the response of the filter will have an appreciable ringing. For the parameter-varying filter, the functions considered for the variation of its parameters are given as in expression (10) with $p_c = 1\ \mathrm{rad/s}$, $G_c = 100$ and $t_c = 1.5\ \mathrm{s}$. Moreover, functions $p_i(t)$ and $G_d(t)$ are given below

$$p_i(t) = e^t \qquad (47a)$$

$$G_d(t) = -90e^{-t}. \qquad (47b)$$

In Fig. 4, the evolution in the phase plane of the homogeneous behavior from $t = 0$ of the LTI filter and the parameter-varying filter is shown. For reference purposes, the initial condition $[x_1(0)\ x_2(0)]^T = [1\ 1]^T$ was considered to define a unique solution for (8). With this particular choice for the initial condition assigned to (8), the evolution of the columns of the transition matrices associated to the aforementioned systems may be better assessed from the figure since they are not scaled. At $t = 0$, the parallelograms formed by the columns of each of the transition matrices have an area equal to 1. As t increases, these parallelograms change in shape and also in area. It can be easily seen that the area of the parallelogram defined by the columns of the transition matrix of (8) with functions $p(t)$ and $G(t)$ as given in expression (10) decreases considerably faster compared to the LTI case. At $t = t_c$, the ratio $R(t)$ of the areas of the parallelograms $A_1(t)$ and $A_2(t)$ defined by the columns of the transition matrix associated to the parameter-varying filter and the LTI filter may be calculated according to the expression given in (46)

$$R(t_c) = e^{-2\int_0^{t_c} e^t dt} \approx 9.459 \cdot 10^{-4}. \qquad (48)$$

In order to verify the stability properties of the parameter-varying filter of this example, the magnitude of the modes of its homogeneous response was computed as described in [21]. A graph of these modes is given in Fig. 5. As it can be seen, the envelope of the modes decrease rapidly from $t = 0$ to $t = t_c$. After this time instant, the envelope of the modes decrease at a exponential constant rate. Therefore, the parameter-varying filter considered in this example shows BIBO stability. The magnitude of the modes of the prototype LTI filter is shown in Fig. 6. As it can be seen, these modes decay exponentially as well. However, their magnitudes are bigger and decay slower from $t = 0$ to $t = t_c$ compared to the modes of the parameter-varying filter.

Finally, Theorem 2 is valid if the state variables of the filters considered in its main statement show exponential asymptotic stability. In order to understand this claim, it should be noted that the system matrix associated to an arbitrary state representation of a second-order LTI system with a small positive eigenvalue and a large negative eigenvalue has a negative trace [26]. Therefore, the area of the parallelogram formed by the columns of the transition matrix of the system will tend to zero for large t. However, the behavior of the system is unstable due to the positive eigenvalue. Fortunately, the proof of Theorem 1 implies that the state variables of the parameter-varying filter show exponential asymptotic stability.

IV. The Amplitude-Based Parameter Control Scheme for the Proposed Parameter-Varying Filter

After discussing the properties of the proposed lowpass parameter-varying filter, it is necessary to consider the problem of the induction of changes in the values of the filter parameters to reduce the duration of its transient behavior. As exposed in Section II, a control scheme is required to automatically vary in time the values of the filter parameters as a response to a sudden variation of the input signal. The aim of this section is to present a modified version of the control presented in Section II for a lowpass parameter-varying filter which may used in conjunction with the proposed filter. This control may be fully implemented using analog blocks as well.

The proposed control is depicted in Fig. 7. In the new control scheme, the ideal delay element present in the control scheme shown in Fig. 1 was approximated with a block characterized

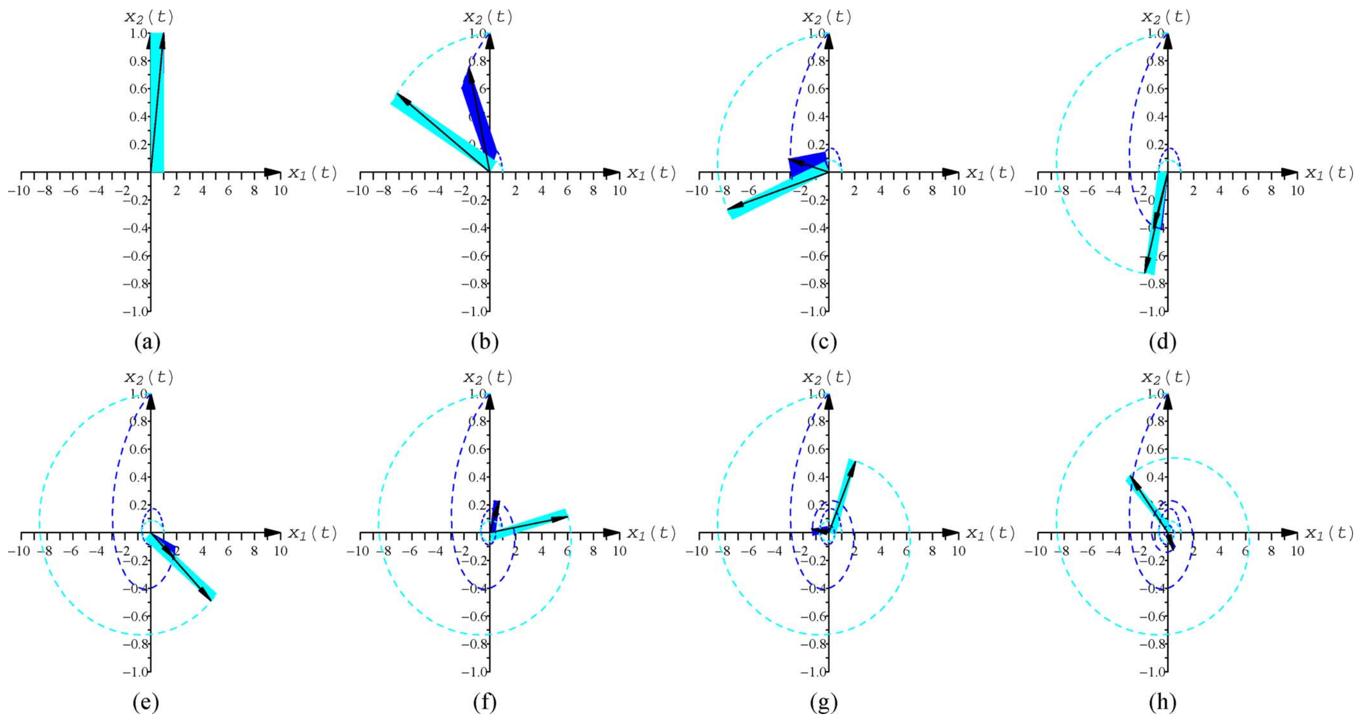

Fig. 4. Evolution in the phase plane of the homogeneous behavior of the parameter-varying filter (blue trace) and the homogeneous behavior of the LTI filter (cyan trace) for the initial condition $[x_1(0)\ x_2(0)]^T = [1\ 1]^T$. (a) $t = 0$ s. (b) $t = 0.1$ s. (c) $t = 0.2$ s. (d) $t = 0.3$ s. (e) $t = 0.4$ s. (f) $t = 0.5$ s. (g) $t = 0.6$ s. (h) $t = 0.7$ s.

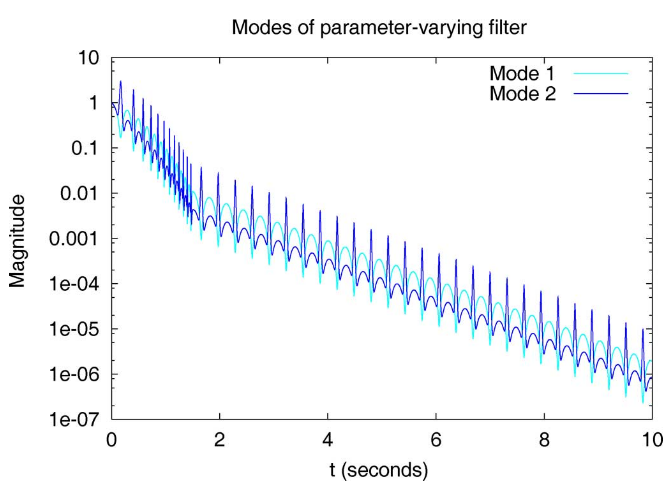

Fig. 5. Magnitude of the modes of the parameter-varying filter.

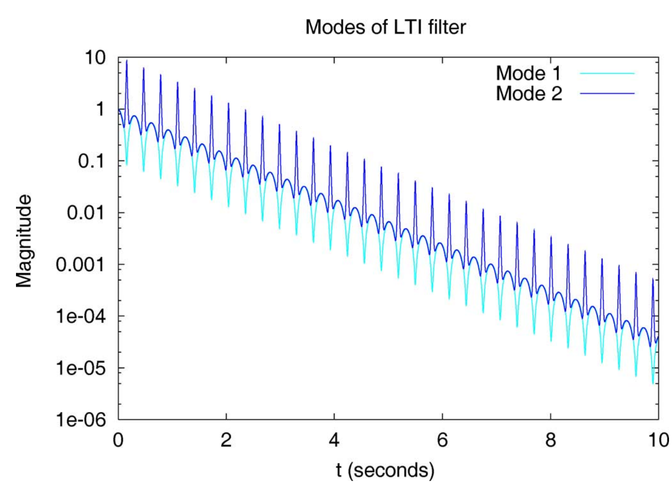

Fig. 6. Magnitude of the modes of the LTI filter.

by a first-order Padé-approximated transfer function. The transfer function $H_d(s)$ of the approximated delay element is given below

$$H_d(s) = \frac{2 - s\tau}{2 + s\tau} \tag{49}$$

where s stands for the complex variable and τ is the required delay. Besides its simplicity, the approximation considered in expression (49) for the delay offers an additional advantage: given that the output of the approximated delay must be subtracted from its input, it is possible to define a new transfer function $H_s(s)$ which will combine both operations. This transfer function is defined as follows:

$$\begin{aligned} H_s(s) &= 1 - \frac{2 - s\tau}{2 + s\tau} \\ &= \frac{2s\tau}{2 + s\tau}. \end{aligned} \tag{50}$$

The resulting transfer function can be readily identified as the transfer function of a highpass filter with a gain of 2. As it will be seen later on, the approximation considered for the delay given

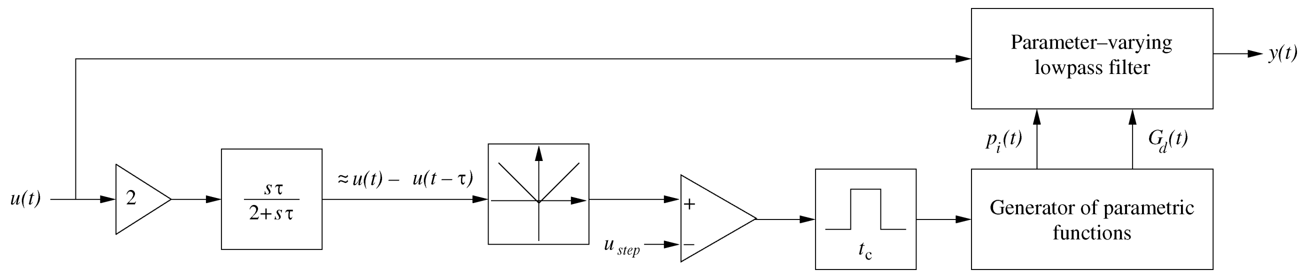

Fig. 7. Amplitude-based parameter control scheme for the proposed parameter-varying filter.

in expression (49) suffices to implement an appropriate control scheme.

The new control scheme includes a non-retriggerable monostable module which is triggered when the output signal of the comparator goes from low state to high state. The monostable circuit will generate a pulse of width t_c. This pulse will establish a time interval in which the amplitude-based parameter control scheme is active and the correction of the transient behavior of the filter is applied. Moreover, the pulse generated by the monostable will serve also as a time base for the generation of functions $p_i(t)$ and $G_d(t)$ present in expression (10).

In order to clarify this last point, consider the system presented in Fig. 8 for the generation of $G_d(t)$. This system is composed by a monostable module, a highpass filter and a half-wave rectifier. Unlike the monostable module present in the control scheme indicated in Fig. 7, the monostable module of Fig. 8 will generate a pulse of width $t_c/2$ for reasons which will be more evident later on. Assuming that this module will generate a pulse (or a sequence of pulses) whose amplitude is equal to one, the highpass filter will generate as a response decaying exponential signals with a time constant equal to t_d as shown in Fig. 9. As it can be seen, when the output of the monostable is equal to one, a decaying signal is generated. However, once the output of the monostable block returns to zero, an increasing signal of negative amplitude is also generated at the output of the highpass filter. In order to eliminate this unwanted signal, a half-wave rectifier is used. Assuming that $t_c/2 > 5t_d$, the non-retriggerable monostable present in the control scheme depicted in Fig. 7 ensures that the output of the highpass filter will be equal to zero before the generation of another decaying exponential. Thanks to this mechanism, a decaying exponential with the same amplitude will be generated whenever it is required.

In order to further simplify the design of the amplitude-based parameter control scheme presented in Fig. 7, it will be assumed that function $p_i(t)$ takes the following form:

$$p_i(t) = (p_{sc} - 1) \cdot p_c. \tag{51}$$

In this expression, p_{sc} stands for a positive constant greater than 1. As it will be seen in the next section, p_{sc} has a fundamental role in the improvement of the transient response of the filter. It should be noticed that the function given in (51) satisfies the requirements established in Theorem 2 to induce a reduction in the duration of the response of the filter represented by (8) compared to the response of the same filter when $p(t) = p_c$ and $G(t) = G_c$.

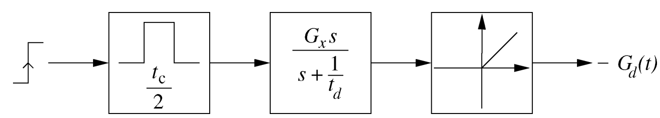

Fig. 8. System for the generation of function $G_d(t)$.

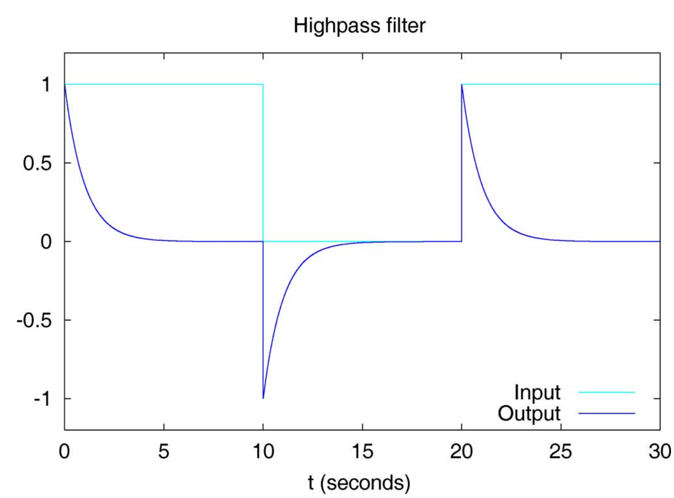

Fig. 9. Response of a highpass filter to a train of pulses. In this example, $t_d = 1$ s.

Finally, the control scheme used to induce variations in the values of the parameters of the lowpass parameter-varying filter will shift in time functions $p(t)$ and $G(t)$ as suggested by the following mathematical relation:

$$p(t) = \begin{cases} p_c & t - t_a < 0 \\ p_c + p_i(t - t_a) & 0 \leq t - t_a \leq t_c \\ p_c & t - t_a > t_c \end{cases} \tag{52a}$$

$$G(t) = \begin{cases} G_c & t - t_a < 0 \\ G_c + G_d(t - t_a) & 0 \leq t - t_a \leq t_c \\ G_c & t - t_a > t_c \end{cases}. \tag{52b}$$

In this expression, constant t_a stands for *each* of the time instants in which a variation in the values of the filter parameters must be induced as a consequence of a sudden change in the amplitude of the input signal.

V. Design Guidelines for the Lowpass Parameter-Varying Filter

The theory presented in the previous section may be now used to formulate some guidelines which may be applied in the design process of a lowpass parameter-varying filter based on the structure presented in Fig. 2. The starting point of the design will be a second-order LTI lowpass filter (or a second-order stage of a LTI lowpass filter of order greater than two) whose transient behavior must be improved by means of the variation of its parameters. Moreover, the poles of this filter must be obtained from a lowpass filter approximation which does not involve zeros in its transfer function (this is the case with Butterworth or Chebyshev approximations, for instance). It is assumed that the second-order lowpass filter whose transient behavior should be improved has complex conjugate poles p_1 and p_2 of the form

$$p_1 = -p_r + p_i j \tag{53a}$$

$$p_2 = -p_r - p_i j \tag{53b}$$

where p_r and p_i are real quantities greater than zero.

A summarized description of the parameters adjusted during the proposed design procedure appears in Table I. The steps involved in their selection will be now outlined:

1) Using expression (9), the values of constants p_c and G_c may be readily determined as follows:

$$p_c = p_r \tag{54a}$$

$$G_c = \left(\frac{p_i}{p_r}\right)^2. \tag{54b}$$

2) Once parameters p_c and G_c have been estimated, the overshoot generated by the original filter must be reduced in magnitude. For this purpose, it will be assumed that the input to the filter will be a unit step function. The maximum amplitude of the overshoot M_o of the LTI filter is given by

$$M_o = e^{-\pi/\sqrt{G_c}}. \tag{55}$$

As it was shown in Section III, the magnitude of the overshoot generated by the lowpass parameter-varying filter can be conveniently reduced by selecting an adequate value for G_x for function $G(t)$. Moreover, the value of constant t_d will also have an impact in the magnitude of the overshoot. Although expression (16) may be used to determine the maximum amplitude of the overshoot of the parameter-varying filter in terms of its defining parameters, the resulting expression is too complicated for its use in a normal design process. For this reason, computer-based graphical methods have to be considered in order to estimate the magnitude of the overshoot for different values of G_x and t_d. In Fig. 10, for instance, a plot of the maximum amplitude of the overshoot is depicted as a function of G_x and t_d. For reference purposes, the values of G_x may be chosen in the range $[0.1G_c, 0.9G_c]$, whereas the values of t_d may be selected from the range $[0.1/p_c, 2/p_c]$.

3) The settling time of the filter has to be adjusted by means of constant p_{sc}. For a second-order LTI lowpass filter with complex-conjugate poles, the location of its poles will determine its settling time $t_s(\epsilon)$ for a given approximation error ϵ. Assuming that the original settling time of the filter designed in Step 2 for a given error ϵ is equal to $t_{s_o}(\epsilon)$, constant p_{sc} may be adjusted such that the response of the parameter-varying filter to a unit step may attain a new settling time $t_{s_n}(\epsilon)$. For this aim

$$p_{sc} = \frac{t_{s_o}(\epsilon)}{t_{s_n}(\epsilon)}. \tag{56}$$

Like in Step 2, it is not possible to derive a simple expression for the settling time of the lowpass parameter-varying filter $t_s(\epsilon)$ for a given approximation error ϵ. For that reason, computer-based methods have to be used again. In Fig. 11, a plot of the settling time of the filter for $\epsilon = 10^{-2}$ as a function of G_x and t_d is presented. As it can be seen, the variation induced in $G(t)$ through the aforementioned parameters also induces a change in the settling time of the parameter-varying filter.

4) As a consequence of the scaling in time of the response of the parameter-varying filter by means of constant p_{sc} in the previous step, constant t_d must be adjusted as well. This step finds its justification in Lemma 4. In Step 2, the value chosen for the aforementioned variable was computed assuming that $p_{sc} = 1$. In other words, no modulation of the time variable was considered yet. Assuming that the original constant chosen in Step 2 is t_{d_o}, the new constant t_{d_n} may be calculated as follows:

$$t_{d_n} = \frac{t_{d_o}}{p_{sc}}. \tag{57}$$

5) Constant t_c must be now adjusted. Function $p(t)$ must be temporarily increased in value during a time interval whose duration is *at least* equal to $t_{s_n}(\epsilon)$ to reduce the duration of the transient behavior of the parameter-varying filter. Therefore, constant t_c should be equal to $t_{s_n}(\epsilon)$. However, if the system depicted in Fig. 8 is used for the generation of $G_d(t)$, the value of this function must approach zero for t sufficiently large. This is required to ensure the adequate operation of the aforementioned system. Therefore, function $G_d(t)$ should be generated during a time interval of at least $5t_{d_n}$. Moreover, an additional wait time interval of at least $5t_d$ should be considered in order to ensure that the output of the highpass filter of Fig. 8 will return to its zero state after the output of the monostable block preceding it goes to zero. Consequently, the latency time required by the system of Fig. 8 to generate two consecutive exponential decaying signals is at least $10t_d$. Given that t_c also establishes a time base for the generation of this signal as indicated in the amplitude-based parameter control scheme of Fig. 7, the following relation holds:

$$t_c = \max\left(t_{s_n}(\epsilon), 10t_{d_n}\right). \tag{58}$$

6) Finally, constants τ and u_{step} must be adjusted. These constants will define the minimum rate of change in time of the input signal v_{th} which will trigger the action of the amplitude-based parameter control scheme. The aforementioned rate of change may be defined as follows:

$$v_{th} = \frac{u_{step}}{\tau}. \tag{59}$$

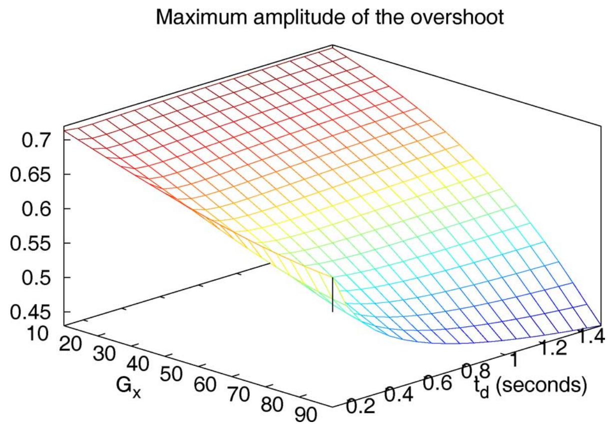

Fig. 10. Maximum amplitude of the overshoot of the parameter-varying filter as a function of parameters t_d and G_x with $p(t) = 1\ \mathrm{rad/s}$ and $G_c = 100$.

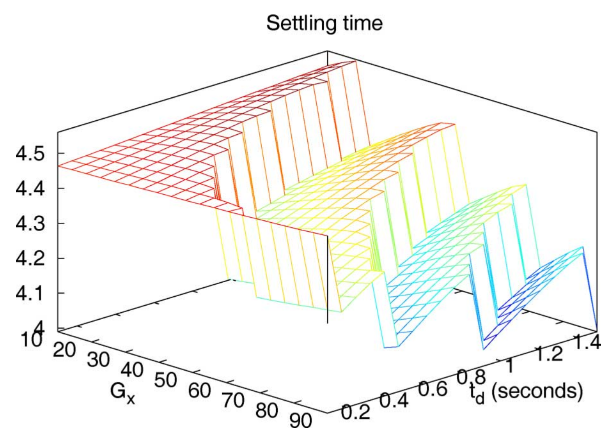

Fig. 11. Settling time of the parameter-varying filter as a function of parameters t_d and G_x with $p(t) = 1\ \mathrm{rad/s}$ and $G_c = 100$.

Simply stated, v_{th} defines the minimum slope (positive or negative) which the input signal should have in order to consider it as a signal which has a relatively fast change in time. This change in time is also coupled with a minimum change in amplitude of the input signal. Constant u_{step} determines such specification. During the design process, parameters v_{th} and u_{step} must be fixed to a given value taking into account the properties in the time domain of the signal to be processed by the parameter-varying filter. Constant τ may be computed from the aforementioned parameters using (59).

The design process described before is relatively simple and it can be easily automated. Moreover, it can be conveniently adapted in the design of filters of order greater than two. As a design example, the design of a fourth-order lowpass Chebyshev filter with a 1-dB ripple in its passband and a cutoff frequency of 1 rad/s will be considered. For this filter, its transient behavior is to be improved by a factor of two by means of the variation of its parameters. In Table II some of the design parameters considered in the design of the LTI filter and the parameter-varying filter are presented.

In the design of a fourth-order lowpass parameter-varying filter based on the cascade of two second-order lowpass parameter-varying filters as indicated in Fig. 12, a somewhat different strategy has been adopted to simplify the design process. First of all, a single parameter p_{sc} is used for the temporary adjustment of the bandwidth for both stages through functions $p_{i_1}(t)$ and $p_{i_2}(t)$. The justification for this step finds once more its origin in Lemma 4. At the same time, a single constant t_d is considered for the control of the overshoot in both stages. Although it is possible to consider different values of this constant for the adjustment of the functions $G_{d_1}(t)$ and $G_{d_2}(t)$ associated to each stage, the selection becomes somewhat more difficult in that particular case.

In this example, parameters G_{x_1} and G_{x_2} present in functions $G_{d_1}(t)$ and $G_{d_2}(t)$ will be assigned different values to optimize the maximum amplitude of the overshoot generated by the filter. For each pair of complex conjugate poles $p_{r_1} \pm p_{i_1}j$ and $p_{r_2} \pm p_{i_2}j$, constants p_{c_1}, G_{c_1}, p_{c_2} and G_{c_2} must be calculated using expression (54). After this, an initial value for parameter t_{d_o} is selected for both stages. In order to simplify the selection process, this constant is chosen taking into account the

TABLE I
PARAMETERS ADJUSTED DURING THE DESIGN PROCESS OF THE PROPOSED SECOND-ORDER PARAMETER-VARYING FILTER.

Parameter(s)	Description
p_c, G_c	These constants determine the response of the filter when its parameters are not subject to change
G_x, t_d	These constants determine the maximum overshoot of the filter
p_{sc}	This constant is used to establish the settling time of the filter
t_c	This constant determines the length of the time interval in which the control presented in Figure 7 is active
u_{step}	This constant defines the minimum variation of amplitude required to induce a variation in the values of the filter parameters
τ	Together with u_{step}, this constant defines the minimum rate of change in time of the input signal required to induce a variation in the values of the filter parameters

TABLE II
DESIGN PARAMETERS AND PERFORMANCE INDEXES TO A UNIT STEP INPUT OF A FOURTH-ORDER LOWPASS PARAMETER-VARYING FILTER BASED ON A CHEBYSHEV FILTER WITH A 1-DB PASSBAND RIPPLE AND A CUTOFF FREQUENCY OF 1 RAD/S.

Design parameter	**Parameter-varying filter**	**Prototype LTI filter**
p_{r_1} (rad/s)	0.13954	0.13954
p_{i_1} (rad/s)	0.98338	0.98338
p_{r_2} (rad/s)	0.33687	0.33687
p_{i_2} (rad/s)	0.40733	0.40733
k_{t_d}	0.5	
t_d (s)	1.79	
G_{x_1}	10	
G_{x_2}	0.4	
$t_{s_{old}}$ (s)	23.06	
p_{sc}	2	
u_{step}	0.5	
v_{th} (s^{-1})	0.05	
τ (s)	10	

Performance index	**Parameter-varying filter**	**Prototype LTI filter**
ϵ	0.01	0.01
$t_s(\epsilon)$ (s)	11.53	25.23
M_o	0.07	0.22

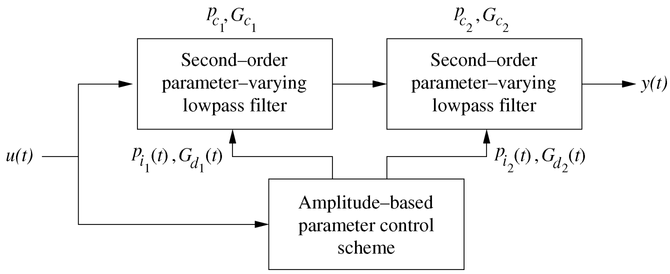

Fig. 12. Fourth-order parameter-varying filter based on the cascade of two second-order stages.

value of the largest time constant associated to the original filter. Constant t_{d_o} may be expressed as follows:

$$t_{d_o} = k_{t_d} \max\left(\frac{1}{p_{c_1}}, \frac{1}{p_{c_2}}\right) \tag{60}$$

where p_{c_1} and p_{c_2} are the parameters calculated in Step 1 of the design process for each of the complex-conjugate pole pairs. In this expression, factor k_{t_d} serves as a scaling constant for the largest time constant of the original filter. The value of this scaling factor is setup by the designer during the design process. For reference purposes, the value of k_{t_d} should be located between 0.1 and 2. It should be noticed that a larger k_{t_d} will lead to a smaller overshoot. However, the time interval in which the control scheme is active will be longer. During this interval, the control scheme is unable to induce another corrective action in the case that the input signal varies abruptly. Once the value of t_{d_o} has been determined, the overshoot of the filter is numerically calculated as a function of parameters G_{x_1} and G_{x_2}. The rest of the design strategy goes on as in Step 3 and further.

The transient behavior of the original LTI filter and the parameter-varying filter to a unit step were simulated using Octave. The simulation results for both systems are shown in Fig. 13. As it can be seen from this figure, the response of the lowpass parameter-varying filter has a faster transient response with a reduced overshoot compared to the response of the prototype LTI lowpass filter. A summary of the performance indexes of both filters to a unit step input may be found in Table II as well. Finally, Fig. 14 shows the response of the parameter-varying filter to a train of pulses. In this figure, the action of the amplitude-based control scheme depicted in Fig. 7 for the variation of the parameters of the lowpass parameter-varying filter can be better appreciated. For this specific example, the edges of the input signal have different slopes. At $t = 50$ s, the input signal decreases to zero with a slope of $-0.1\ \mathrm{s}^{-1}$, whereas at $t = 100$ s, the input signal goes to one with a slope of $0.05\ \mathrm{s}^{-1}$. Moreover, parameters v_{th} and u_{step} were chosen to be equal to $0.05\ \mathrm{s}^{-1}$ and 0.5, respectively. As it can be seen from the simulation results presented in Fig. 14, the transient response of the parameter-varying filter is compensated only when the absolute value of the slope of the input signal is larger than $0.05\ \mathrm{s}^{-1}$.

VI. Conclusions

In this article, a mathematical model for a second-order lowpass parameter-varying filter based on the cascade connection of two identical first-order lowpass parameter-varying stages and a global negative feedback loop was presented. The stability

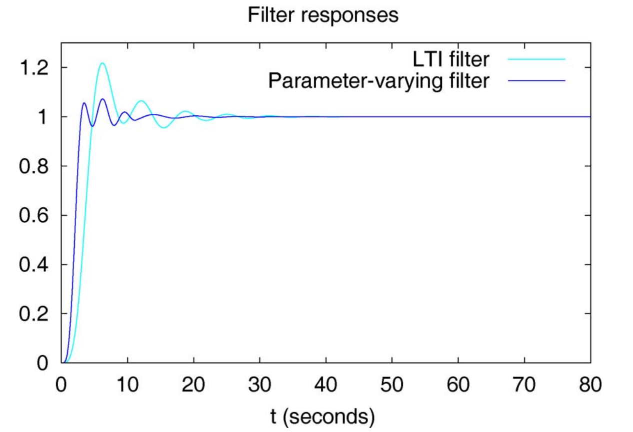

Fig. 13. Response to a unit step input of a fourth-order LTI reference lowpass filter and a fourth-order parameter-varying filter based on the scheme presented in Fig. 12.

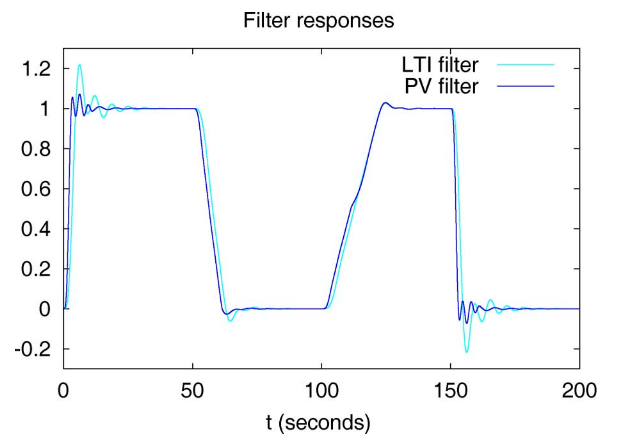

Fig. 14. Response to a train of pulses with different slopes for its edges of a fourth-order LTI reference lowpass filter and the proposed fourth-order lowpass parameter-varying filter with the amplitude-based parameter control scheme of Fig. 7.

properties of the filter guarantee that it will show a bounded response to a bounded input. Moreover, the stability properties of the filter are not compromised if functions $p(t)$, $G(t)$ and $G_d(t)$ considered to generate a variation of its parameters show a deviation in value with respect to their coefficients in the original model. This last property makes this model attractive for its implementation as an analog system with circuit blocks which do not need to be accurate. Finally, simulation results demonstrate that the proposed model for the second-order lowpass parameter-varying filter has an improved transient response compared to a reference LTI lowpass filter.

Acknowledgment

The authors thank P.H. Duran, J. Piskorowski, R. Kaszyński, A. Sarmiento Reyes and J.C. Sánchez García for useful discussions regarding this work. The authors are particularly indebted to the anonymous reviewers who contributed with their comments to the overall improvement of this article.

References

[1] R. Kaszyński, "A proposal of non-stationary low-pass Chebyshev's filters," in *Proc. 1996 IEEE Int. Conf. Emerging Technologies Factory Automation (EFTA'96)*, Kauai, HI, 1996, vol. 2, pp. 759–762.

[2] R. Kaszyński, "Stability of parametric, analog low-pass filters," in *Proc. 1999 7th IEEE Int. Conf. Emerging Technologies Factory Automation (EFTA'99)*, Barcelona, Spain, 1999, vol. 1, pp. 579–582.

[3] J. Piskorowski, "A concept of Q-varying continuous-time notch filter with improved dynamic behavior," in *Proc. IEEE Instrum. Meas. Tech. Conf.*, Singapore, 2009, pp. 913–917.

[4] J. Piskorowski and M. A. Gutiérrez de Anda, "A new concept of continuous-time narrow bandpass Q-varying filter with transient suppression," in *Proc. 2010 IEEE Int. Symp. Circuits Syst.*, Paris, France, 2010, pp. 1272–1275.

[5] M. Jaskula and R. Kaszyński, "Using the parametric time-varying analog filter to average evoked potential signals," *IEEE Trans. Instrum. Meas.*, vol. 53, no. 6, pp. 709–715, Jun. 2004.

[6] A. Kaszkin, R. Kaszyński, P. Pietrzak, W. Stiel, and , Siemens AG, "Verfahren und Vorrichtung zum Gravimetrischen Dosieren von Schütt- oder Fließfähigen Wägegut (Method and Apparatus for Gravimetrically Metering Pourable or Flowable Material to be Weighted)," European Patent Appl. No. 06012230.6, Jun. 2006.

[7] J. Piskorowski and T. Barciński, "Dynamic compensation of load cell response: A time-varying approach," *Mech. Syst. Signal Process.*, vol. 22, no. 7, pp. 1694–1704, Oct. 2008.

[8] A. Kaszkin, W. Stiel, P. Pietrzak, and , Siemens AG, "Kontrollwaage und Verfahren zum Kontrollwiegen (Checkscales and Control for Checkweighing)," European Patent Appl. No. 06012226.4, Jun. 2006.

[9] P. Pietrzak, "Fast filtration method for static automatic catchweighting instruments using a non-stationary filter," *Metrol. Meas. Syst.*, vol. 16, no. 4, pp. 669–676, Oct.–Dec. 2009.

[10] P. Pietrzak, "Dynamic mass measurement using a discrete time-variant filter," in *Proc. IEEE 26th Conv. Elect. Electron. Eng. Israel (IEEEI)*, Eilat, Israel, 2010, pp. 000151–000155.

[11] J. Walczak and A. Romanowska, "Second order LTV sections with simultaneous variation of both parameters," *Elektryka*, vol. 206, no. 2, pp. 53–66, 2008.

[12] M. A. Gutiérrez de Anda, I. Meza Dector, and J. C. Sánchez García, "A parameter-varying lowpass filter with reduced transient response," in *Proc. Eur. Conf. Circuit Theory Design*, Antalya, Turkey, 2009, pp. 149–152.

[13] M. A. Gutiérrez de Anda, I. Meza Dector, J. C. Sánchez García, R. Kaszyński, and J. Piskorowski, "A first-order parameter-varying filter using dynamic translinear techniques," presented at the 14th Int. IEEE/IFAC Conf. Methods Models Autom. Robot., Miedzyzdroje, Poland, 2009.

[14] L. A. Zadeh and C. A. Desoer, *Linear System Theory: The State Space Approach*. New York: McGraw Hill, 1963.

[15] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice Hall, 1980.

[16] W. J. Rugh, *Linear System Theory*. Englewood Cliffs, NJ: Prentice Hall, 1996.

[17] L. Y. Adrianova, *Introduction to Linear Systems of Differential Equations*. Providence, RI: American Mathematical Society, 1995.

[18] P. J. Antsaklis and A. N. Michel, *A Linear Systems Primer*. Boston, MA: Birkhäuser, 2007.

[19] B. D. O. Anderson, "External and internal stability of linear systems – A new connection," *IEEE Trans. Autom. Control*, vol. 17, no. 1, pp. 107–111, Feb. 1972.

[20] L. M. Silverman and B. D. O. Anderson, "Controllability, observability and stability of linear systems," *SIAM J. Contr.*, vol. 6, no. 1, pp. 121–130, Feb. 1968.

[21] J. Piskorowski and M. A. Gutiérrez de Anda, "A new class of continuous-time delay-compensated parameter-varying lowpass elliptic filters with improved dynamic behavior," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 56, no. 1, pp. 179–189, Jan. 2009.

[22] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*. Boca Raton, FL: CRC Press, 2003.

[23] W. P. Torres, A. V. Oppenheim, and R. R. Rosales, "Generalized frequency modulation," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 48, no. 12, pp. 1405–1412, Dec. 2001.

[24] G. N. Watson, *A Treatise on the Theory of Bessel Functions*. Cambdridge, U.K.: Cambridge Univ. Press, 1995.

[25] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1965.

[26] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2000.

Miguel Ángel Gutiérrez de Anda (M'07) was born in Puebla, Mexico, in 1973. He received the M.Sc. degree in electronics from the National Institute for Astrophysics, Optics and Electronics (INAOE), Tonantzintla, Mexico in 1998, and the Ph.D. degree in electrotechnical engineering from Delft University of Technology, Delft, The Netherlands, in 2003.

In 1998 he was a Summer Intern with Nokia Telecommunications, Haukipudas, Finland. Between 2003 and 2004, he was a Postdoctoral Scholar in the frame of the FP5 European Project SEWING with Warsaw University of Technology, Warsaw, Poland. From 2005 to 2006 he was a Visiting Professor at the Autonomous Metropolitan University, Iztapalapa, Mexico. Since 2007, he has been an Associate Researcher with the Electronics Department, INAOE. He was a Guest Scholar at West Pomeranian University of Technology (formerly known as Szczecin University of Technology), Szczecin, Poland, for six short periods between 2007 and 2010. He was also a Guest Scholar at Warsaw University of Technology for a short period in 2010. His current areas of interest are linear time-varying circuits and systems, nonlinear circuits and systems, design of analog circuits and applied mathematical methods for the analysis and design of electronic circuits.

Dr. Gutiérrez de Anda was granted a scholarship funded by the Republic of Poland to carry out a postdoctoral stay at Warsaw University of Technology during the academic year 2003–2004. Moreover, he was awarded in 2008 with a fellowship funded by the Foundation for Polish Science (Fundacja na rzecz Nauki Polskiej – Kasa im. Józefa Mianowskiego) to carry out scientific research at West Pomeranian University of Technology during 2009.

Isabelo Meza Dector (S'09) was born in Cañada Morelos, Mexico, in 1984. He received the B.Sc. degree in electronics science from BUAP, Puebla, Mexico, in 2008, and the M.Sc. degree in microelectronics engineering from the National Polytechnic Institute (IPN, ESIME-Culhuacán), Mexico City, Mexico, in 2010. He is currently pursuing the Ph.D. degree at the National Institute for Astrophysics, Optics and Electronics (INAOE), Tonantzintla, Mexico, in the field of analog integrated circuit design

He was a Guest Researcher at West Pomeranian University of Technology, Szczecin, Poland, for two short periods in 2009 and 2010. In 2010 he was a Research Assistant with the Electronics Department, INAOE, Tonantzintla, Mexico. His current areas of interest are design of analog and mixed mode circuits, continuous-time adaptive filters, FPGA-based circuit design, microcontroller-based embedded systems and PCB design.

Mr. Meza Dector was the holder of a scholarship granted by Mexican CONACyT during the course of his M.Sc. degree studies.