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Exact solution of the ion-laser interaction in all regimes

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We show that in the trapped ion-laser interaction all the regimes may be considered analytically. We may solve not only for different laser intensities, but also away from resonance and from the Lamb-Dicke regime. It is found a dispersive Hamiltonian for the high intensity regime, that, being diagonal, its evolution operator may be easily calculated.

1 Introduction

The ion-laser interaction may be easily solved in the low intensity regime (LIR) [1–7], but besides the condition that the laser intensity is much lower than the vibrational frequency, we set the condition that the detuning between the laser and the atomic transition frequency is an integer multiple of the vibrational frequency. Then some questions arise: Is it possible not to consider integer multiples of the vibrational frequency? Is it possible to solve for high and middle intensities?

Indeed, it is possible to find solutions for any set of parameters, i.e. in all regimes [8], however the solutions are not general because the set of eigenstates that may be found can not expand all possible (general) states.

It has been shown already that for low intensities it is possible also to consider the ion micromotion [9], and by using Ermakov-Lewis invariant methods [10] it was possible to *linearize* the ion-laser Hamiltonian when the micromotion was included [11]. Here we would like to show how it is possible to solve the interaction in different regimes, including high intensity and medium intensity. the method allows also not to consider multiple integers of the vibrational frequency.

2 Ion-laser interaction

The Hamiltonian for the ion-laser dipole interaction, with no approximations can be written as (we set $\hbar = 1$)

$$\hat{H} = \nu\hat{n} + \frac{\omega_a}{2}\hat{\sigma}_z + \Omega\left(\hat{\sigma}_+ + \hat{\sigma}_-\right) \times \left(\mathrm{e}^{[i\eta(a+a^\dagger)-\omega_L t]} + \mathrm{e}^{-i[\eta(a+a^\dagger)-\omega_L t]}\right), \quad (1)$$

where ν is the harmonic trapping frequency, ω_a is the atomic transition frequency, ω_L is the field frequency, Ω the (real) Rabi frequency of the ion-laser coupling and η the Lamb-Dicke parameter. The operators $a^\dagger$ and a are the creation and annihilation operators for the vibrational motion of the ion, and the σ's are the Pauli spin operators.

By doing the transformation $\hat{T}|\psi\rangle$ with $\hat{T} = \exp(-i\frac{\omega_a+\delta}{2}\hat{\sigma}_z t)$ and performing the optical rotating wave approximation (RWA) [12] we arrive at the well-known Hamiltonian

$$\hat{H}_{\mathrm{ion}} = \nu\hat{n} + \frac{\delta}{2}\hat{\sigma}_z + \Omega\left(\hat{\sigma}_+\hat{D}(i\eta) + \hat{\sigma}_-\hat{D}^\dagger(i\eta)\right), \quad (2)$$

where $\hat{D}(i\eta) = \mathrm{e}^{i\eta(a+a^\dagger)}$ is the Glauber displacement operator [13], and $\delta = \omega_a - \omega_L$ the laser-ion detuning.

2.1 Low intensity regime

The low intensity regime is the well-known regime, where several effects like multi-phonon transitions, Jaynes-Cummings (JC) and anti-JC interactions may be engineered. To solve this regime, we follow first the traditional approach. We start by using the Baker-Hausdorff formula [14] to factor the displacement operators in Eq. (2) into a product of exponentials and consider $\delta =$

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$k\nu$, i.e. an integer multiple of ν with $k = 0, \pm1, \pm2, \ldots$, we then obtain

$$\hat{H}_{\rm ion} = \nu\hat{n} + \frac{k\nu}{2}\hat{\sigma}_z + \Omega e^{-\eta^2/2} \times \left(\hat{\sigma}_+ e^{i\eta a^\dagger} e^{i\eta a} + \hat{\sigma}_- e^{-i\eta a^\dagger} e^{-i\eta a}\right). \quad (3)$$

Now we expand the exponentials of the annihilation and creation operators in Taylor series and get rid off the free Hamiltonians via a transformation to the interaction picture to obtain the Hamiltonian

$$\hat{H}_I = \Omega e^{-\eta^2/2} \times \left(\hat{\sigma}_- \sum_{n,m=0}^{\infty} \frac{(-i\eta)^{n+m}}{n!m!} a^{\dagger n} a^m e^{i\nu t(n-m+k)} + H.c\right). \quad (4)$$

We use the fact that are in the LIR, $\nu \gg \Omega$ and make the RWA, i.e. we only keep time independent terms in the above Hamiltonian to end up with

$$\hat{H}_I = \Omega e^{-\eta^2/2}\left(a^{\dagger k}(-i\eta)^k \hat{\sigma}_- \frac{\hat{n}!}{(\hat{n}+k)!} L_{\hat{n}}^{(k)}(\eta^2) + H.c\right), \quad (5)$$

with $L_{\hat{n}}^{(k)}(x)$ the associated Laguerre polynomials of order (operator) $\hat{n} = a^\dagger a$. The Hamiltonian above is now readily solvable, so that we may find easily the evolution operator, $\hat{U}_I = \exp(-i\hat{H}_I t)$, associated to it.

3 Other regimes

Although the atom-field and ion-laser interactions appear to be physically and mathematically quite distinct, they are in fact exactly equivalent. This may be verified when we try to solve the Schrödinger equation (SE)

$$i\frac{d|\psi(t)\rangle}{dt} = \hat{H}_{\rm ion}|\psi(t)\rangle, \quad (6)$$

we can perform the transformation $|\psi(t)\rangle = R^\dagger|\tilde{\psi}(t)\rangle$, with

$$\hat{R} = e^{i\hat{n}\frac{\pi}{2}} e^{\frac{\pi}{4}(\hat{\sigma}_+ - \hat{\sigma}_-)} e^{-i\frac{\eta}{2}(\hat{a}+\hat{a}^\dagger)\sigma_z} \quad (7)$$

such that $\hat{\mathscr{H}}_{\rm ion} = \hat{R}\hat{H}_{\rm ion}\hat{R}^\dagger$

$$\hat{\mathscr{H}}_{\rm ion} = \nu\hat{n} + \Omega\hat{\sigma}_z + \frac{\eta\nu}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)\left(\hat{a} + \hat{a}^\dagger\right) + \frac{\delta}{2}(\hat{\sigma}_+ + \hat{\sigma}_-) + \frac{\nu\eta^2}{4}, \quad (8)$$

and the SE for the transformed Hamiltonian reads

$$i\frac{d|\tilde{\psi}(t)\rangle}{dt} = \hat{\mathscr{H}}_{\rm ion}|\tilde{\psi}(t)\rangle. \quad (9)$$

Therefore we have *linearized* the ion-laser interaction in an exact way, this by means of a unitary transformation, i.e. both Hamiltonians, $\hat{H}_{\rm ion}$ and $\hat{\mathscr{H}}_{\rm ion}$ are equivalent. In the following we will neglect the term $\frac{\nu\eta^2}{4}$ because it only represents a constant shift of all the eigenenergies.

Of course, transformation (7) has to be applied to an initial condition for the internal state of the ion and its vibrational motion wavefunction. Let us assume that we have the initial state

$$|\psi(0)\rangle = |i\alpha\rangle|e\rangle, \quad (10)$$

where $|i\alpha\rangle$ is a coherent state, and for simplicity we take α a real number (to avoid extra phases later, but the calculation may be done for complex α). Then we have that the initial wave function associated with the transformed Hamiltonian (8) is

$$|\tilde{\psi}(0)\rangle = R|\psi(0)\rangle. \quad (11)$$

If we write the initial wave function in terms of 2×2 matrices we obtain

$$|\tilde{\psi}(0)\rangle = \frac{1}{\sqrt{2}} e^{i\hat{n}\frac{\pi}{2}} \begin{pmatrix} \hat{D}^\dagger(i\eta/2) & \hat{D}(i\eta/2) \\ -\hat{D}^\dagger(i\eta/2) & \hat{D}(i\eta/2) \end{pmatrix} \begin{pmatrix} |i\alpha\rangle \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} |-(\alpha-\eta/2)\rangle \\ -|-(\alpha-\eta/2)\rangle \end{pmatrix}, \quad (12)$$

this is, we have changed the complicated Hamiltonian (2) by the linear Hamiltonian (8) via a unitary transformation. The small prize we have to pay, is that in the initial wave function the coherent state is displaced and the ion is initially (in the new frame) in a superposition of ground and excited states.

3.1 Medium intensity regime (MIR)

We now consider the case where the vibrational frequency is of the order of (twice) the field intensity (Rabi frequency). We also consider the Lamb-Dicke regime, i.e. $\eta \ll 1$. For simplicity we will set $\delta = 0$ to show the different possibilities we have now. However it is not difficult to produce effective Hamiltonians also in the off-resonance case. In this case the Hamiltonian (8) may be casted into

$$\hat{\mathscr{H}}_{MIR} = \nu\hat{n} + \Omega\hat{\sigma}_z + \frac{\eta\nu}{2}\left(\hat{\sigma}_+\hat{a} + +\hat{a}^\dagger\hat{\sigma}_-\right) \quad (13)$$

which is a Hamiltonian that has been extensively studied [15, 16], therefore, we will not add more here, except the fact that for the medium intensity regime the Hamiltonian (2) may be exactly expressed as a JCM Hamiltonian via a unitary transformation and the RWA (for $\eta \ll 1$,

i.e., in the Lamb-Dicke regime), without extra approximations.

3.2 Low and high intensity regimes (HIR)

We have shown in Sect. 2 how to solve the LIR case. The solution for this case has been known for several years [17, 18]. Here we will show a different method that is also valid for the HIR.

Just for the matter of qualitative analysis, let us take $\delta = 0$. Consider now $\Omega \ll \nu$ (LIR) or $\Omega \gg \nu$ (HIR) in Eq. (8). As this Hamiltonian for $\delta = 0$ is equivalent to the atom-field interaction, we can borrow knowledge from such interaction: we know that when the field and atomic transition frequencies are very different (in our case, it is translated in the equation $|\nu - 2\Omega| \ll \eta\nu/2$, that may happen in either of both regimes, HIR or LIR) atom and field stop to exchange energy and we obtain a dispersive Hamiltonian [19]. The same happens in the ion-laser interaction, and via an small rotation approach [20], we will be able to cast Hamiltonian (8) as an effective (dispersive) Hamiltonian.

By transforming the Hamiltonian (8) with the unitary operators

$$\hat{U}_1 = \mathrm{e}^{\xi_1(\hat{a}^\dagger\hat{\sigma}_+ - \hat{a}\hat{\sigma}_-)}, \qquad \hat{U}_2 = \mathrm{e}^{\xi_2(\hat{a}\hat{\sigma}_+ - \hat{a}^\dagger\hat{\sigma}_-)}, \tag{14}$$

i.e.

$$\hat{\mathscr{H}}_{\mathrm{eff}} = \hat{U}_1\hat{U}_2\hat{\mathscr{H}}_{\mathrm{ion}}\hat{U}_1^\dagger\hat{U}_2^\dagger \tag{15}$$

with $\xi_1, \xi_2 \ll 1$, we can remain up to first order in the expansion [20] $\mathrm{e}^{\xi A}B\mathrm{e}^{-\xi A} = B + \xi[A,B] + \frac{\xi^2}{2!}[A,[A,B]] + \ldots \approx B + \xi[A,B]$ and obtain the effective Hamiltonian

$$\hat{\mathscr{H}}_{\mathrm{eff}} = \nu\hat{a}^\dagger\hat{a} + \Omega\hat{\sigma}_z - \chi_{\mathrm{ion}}\hat{\sigma}_z \times\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \frac{\delta}{2}(\sigma_+ + \sigma_-) + \frac{\kappa}{2}\hat{\sigma}_z\left(\hat{a}^\dagger + \hat{a}\right), \tag{16}$$

that for $\delta = 0$ is known as the dispersive Hamiltonian.

We have used

$$\xi_1 = \frac{\eta\nu}{2(\nu + 2\Omega)}, \qquad \xi_2 = \frac{\eta\nu}{2(2\Omega - \nu)}. \tag{17}$$

Note that, just as in the atom-field case, there is no need to transform the (already transformed) initial state (12) as an small rotation has been applied. We can see that in fact $\xi_1, \xi_2 \ll 1$ either in the LIR (in this case we have also to consider $\eta \ll 1$) or in the HIR (no constrain for η), which justifies completely the approximation for the above Hamiltonian. For the resonant case, $\delta = 0$, it becomes diagonal and we can solve

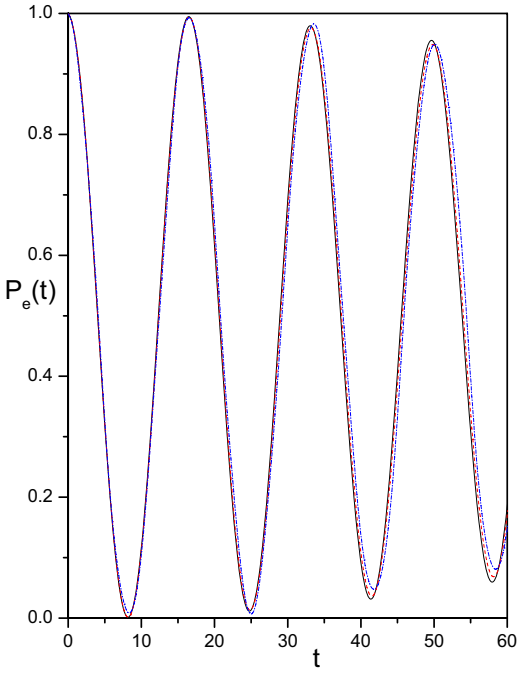

Figure 1 (online color at: www.ann-phys.org) Plot of $P_e(t)$ as a function of t for $k = 0$, $\nu = 1$, $\Omega = 0.2$ and $\eta = 0.1$. The vibrational motion of the ion is considered to be in a coherent state, $|\alpha|^2 = 4$ and the ion in its excited state. Solid line represents the numerical (exact) solution, dashed line the solution from Sect. 2 and the dot-dashed line the solution for the dispersive Hamiltonian of Sect. 3.

it in an easy way. In Fig. 1 we show a plot for the probability to find the ion in its excited state, $P_e(t) = \langle\psi(0)|\hat{R}^\dagger\exp(i\,t\hat{\mathscr{H}}_{\mathrm{eff}})\hat{R}|e\rangle\langle e|\hat{R}^\dagger\exp(-i\,t\hat{\mathscr{H}}_{\mathrm{eff}})\hat{R}|\psi(0)\rangle$ as a function of time for $k = 0$, $\nu = 1$, $\Omega = 0.2$ and $\eta = 0.1$. The vibrational motion of the ion is considered to be in a coherent state, $|\alpha|^2 = 4$ and the ion in its excited state. The three curves in the figure correspond to the exact numerical solution (solid line), the solution form Hamiltonian (5) (dashed line) and the solution for the dispersive Hamiltonian (16). We can see excellent agreement among the three plots for the LIR. Now, for the HIR in Fig. 2 we show a plot also of $P_e(t)$ as a function of time, but now with the parameters for $k = 0$, $\Omega = 1$, $\nu = 0.2$ and $\eta = 0.1$, for the exact numerical solution (solid line) and our solution from this section (dashed line). Again it may be noticed an excellent agreement between both curves. The neglected terms in the dispersive Hamiltonian obtained from an small rotation approach seem not to play any role in both figures. We should stress that there is no other analytical solution to compare with, as

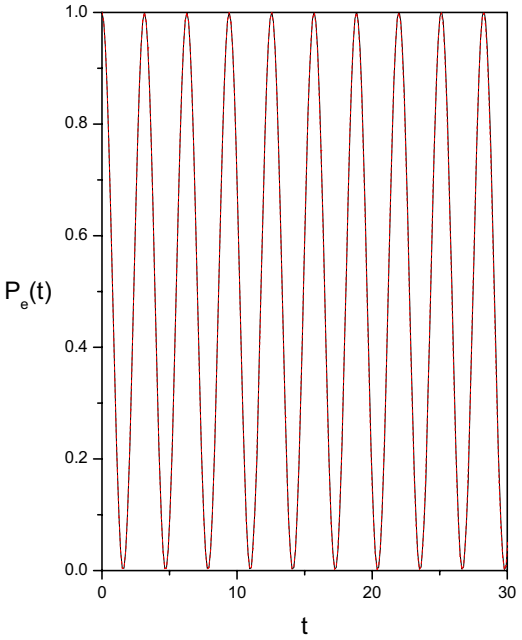

Figure 2 (online color at: www.ann-phys.org) Plot of $P_e(t)$ as a function of t for $k = 0$, $\Omega = 1$, $\nu = 0.2$ and $\eta = 0.1$. The vibrational motion of the ion is considered to be in a coherent state, $|\alpha|^2 = 4$ and the ion in its excited state. Solid line represents the numerical (exact) solution, dashed line the solution for the dispersive Hamiltonian of Sect. 3.

ours is the first analytical solution in this regime (also in the medium regime). The new interaction constants in the effective Hamiltonian (16) have the form

$$\chi_{\rm ion} = \frac{2\eta^2\nu^2\Omega}{4\Omega^2-\nu^2}, \quad \kappa = \frac{\delta\eta\nu^2}{4\Omega^2-\nu^2}. \tag{18}$$

In the resonant case and high intensity regime, $\Omega \gg \nu$, it is easy to show that

$$\chi_{\rm ion} \to \chi^{\rm high} = \frac{2\eta^2\nu^2}{4\Omega}\frac{1}{1-\frac{\nu^2}{4\Omega^2}} \approx \frac{\eta^2\nu^2}{2\Omega}. \tag{19}$$

while in the low intensity regime, $\Omega \ll \nu$, we will have the same Hamiltonian but χ will change to

$$\chi_{\rm ion} \to \chi_{\rm low} = -2\eta^2\Omega\frac{1}{1-\frac{4\Omega^2}{\nu^2}} \approx -2\eta^2\Omega \tag{20}$$

If in Eq. (16) we take the detuning δ different from zero, we could get the usual blue and red side-bands interactions (see for instance [7]). This is done by choosing the value $\delta = \pm\nu$. The only case in which we can obtain such regimes is the low intensity case, where one can perform the RWA to the Hamiltonian (16), which agrees with the usual procedure for obtaining such blue and red side-band regimes. The high intensity case, $\Omega \gg \nu$ does not allow such side-bands because in the Hamiltonian (16) the interaction constants multiplying the different terms may be of the same order.

4 Exact numerical analysis

The numerical propagation of phase-space function is carried out by split-operator method, which, when $\epsilon\Delta t \ll 1$ (ϵ being the relevant frequency depending if we are in the LIR or HIR case) approximates the propagator $\exp(-i\Delta t\hat{H})$ in the form [21]:

$$\exp(-i\Delta t\hat{H}) \approx \exp\left(-i\frac{\Delta t}{2}\hat{n}\right)\exp\left(-i\Delta t\hat{B}\right)\exp\left(-i\frac{\Delta t}{2}\hat{n}\right), \tag{21}$$

where $\hat{B} = \delta\hat{\sigma}_z/2 + \Omega\left[\hat{\sigma}_+\hat{D}(i\eta) + \hat{\sigma}_-\hat{D}^\dagger(i\eta)\right]$. The first and third terms in the right-hand side are the expression for the harmonic oscillator propagator; the middle term may be written as

$$\hat{U} = \exp(-i\Delta t\hat{B}) = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \tag{22}$$

with

$$\begin{aligned} U_{11} &= \cos\Lambda\Omega\Delta t - i\frac{\delta}{2\Lambda}\sin\Lambda\Omega\Delta t\,, \\ U_{22} &= \cos\Lambda\Omega\Delta t + i\frac{\delta}{2\Lambda}\sin\Lambda\Omega\Delta t\,, \\ U_{12} &= -\frac{i}{\Lambda}\exp[i\eta(\hat{a}+\hat{a}^\dagger)]\sin\Lambda\Omega\Delta t\,, \\ U_{21} &= -\frac{i}{\Lambda}\exp[-i\eta(\hat{a}+\hat{a}^\dagger)]\sin\Lambda\Omega\Delta t\,, \end{aligned} \tag{23}$$

where $\Lambda = \sqrt{1+\delta^2/4}$. It is worth mentioning that the numerical method based on split-operator method, makes use of Fast Fourier Transform routine, the specific details are shown in reference [21].

5 Conclusions

We have shown that it is possible to solve analytically the ion-laser Hamiltonian in different intensity regimes, from low to high. For the MIR we have casted the ion laser Hamiltonian into a JCM Hamiltonian (for the on-resonant case) that allows easy solution. For the HIR we

have found a dispersive Hamiltonian, which, being diagonal, it is direct to solve. We have found excellent agreement between the exact (numerical) solutions and our proposed solutions.

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Key words. Ion-laser interaction, rotating wave approximation, effective Hamiltonians.

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