АСТРОФИЗИКА

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AN IMPROVED VERSION OF THE IMPLICIT INTEGRAL METHOD TO SOLVING RADIATIVE TRANSFER PROBLEMS

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Radiative transfer (RT) problems in which the source function includes a scattering-like integral are typical two-points boundary problems. Their solution via differential equations implies to make hypotheses on the solution itself, namely the specific intensity $I(\tau; n)$ of the radiation field. On the contrary, integral methods require to make hypotheses on the source function $S(\tau)$. It looks of course more reasonable to make hypotheses on the latter because one can expect that the run of $S(\tau)$ with depth be smoother than that of $I(\tau; n)$. In previous works we assumed a piece-wise parabolic approximation for the source function, which warrants the continuity of $S(\tau)$ and its first derivative at each depth point. Here we impose the continuity of the second derivative $S''(\tau)$. In other words, we adopt a cubic spline representation to the source function, which highly stabilize the numerical processes.

Key words: *numerical methods:radiative transfer - stars:atmospheres*

1. *Introduction*. Some years ago we proposed a new algorithm, the Implicit Integral Method (IIM), to solving those radiative transfer problems in which the specific source functions (one for each frequency and direction pair) depend linearly on the radiation field via a single quantity independent of both frequency and direction. In the paradigm instance of radiative transfer through an ideal medium formed by atoms with only two energy levels (Two-Level Atom model), this quantity is the integral over frequencies of the mean specific intensity of the radiation field, weighted with the spectral profile (Paper I [1]).

Because it is independent of both frequency and direction, such a quantity constitutes a single scalar coupling for all the specific RT equations, and can be chosen in a natural way as the protagonist variable for the numerical solution of the RT problem. This choice is the distinctive and essential feature of our IIM: to work with a quantity which is independent of both frequency and direction brings about that the method does not require to store and invert huge matrices like in the customary numerical algorithms employed in RT problems. We have already remarked in Paper I [1] that our algorithm is a mere phenomenological representation of the actual physical process. Because of that and due to the lack of a matricial structure, the advantages of the IIM in terms of reliability, accuracy and robustness could be self-evident, as well as the conspicous saving of both computational time and memory storage it makes possible.

The aforesaid advantages suggested us the possibility to employ the IIM also in the computation of stellar atmospheres models, where we must solve many (some hundreds) RT equations, one for each frequency. The source function of each specific RT equation is here the weighted mean of a term that includes the mean specific intensity of the radiation field through a scattering-like integral with a thermal contribution given by the Planck function $B_\nu(T)$. The paradigm problem of the self-consistent temperature correction when computing stellar atmosphere models was considered in Crivellari and Simonneau [2].

First of all, we must recognize that the geometrical structure of the system, that is the sequence of the discrete atmospheric layers, must be necessarily the same for all the frequencies. But we must also recognize that for any given frequency some layers do not contribute to the formation of the spectrum. They do not take an effective part in the radiative transfer process because either they are exceedingly transparent (i.e. $\exp\{-\Delta\tau_\nu\} \simeq 1$) or they correspond to optically very deep regions (i.e. $\exp\{-\Delta\tau_\nu\} \simeq 0$). The layers intermediate between the above two groups constitute the specific spectral formation region. However all the layers of the structure must be taken into account in the numerical algorithm, irrespectively of the frequency considered. Yet, due to the dramatic difference among the values of the opacity with frequency, different spectral intervals form in very different geometrical regions. That compels us to divide the atmosphere into very many layers in order to cover properly all the spectral formation intervals. On the other hand, it is matter of the run with depth of the data that are common to radiative transfer at all the frequencies. As an example, given a temperature distribution on the discrete atmospheric layers, the variation with depth of the numerical values of the Planck function $B_\nu(T)$, i.e. the monochromatic thermal sources, may vary enormously frequency by frequency. For instance, in the case of a solar-like star, from the bottom to the top of the atmosphere $B_\nu(T)$ varies by a factor of the order of 10^3 for frequencies in the visible part of the spectrum, while this factor can be of the order of 10^{14} for frequencies in the range of Lyman α. Therefore a set of depth points suitable for a good description of the mathematical behaviour of the source function at some frequencies cannot be adequate at other frequencies. Again very many common discrete depth points are necessary in order to provide a proper distribution of the data for the adequate treatment of each monochromatic RT equation.

The foregoing requirements make it impossible in the practice to replace derivatives by finite differences, as in the outermost layers the optical thickness is almost zero for many frequencies. The use of integral methods may seem to be the only advisable way out, but the very large number of discrete optical depth points, necessary to warrant the proper treatment of the RT process at all the frequencies, does not advise to employ global integral methods, too.

We can get rid of the difficulties brought about by the introduction of very many layers on the one hand by employing our IIM, which allows us to take into consideration as many geometrical depths as necessary because, as already said, it does not require the storage and inversion of huge matrices. Moreover, on the other hand, we can introduce a better mathematical representation of each monochromatic source function $S_\nu(\tau_\nu)$ in order to account for the possible rapid variation of both the branching parameter ε_ν (see eq. [2] later) and $B_\nu(T)$ with respect to each specific optical depth τ_ν. In such a way we can optimize the treatment of all the individual frequencies.

In the original formulation of the IIM (see the above references) we considered models that comprised 150-200 discrete layers between the surface and the bottom of the atmosphere. Inside each of them we approximated each specific source function $S_\nu(\tau_\nu)$ by an arc of parabola and imposed the continuity of $S_\nu(\tau_\nu)$ and $S'_\nu(\tau_\nu)$ at all the NL dividing points. This piecewise parabolic approximation yielded excellent results in many cases (see the above references). However under extreme conditions, (e.g. for Lyman α frequencies in cool stars) such an approximation can introduce numerical instabilities that spoil the computation of the model.

To impose also the continuity of the second derivative of $S_\nu(\tau_\nu)$ at all the NL dividing points can remove the foregoing instabilities. Consequently we propose here a cubic spline model for each specific source function. In some way this model constitutes a regularization of the process to computing the values of the source functions. The formalism of the cubic spline approximation (namely a two-point boundary value problem developed to interpolate among the NL explicitly known values of a given function) can be employed in the present case although the NL values of $S_\nu(\tau_\nu)$ are yet unknown.

To employ the cubic spline approach in order to describe the behaviour of the source function in typical RT problems, where a scattering term appears in the source function, is the best (may be the unique) correct choice for both theoretical and numerical reasons. A theoretical reason is brought about by the non-local nature of the problem: the specific intensities and consequently the source function at a given depth point depend via the RT process on the values of the source function at all the other points of the system. Thus the numerical values of the source function must be computed simultaneously at all the depth points. Therefore such a non-local character of the physical problem must be represented by means of a non-local mathematical structure. Also the derivatives of the source function at any depth point must be formulated as a linear relation including the implicit values of the source function at all the depth points, not only as a linear relation of the implicit values of the source function at each triad of consecutive depth points.

The pratical reason is for the sake of the stability of the computational algorithm. The cubic spline model minimizes the strain energy integral, that is the integral of the squared values of the second derivative of the protagonist function, namely of the variation of its curvature - i.e. the oscillations. (See, e.g., Rivlin [3]). That is, the use of the cubic spline approximation to the source function minimizes the risk of destabilizing oscillations.

From the algorithmical stand point, the kernel of the original IIM is a forward-elimination scheme that links the so far unknown values of the source function at each pair of consecutive optical depth points (τ_L, τ_{L+1}) by mean of a linear relation with known coefficients. The latter are determined by taking into account the RT equations that describe layer by layer the propagation of both the downgoing and the upgoing specific intensities. Now we realized that, by using the cubic spline formalism the same forward-elimination scheme can also be employed to link the unknown values of the second derivatives of the source functions, again by means of a linear relation.

Once attained the deepest optical depth point τ_{NL} at the end of the forward-elimination, we can impose the bottom boundary condition (eq. (5) later on) to both the RT process and the cubic spline chain; in other words we can close the linear relation between $S_\nu(\tau_{NL-1})$ and $S_\nu(\tau_{NL})$ on the one hand, between $S''_\nu(\tau_{NL-1})$ and $S''_\nu(\tau_{NL})$ on the other. This allows us to recover the numerical values of the source functions and their second derivatives at the bottom, as well as those of the set of the incident upgoing specific intensities $\{I^+(\tau_{NL}, \mu_J), J = 1, ND\}$. Then, in a succesive back-substitution scheme, we are in a position to compute at each depth point the numerical values of the source functions and their second derivatives by using the above linear relations, whose coefficients have been stored during the previous forward-elimination.

Thanks to that we have at hand a unique algorithm to solve each specific RT problem under the imposed constraint that the specific source functions as well as their first and second derivatives be continuous at all the NL points of the grid chosen for the geometrical representation of the stellar atmosphere. In such a way we can get rid of the instabilities that may arise in the case of extreme variations of the source functions without paying any extra computational cost.

2. *The mathematical background*. For the sake of an easier presentation of the new more precise version of the IIM announced in Section 1, we will consider the simplest instance that yet contains all the difficulties intrinsic to RT astrophysical problems, namely the transport of monochromatic radiation through a plane-parallel medium in which matter particles can scatter, absorb and emit photons. In the previous works above quoted the original formulation was applied to much more general instances. The version presented

here can be easily applied to such cases.

Following the customary notation, the RT equations that describe the evolution of the upgoing intensities $I^+(\tau,\mu)$ and the downgoing intensities $I^-(\tau,\mu)$ are

$$\pm\mu\frac{d}{d\tau}I^\pm(\tau,\mu)=I^\pm(\tau,\mu)-S(\tau), \tag{1}$$

where τ denotes the optical depth and μ is the cosine of the angle formed by the direction of propagation with the perpendicular to the plane-parallel layers ($\mu\equiv\cos\theta, 0\le\mu\le 1$).

The source function is a weighted mean between the thermal source $B(\tau)$ and the mean intensity $J(\tau)$, namely

$$S(\tau)=\varepsilon B(\tau)+(1-\varepsilon)J(\tau). \tag{2}$$

The branching parameter $\varepsilon=\varepsilon(\tau)$ is the ratio of the absorption coefficient to the total opacity (i.e. the sum of the absorption and the scattering coefficient). The latter defines the scale of the optical depth τ; $(1-\varepsilon)$ is customarily called the albedo. In terms of the upgoing and the downgoing intensities the mean intensity is given by

$$J(\tau)=\int_0^1\left[I^+(\tau,\mu)+I^-(\tau,\mu)\right]d\mu. \tag{3}$$

The integral in eq. (3) is representative of any scattering integral, which may be different for the application of the IIM to different instances.

In the discrete ordinates approximation the integral in eq. (3) is replaced by the sum of the intensities corresponding to a finite number of *ND* directions. Then

$$J(\tau)\approx\sum_{J=1}^{ND}\mathrm{w}_J\left[I^+(\tau,\mu_J)+I^-(\tau,\mu_J)\right]. \tag{4}$$

For most RT problems in plane-parallel geometry (at least for stellar atmosphere models computations) a five-points Gauss division of the interval $0\le\mu\le 1$ is more than enough. The w_J's are the corresponding integration weights.

The numerical solution requires the discretization of the optical depth variable τ, too. The stellar atmosphere must be sliced into a set of *NL* plane-parallel horizontal layers, divided by the set of $NL+1$ optical depths points $\{\tau_0,\tau_1,\tau_2,\dots,\tau_{NL}\}$. The value $\tau_0=0$ corresponds to the surface and τ_{NL} to the bottom of the atmosphere. The computation of a fairly good model require that *NL* be of the order of two hundred.

The values of the incident intensities onto the top surface, i.e. the downgoing intensities $I^-(\tau_0,\mu_J)$ and those of the incident intensities onto the bottom surface, i.e. the upgoing intensities $I^+(\tau_{NL},\mu_J)$, must be known; they are data of the RT problem. In the case of a stellar atmosphere $I^-(\tau_0,\mu)$ is usually assumed to be zero, that is there is not radiation incident onto the stellar surface. We will show later that the method can equally work also under more general conditions. For the upgoing intensities at the bottom of the atmosphere we can assume that the diffusion approximation holds valid, that is

$$I^+(\tau_{NL},\mu_J)=S(\tau_{NL})+S'(\tau_{NL})\mu_J+S''(\tau_{NL})\mu_J^2+S'''(\tau_{NL})\mu_J^3\,, \tag{5}$$

which is brought about by the cubic polynomial behaviour of $S(\tau)$ at depths immediately greater than τ_{NL}. These two families of boundary conditions are sufficient to ensure that the RT problem is self-consistent.

The link between the values of the specific intensities at any pair of consecutive optical depth points (τ_L,τ_{L+1}), namely any single link of the whole RT chain, is given by the corresponding RT equations in the integral form, that is

$$I^+(\tau_L,\mu_J)=I^+(\tau_{L+1},\mu_J)\exp\left(-\frac{\Delta\tau_L}{\mu_J}\right)+\int_{\tau_L}^{\tau_{L+1}}S(t)\exp\left(-\frac{t-\tau_L}{\mu_J}\right)dt \tag{6}$$

and

$$I^-(\tau_{L+1},\mu_J)=I^-(\tau_L,\mu_J)\exp\left(-\frac{\Delta\tau_L}{\mu_J}\right)+\int_{\tau_L}^{\tau_{L+1}}S(t)\exp\left(-\frac{\tau_{L+1}-t}{\mu_J}\right)dt\,, \tag{7}$$

where $\Delta\tau_L\equiv\tau_{L+1}-\tau_L$. Equations (6) and (7) are the straightforward representation of the RT process.

At the surface (i.e. for $\tau_0=0$) the set of values $\{I^-(0,\mu_J),J=1,ND\}$ are the initial conditions for the inward RT problem, while the set $\{I^+(0,\mu_J),J=1,ND\}$ is the solution of the outward RT problem, i.e. the emergent intensities. At the bottom the set $\{I^+(\tau_{NL},\mu_J),J=1,ND\}$ yields the upgoing initial conditions (cf. eq. (5)); the set $\{I^-(\tau_{NL},\mu_J),J=1,ND\}$ is the result of the inward RT process.

Let us now turn our attention on the cubic spline approximation to the source function $S(\tau)$. That is, we will assume a cubic polynomial approximation inside each particular interval (τ_L,τ_{L+1}), defined by two consecutive optical depth points, as the single link of the spline chain. Anyone of these arcs of cubic is uniquely determined by the values of the source function and those of its second derivative at the end points (knots) of each interval.

To impose the continuity of the source function as well as that of its first and second derivative at the end points of each interval (τ_L,τ_{L+1}) leads to the cubic spline condition

$$\begin{aligned}&\frac{1}{\Delta\tau_L}S(\tau_{L-1})-\left(\frac{1}{\Delta\tau_L}+\frac{1}{\Delta\tau_{L+1}}\right)S(\tau_L)+\frac{1}{\Delta\tau_{L+1}}S(\tau_{L+1})=\\&=\frac{\Delta\tau_L}{6}S''(\tau_{L-1})+\left(\frac{\Delta\tau_L}{3}+\frac{\Delta\tau_{L+1}}{3}\right)S''(\tau_L)+\frac{\Delta\tau_{L+1}}{6}S''(\tau_{L+1}),\end{aligned} \tag{8}$$

where $\Delta\tau_L\equiv\tau_L-\tau_{L-1}$ and $\Delta\tau_{L+1}\equiv\tau_{L+1}-\tau_L$. Likewise, as a consequence of the cubic behaviour of $S(\tau)$ between τ_L and τ_{L+1}, it will hold that

$$S'(\tau_L)=\frac{1}{\Delta\tau_L}[S(\tau_{L+1})-S(\tau_L)]-\frac{\Delta\tau_L}{3}S''(\tau_L)-\frac{\Delta\tau_L}{6}S''(\tau_{L+1}), \tag{9}$$

$$S'(\tau_{L+1})=\frac{1}{\Delta\tau_L}[S(\tau_{L+1})-S(\tau_L)]+\frac{\Delta\tau_L}{6}S''(\tau_L)+\frac{\Delta\tau_L}{3}S''(\tau_{L+1}) \tag{10}$$

and

$$S'''(\tau_L)=\frac{1}{\Delta\tau_L}[S''(\tau_{L+1})-S''(\tau_L)]. \tag{11}$$

By means of eq. (8) we are in a position to join the neighbouring links of the spline chain while ensuring the required continuity at the knots.

Like for the RT chain, also for the cubic spline chain we need two bundary conditions. Customarily these are $S''(\tau_0)=S''(\tau_1)$ and $S''(\tau_{NL})=S''(\tau_{NL-1})$. In the present study we assume that $S''(\tau_0)=S''(\tau_1)$ at the surface, namely that the first arc of the spline chain is a parabola. On the contrary, the boundary condition for the spline chain at the bottom must be consistent with the diffusion approximation for the incident upgoing intensities $I^+(\tau_{NL},\mu_J)$, given by eq. (5), which is a consequence of having assumed also a cubic polynomial behaviour for $S(\tau)$ at depths greater than τ_{NL}. This condition is in agreement with the cubic polynomial behaviour of $S(\tau)$ inside the last layer (τ_{NL-1},τ_{NL}). Hence we cannot introduce now a different approach to $S(\tau)$. However we can derive the formal value of the first derivative $S'(\tau)$ from eq. (2), that is

$$S'(\tau)=[\varepsilon(\tau)B(\tau)]'+[1-\varepsilon(\tau)]'J(\tau)+[1-\varepsilon(\tau)]J'(\tau), \tag{12}$$

where

$$J'(\tau_L)=\sum_{J=1}^{ND}\mathrm{wd}_J\,\frac{I^+(\tau_L,\mu_J)-I^-(\tau_L,\mu_J)}{\mu_J}. \tag{13}$$

Equation (12), evaluated at the deepest optical depth point τ_{NL} will then yield the required lower boundary condition, as will be shown later.

Let us get back now to eqs (6) and (7). For any interval (τ_L,τ_{L+1}) the arc of cubic approximating to $S(\tau)$ is given by

$$S(\tau)=S(\tau_L)+S'(\tau_L)(\tau-\tau_L)+\frac{1}{2}S''(\tau_L)(\tau-\tau_L)^2+\frac{1}{6}S'''(\tau_L)(\tau-\tau_L)^3. \tag{14}$$

By replacing eq. (14) in eqs (6) and (7), and taking into account eqs (9) through (11), we get eventually

$$\begin{aligned}I^+(\tau_L,\mu_J)&=I^+(\tau_{L+1},\mu_J)\exp-\left(\frac{\Delta\tau_L}{\mu_J}\right)+\\&+\mathrm{ws}_1(J)S(\tau_L)+\mathrm{ws}_2(J)S(\tau_{L+1})+\mathrm{wd}_1(J)S''(\tau_L)+\mathrm{wd}_2(J)S''(\tau_{L+1})\end{aligned} \tag{15}$$

and

$$\begin{aligned}I^-(\tau_{L+1},\mu_J)&=I^-(\tau_L,\mu_J)\exp-\left(\frac{\Delta\tau_L}{\mu_J}\right)+\\&+\mathrm{ws}_2(J)S(\tau_L)+\mathrm{ws}_1(J)S(\tau_{L+1})+\mathrm{wd}_2(J)S''(\tau_L)+\mathrm{wd}_1(J)S''(\tau_{L+1}).\end{aligned} \tag{16}$$

The quadrature weights $\mathbf{ws}_1(J)$, $\mathbf{ws}_2(J)$, $\mathbf{wd}_1(J)$ and $\mathbf{wd}_2(J)$ are computed straightforwardly by taking into account eqs (9) through (11) to yield

$$\mathbf{ws}_1(J)=\left[1-\frac{1}{\delta}\right]+\frac{1}{\delta}e^{-\delta}, \tag{17}$$

$$\mathbf{ws}_2(J)=\frac{1}{\delta}-\left[1+\frac{1}{\delta}\right]e^{-\delta}, \tag{18}$$

$$\mathbf{wd}_1(J)=\mu^2\left[\left(1-\frac{\delta}{3}-\frac{1}{\delta}\right)-\left(\frac{\delta}{6}-\frac{1}{\delta}\right)e^{-\delta}\right], \tag{19}$$

$$\mathbf{wd}_2(J)=\mu^2\left[\left(-\frac{\delta}{6}+\frac{1}{\delta}\right)-\left(1+\frac{\delta}{3}+\frac{1}{\delta}\right)e^{-\delta}\right], \tag{20}$$

where $\delta=\Delta\tau_L/\mu_J$.

Sometimes, when $\delta \ll 1$, for sake of numerical percision it may be necessary to recast the foregoing weights into the form

$$\mathbf{ws}_1(J)=\frac{1}{2}\delta-\frac{1}{6}\delta^2+\frac{1}{24}\delta^3-\frac{1}{120}\delta^4+\frac{1}{720}\delta^5-\frac{1}{5040}\delta^6+\frac{1}{40320}\delta^7, \tag{21}$$

$$\mathbf{ws}_2(J)=\frac{1}{2}\delta-\frac{1}{3}\delta^2+\frac{1}{8}\delta^3-\frac{1}{30}\delta^4+\frac{1}{144}\delta^5-\frac{1}{840}\delta^6+\frac{1}{5760}\delta^7, \tag{22}$$

$$\mathbf{wd}_1(J)=-\mu^2\left(\frac{1}{24}\delta^3-\frac{7}{360}\delta^4+\frac{1}{180}\delta^5-\frac{1}{840}\delta^6+\frac{5}{24192}\delta^7\right), \tag{23}$$

$$\mathbf{wd}_2(J)=-\mu^2\left(\frac{1}{24}\delta^3-\frac{1}{45}\delta^4+\frac{1}{144}\delta^5-\frac{1}{630}\delta^6+\frac{1}{3456}\delta^7\right). \tag{24}$$

To conclude, eqs (15) and (16) together with eqs (17) through (20) allow us to write explicitly for each direction μ_J the relations between $I^+(\tau_L,\mu_J)$ and $I^+(\tau_{L+1},\mu_J)$ on the one hand, between $I^-(\tau_{L+1},\mu_J)$ and $I^-(\tau_L,\mu_J)$ on the other. These relations are linear functions of the unknown values of $S(\tau_L)$, $S(\tau_{L+1})$, $S''(\tau_L)$ and $S''(\tau_{L+1})$, which will play a protagonist role in the numerical algorithm. The cubic spline condition, given by eq. (8), impose a further relation between $S(\tau_L)$ and $S''(\tau_L)$ at each knot τ_L.

We recall that for any frequency the specific source function is approximated by an arc of cubic inside each interval (τ_L,τ_{L+1}). Therefore in the layers deeper than the corresponding spectral formation region, where $\exp\{-\Delta\tau\}$ is pratically null, the form of the weights $\mathbf{ws}_1(J)$, $\mathbf{ws}_2(J)$, $\mathbf{wd}_1(J)$ and $\mathbf{wd}_2(J)$ given by eqs (17) through (20) warrants that the intensities $I^+(\tau,\mu_J)$ recover there the form of eq. (5), originally assigned at the bottom of the atmosphere (i.e. at τ_{NL}). That is to say, the boundary condition, initially assigned at the bottom, is transported up to the end of the spectral formation region, keeping its form in a natural way. On the other hand, in the outer layers beyond the region of formation, where $\exp\{-\Delta\tau\}$ approaches unity, eqs (6) and (7) warrant that $I^+(\tau,\mu_J)$ and $I^-(\tau,\mu_J)$ keep constant. Thus, albeit the total number NL

of layers exceed that required by the proper physical treatment of the formation region for each single frequency, such an excess does not affect the numerical computation of the protagonist variables. That is to say, frequency by frequency the effective transport of the specific intensities is performed in a natural way inside its own region of formation, provided that care has be taken to select the geometrical width of the stellar atmosphere system so that, as already stressed in the *Introduction*, the former include the region of formation for all the frequencies.

We have then at hand all the mathematical tools that will allow us to solve the global RT problem in the same way as in the original IIM scheme (see Paper I [1]).

However only to warrant the continuity of the two first derivatives of the source function is not enough to avoid the occurence of instabilities. As in the cubic spline fundamental equation (eq. (8)) the protagonist variables are the function itself and its second derivative (both tied through their values at any set of three consecutive points), also in the RT elimination scheme the source function and its second derivative must be the protagonist variables.

In a previous attempt we formulated the equations (15) and (16), which describe the propagation of the upgoing and downgoing intensities, in terms of $S(\tau_L)$, $S(\tau_{L+1})$, $S'(\tau_L)$ and $S'(\tau_{L+1})$ after the elimination of $S''(\tau_L)$ and $S''(\tau_{L+1})$ given as functions of $S(\tau_L)$, $S(\tau_{L+1})$, $S'(\tau_L)$ and $S'(\tau_{L+1})$ thanks to the cubic behaviour of $S(\tau)$. In the actual version we describe the propagation of the aforesaid intensities by means of $S(\tau_L)$, $S(\tau_{L+1})$, $S''(\tau_L)$ and $S''(\tau_{L+1})$, again by means of the cubic behaviour of $S(\tau)$. From the mathematical point of view both representations should yield the same results, but from the numerical standpoint it looks much better to work directly with the second derivatives $S''(\tau_L)$ and $S''(\tau_{L+1})$, because the fundamental equation (8), that links the sequence of succesive layers in the cubic spline scheme, requires the variables $S(\tau)$ and $S''(\tau)$.

In the present formulation of the propagation equations (15) and (16) the integration weights $\mathbf{ws}_1(J)$ and $\mathbf{ws}_2(J)$, given by eqs (17) and (18), account strictly for the linear piecewise approximation to any monochromatic source function $S_\nu(\tau)$. The remaining weights, $\mathbf{wd}_1(J)$ and $\mathbf{wd}_2(J)$, account for the deviation from the linear behaviour, either parabolic or cubic. Whenever $\mathbf{wd}_1(J)$ and $\mathbf{wd}_2(J)$ take on small values, the linear approximation is more than enough. This is the case in the outermost layers, where it holds that $\delta < 1$ and $\delta^3 << 1$; the linear approximation is automatically recovered, as only the weights $\mathbf{ws}_1(J)$ and $\mathbf{ws}_2(J)$ account for the variation of the source function in optically thin layers. That is to say, in the practice only $S(\tau_L)$ and $S(\tau_{L+1})$ take part in the elimination scheme. In other words, the effects of a non-linear behaviour play the role of a perturbation of the linear behaviour.

In the original formulation of the IIM (Paper I), we employed the variables $S(\tau_L)$, $S(\tau_{L+1})$, $S'(\tau_L)$ and $S'(\tau_{L+1})$, together with the corresponding integration weights, in order to describe the propagation of the upgoing and downgoing intensities between any pair of optical depth points τ_L and τ_{L+1}. Whatever their behaviour (linear, quadratic or cubic), all the four variables and the relevant integration weights took an active part in the elimination scheme, both from the theoretical and the numerical standpoint. This can have been at the origin of the instabilities that showed up, above all in the regions of small optical depth. The actual version of the IIM, due to the above mentioned reasons, results certainly more reliable.

3. *The Forward-Elimination/Back-Substitution scheme*. As already said, we will work with a set of fundamental variables whose values are unknown: the upgoing and downgoing specific intensities $I^{\pm}(\tau_L,\mu_J)$, the corresponding source functions $S(\tau_L)$ and their second derivatives $S''(\tau_L)$. The major aim of this section is to derive linear relations among the values of the foregoing fundamental variables at the two consecutive optical depth points τ_L and τ_{L+1} that delimitate each of the layer (τ_L,τ_{L+1}) succesively under study. The coefficients of these relations are easily computed, and will be denoted in the following by bold face symbols.

3.1. *The algorithmic representation of the upper boundary conditions*. We start necessarily with only one half of the data of the problem, namely the set of the downgoing intensities incident onto the upper boundary layer at $\tau_0=0$, i.e. $\{I^-(\tau_0,\mu_J), J=1,ND\}$, that we will write in its most general form as

$$I^-(\tau_0,\mu_J)=\mathbf{cm0}(J)+\mathbf{cms1}(J)S(\tau_0)+\mathbf{cms2}(J)S(\tau_1)+ \\ +\mathbf{cmds1}(J)S''(\tau_0)+\mathbf{cmds2}(J)S''(\tau_1)+\sum_{J'=1}^{ND}\mathbf{R}(J,J')I^+(\tau_0,\mu_{J'}). \tag{25}$$

The coefficient $\mathbf{cm0}(J)$ accounts for the numerical value of the incident intensity $I^-(\tau_0,\mu_J)$, which is usually null. The reflexion matrix $\mathbf{R}(J,J')$ takes into account the possible effects of backscattering outside the stellar surface. Under usual conditions it holds that also $\mathbf{R}(J,J')=0$. On physical grounds it is hard to justify the dependence of $I^-(\tau_0,\mu_J)$ on the values of the source function and its second derivative at points τ_0 and τ_1 through the coefficients $\mathbf{cms1}(J)$, $\mathbf{cms2}(J)$, $\mathbf{cmds1}(J)$ and $\mathbf{cmds2}(J)$. It is rather an algorithmical requirement, as these coefficients allow us to link linearly the values of the protagonist variables between two consecutive optical depth points. Consistently with the upper boundary conditions, the latter coefficients have to be set equal to zero.

At the end of the treatment of radiative transfer in the first layer (as well as in the succesive ones) some of these coefficients will take on values different from zero. These new values can overrun the previous memory storage, because the current relation for the downgoing intensities at $\tau_0 = 0$ will not be necessary any longer.

Inside the forward-elimination scheme for the RT process we must propagate not only the upgoing and downgoing specific intensities (which brings about the propagation of the source function as defined by eqs (2) and (3)), but also the second derivative $S''(\tau)$ of the source function in the cubic spline scheme.

As already said, we assume that in the first layer (τ_0, τ_1) the source function $S(\tau)$ can be approximated by an arc of parabola, which implies that $S''(\tau_0) = S''(\tau_1)$. This condition will be included in the coefficients of the relation

$$S''(\tau_0) = \mathbf{cds0} + \mathbf{cds1}\, S(\tau_0) + \mathbf{cds2}\, S(\tau_1) + + \mathbf{cdds1}\, S''(\tau_0) + \mathbf{cdds2}\, S''(\tau_1) + \sum_{J=1}^{ND} \mathbf{cdi}(J)\, I^+(\tau_0, \mu_J), \tag{26}$$

where the values of $S(\tau_0)$, $S(\tau_1)$, $S''(\tau_0)$, $S''(\tau_1)$ and the set $\{ I^+(\tau_0, \mu_J),\ J = 1, ND \}$ are unknown. In order to fulfill the above boundary condition, all the coefficients in eq. (26) must be equal to zero, excepted **cdss2** that must be set equal to one. To express here $S''(\tau_0)$ as a function of $S(\tau_0)$, $S(\tau_1)$, $S''(\tau_0)$ itself, $S''(\tau_1)$ and the set $\{ I^+(\tau_0, \mu_J), J = 1, ND \}$ is just for algorithmical ease. When convenient, we will solve for $S(\tau_0)$ - and for $S''(\tau_0)$ - in terms of $S(\tau_1)$, $S''(\tau_1)$ and $\{ I^+(\tau_1, \mu_J) \}$.

3.2. *The layer by layer elimination*. We are going to show here how the treatment of the first layer (τ_0, τ_1), labelled by $L = 1$, will yield the coefficients of the relation

$$S(\tau_0) = \mathbf{cbs0}(1) + \mathbf{cbss}(1) S(\tau_1) + \mathbf{cbsd}(1) S''(\tau_1) + \sum_{J=1}^{ND} \mathbf{cbsi}(1, J)\, I^+(\tau_1, \mu_J) \tag{27}$$

and those of the relation

$$S''(\tau_0) = \mathbf{cbd0}(1) + \mathbf{cbds}(1) S(\tau_1) + \mathbf{cbdd}(1) S''(\tau_1) + \sum_{J=1}^{ND} \mathbf{cbdi}(1, J)\, I^+(\tau_1, \mu_J). \tag{28}$$

These coefficients will be stored in order to compute $S(\tau_0)$ and $S''(\tau_0)$ in the succesive backsubstitution process, once the values of $S(\tau_1)$, $S''(\tau_1)$ as well as the set $\{ I^+(\tau_1, \mu_J), J = 1, ND \}$ have been determined. The above relations link any pair of succesive layers. As already said, the determination of these relations constitutes the aim of this section.

In parallel we are going to show also how to recover the initial conditions for $I^-(\tau_1, \mu_J)$ and $S''(\tau_1)$, i.e. the values of the coefficients of the relations equivalent to eqs (25) and (26), now for τ_1.

Let us detail our foregoing purpose. At the beginning of the study of each succesive layer - here the first one - we must consider the implicit computation of the corresponding source function at the upper limiting optical depth, here τ_0. The form of the incident downgoing intensities at τ_0, given by eq. (25), together with the implicit values of the set $\{I^+(\tau_0,\mu_J), J=1, ND\}$ allow us to compute from eq. (4) the coefficients of a linear relation among $J(\tau_0)$ and $S(\tau_0)$, $S(\tau_1)$, $S''(\tau_0)$, $S''(\tau_1)$ and the set $\{I^+(\tau_0,\mu_J), J=1, ND\}$. Then eq. (2), where $\varepsilon(\tau_0)$ and $B(\tau_0)$ are given, will yield the coefficients of the linear relation

$$S(\tau_0)=\mathbf{cs0}+\mathbf{css}\,S(\tau_0)+\mathbf{css2}\,S(\tau_1)+ \\ +\mathbf{csds1}\,S''(\tau_0)+\mathbf{csds2}\,S''(\tau_1)+\sum_{J=1}^{ND}\mathbf{csi}(J)I^+(\tau_0,\mu_J), \tag{29}$$

where we have not solved for $S(\tau_0)$ again for the sake of algorithmical ease.

We compute now for each direction μ_J the quadrature weights $\mathbf{ws1}(J)$, $\mathbf{ws2}(J)$, $\mathbf{wd1}(J)$ and $\mathbf{wd2}(J)$ according to eqs (17) through (20) - or alternatively eqs (21) through (24) - for $\Delta\tau_1=\tau_1-\tau_0$. These weights allow us an implicit quadrature of the source function in the description of the propagation of the upgoing intensities from $I^+(\tau_1,\mu_J)$ to $I^+(\tau_0,\mu_J)$, and later of the downgoing intensities from $I^-(\tau_0,\mu_J)$ to $I^-(\tau_1,\mu_J)$.

At this point we can introduce the implicit form for $I^+(\tau_0,\mu_J)$ in terms of $S(\tau_0)$, $S(\tau_1)$, $S''(\tau_0)$, $S''(\tau_1)$ and the set $\{I^+(\tau_1,\mu_J), J=1, ND\}$, given by eq. (15), in eq. (25) for $I^-(\tau_0,\mu_J)$, which is the initial condition for the study of the layer (τ_0,τ_1). By re-arrangement of the coefficients we can write

$$I^-(\tau_0,\mu_J)=\mathbf{cm0}(J)+\mathbf{cms1}(J)S(\tau_0)+\mathbf{cms2}(J)S(\tau_1)+ \\ +\mathbf{cmds1}(J)S''(\tau_0)+\mathbf{cmds2}(J)S''(\tau_1)+\sum_{J'=1}^{ND}\mathbf{R}(J,J')I^+(\tau_1,\mu_{J'}) \tag{30}$$

for any direction μ_J. These new values of the coefficients can overrun the memory places of the previous ones, corresponding to the intial condition given by eq. (25).

We repeat the same exercice, namely to employ eq. (15) inside both the functional form for $S(\tau_0)$, given by eq. (29), and that for $S''(\tau_0)$, given by eq. (26), in order to recover the previous form for both of them, but now as a function of the upgoing intensities at τ_1 insted of τ_0, hence with different coeffficients. That is

$$S(\tau_0)=\mathbf{cs0}+\mathbf{css1}\,S(\tau_0)+\mathbf{css2}\,S(\tau_1)+ \\ +\mathbf{csds1}\,S''(\tau_0)+\mathbf{csds2}\,S''(\tau_1)+\sum_{J=1}^{ND}\mathbf{csi}(J)I^+(\tau_1,\mu_J) \tag{31}$$

and

$$S''(\tau_0)=\mathbf{cds0}+\mathbf{cds1}\,S(\tau_0)+\mathbf{cds2}\,S(\tau_1)+ \\ +\mathbf{cdds1}\,S''(\tau_0)+\mathbf{cdds2}\,S''(\tau_1)+\sum_{J=1}^{ND}\mathbf{cdi}(J)I^+(\tau_1,\mu_J). \tag{32}$$

Now, just by solving for $S(\tau_0)$ and $S''(\tau_0)$ we obtain the coefficients of the relations (27) and (28), earlier announced at the beginning of Section 3.2. These coefficients must be stored for further use. In such a way we have achieved part of out goal.

At this point let us describe the propagation of the downgoing intensities from $I^-(\tau_0, \mu_J)$, given by eq. (30), to $I^-(\tau_1, \mu_J)$ according to eq. (16). By re-arrangement of the coefficients we get the new values corresponding to the relation

$$I^-(\tau_1, \mu_J) = \mathbf{cm0}(J) + \mathbf{cms1}(J) S(\tau_0) + \mathbf{cms2}(J) S(\tau_1) + \\ + \mathbf{cmds1}(J) S''(\tau_0) + \mathbf{cmds2}(J) S''(\tau_1) + \sum_{J'=1}^{ND} \mathbf{R}(J, J') I^+(\tau_1, \mu_{J'}) \tag{33}$$

for all the directions μ_J. Again these new values of the coefficients can overrun the previous ones, corresponding to eq. (30).

If we introduce the foregoing eqs (27) and (28), whose coefficients we have just computed, in the functional form of $I^-(\tau_1, \mu_J)$ given by eq. (33), by re-arrangement of the previous coefficients we derive the new ones for the relation

$$I^-(\tau_1, \mu_J) = \mathbf{cm0}(J) + \mathbf{cms1}(J) S(\tau_1) + \mathbf{cmds1}(J) S''(\tau_1) + \sum_{J'=1}^{ND} \mathbf{R}(J, J') I^+(\tau_1, \mu_{J'}), \tag{34}$$

which we will cast into the form required by eq. (25) by setting equal to zero the coefficients $\mathbf{cms2}(J)$ and $\mathbf{cmds2}(J)$. We have thus determined the coefficients of the linear relation required as the initial condition at τ_1, that will be necessary to study the propagation of the downgoing intensities in the succesive layer (τ_1, τ_2).

We have still to determine the initial condition for the propagation of $S''(\tau)$, that is to say a linear relation like eq. (26), now for τ_1. It is matter of recovering the functional form of $S''(\tau_1)$ in ordert to start the study of the spline chain in the layer (τ_1, τ_2). We have at hand the fundamental relation for the cubic spline, namely eq. (8) that links linearly $S''(\tau_0)$, $S''(\tau_1)$ and $S''(\tau_2)$ with $S(\tau_0)$, $S(\tau_1)$ and $S(\tau_2)$.

By introducing in eq. (8) the formal expressions for $S(\tau_0)$ and $S''(\tau_0)$, given by eqs (27) and (28), we get easily the coefficients of the equation

$$S''(\tau_1) = \mathbf{cds0} + \mathbf{cds1}\, S(\tau_1) + \mathbf{cds2}\, S(\tau_2) + \\ + \mathbf{cdds1}\, S''(\tau_1) + \mathbf{cdds2}\, S''(\tau_2) + \sum_{J=1}^{ND} \mathbf{cdi}(J) I^+(\tau_1, \mu_J), \tag{35}$$

akin to eq. (26), the bootstrap at τ_0, that was the initial condition to studying the layer (τ_0, τ_1). Equation (35), together with (33) that is the initial condition for the treatment of radiative transfer, will allow us to repeat the foregoing procedure for the layer (τ_1, τ_2). This scheme is then iterated layer by layer till the bottom of the atmosphere.

3.3. *The solution at the bottom and the Back-Substitution.* At the end of the forward-elimination scheme we have at hand the full set of coefficients of eqs (27) and (28) for each optical depth of the set $\{\tau_0, \tau_1, ..., \tau_{NL-1}\}$. The explicit values of $S(\tau_L)$ and $S''(\tau_L)$ as well as those of the set of the outgoing intensities $\{I^+(\tau_L, \mu_J), J = 1, ND\}$ have now to be computed in the back-substitution scheme.

For the sake of a more clear exposition of the mathematical solution at the bottom we will rewrite eqs (27) and (28) for τ_{NL}, that is

$$\begin{aligned} S(\tau_{NL-1}) &= \mathbf{cbs0}(NL-1) + \mathbf{cbss}(NL-1)S(\tau_{NL}) + \\ &+ \mathbf{cbsd}(NL-1)S''(\tau_{NL}) + \sum_{J=1}^{ND} \mathbf{cbsi}(NL-1, J)I^{+}(\tau_{NL}, \mu_J) \end{aligned} \tag{36}$$

and

$$\begin{aligned} S''(\tau_{NL-1}) &= \mathbf{cbd0}(NL-1) + \mathbf{cbds}(NL-1)S(\tau_{NL}) + \\ &+ \mathbf{cbdd}(NL-1)S''(\tau_{NL}) + \sum_{J=1}^{ND} \mathbf{cbdi}(NL-1, J)I^{+}(\tau_{NL}, \mu_{J'}). \end{aligned} \tag{37}$$

Also, at the end of the forward-elimination scheme, the current values of the coefficients of the equation for $I^{-}(\tau_{NL}, \mu_J)$, that is

$$\begin{aligned} I^{-}(\tau_{NL}, \mu_J) &= \mathbf{cm0}(J) + \mathbf{cms1}(J)S(\tau_{NL}) + \mathbf{cms2}(J)S(\tau_{NL+1}) + \\ &+ \mathbf{cmds1}(J)S''(\tau_{NL}) + \mathbf{cmds2}(J)S''(\tau_{NL+1}) + \sum_{J'} R(J, J')I^{+}(\tau_{NL}, \mu_{J'}) \end{aligned} \tag{38}$$

are still stored in the scratch memory. For the sake of a homogeneous algorithm we had kept the dependence on $S(\tau_{NL+1})$ and $S''(\tau_{NL+1})$ through the coefficients $\mathbf{cms2}(J)$ and $\mathbf{cmds2}(J)$. But these coefficients are null so that $S(\tau_{NL+1})$ and $S''(\tau_{NL+1})$ do not play any active role. The same algorithmical requirement compelled us to introduce the dummy supplementary optical depth τ_{NL+1}.

At this point we can apply the lower boundary condition for the radiative transfer, i.e. the formal expression for $I^{+}(\tau_{NL}, \mu_J)$ given by eq. (5). If we replace this expression in the previous eqs (36), (37) and (38), by re-arrangement of terms we obtain the explicit values of the coefficients of the two linear relations for $S(\tau_{NL-1})$ and $S''(\tau_{NL-1})$as a function of $S(\tau_{NL})$, $S'(\tau_{NL})$, $S''(\tau_{NL})$ and $S'''(\tau_{NL})$, that we will write as

$$S(\tau_{NL-1}) = \mathcal{LR}\{S(\tau_{NL}), S'(\tau_{NL}), S''(\tau_{NL}), S'''(\tau_{NL})\} \tag{39}$$

and

$$S''(\tau_{NL-1}) = \mathcal{LR}\{S(\tau_{NL}), S'(\tau_{NL}), S''(\tau_{NL}), S'''(\tau_{NL})\}. \tag{40}$$

Likewise, if we take into account the aforesaid expression for $I^{-}(\tau_{NL}, \mu_J)$, whose coefficients $\mathbf{cms2}(J)$ and $\mathbf{cmds2}(J)$ are null, we get also the coefficients of the linear relations

$$I^{-}(\tau_{NL}, \mu_J) = \mathcal{LR}\{S(\tau_{NL}), S'(\tau_{NL}), S''(\tau_{NL}), S'''(\tau_{NL})\} \tag{41}$$

for each direction μ_J.

In the forward-elimination, at the beginning of the study of each layer (τ_L, τ_{L+1}), we have formally computed the mean intensity $J(\tau_L)$ and the corresponding source function $S(\tau_L)$ at the upper optical depth τ_L. That is to say, we have not yet used the relation given by eq. (2) at the last optical depth τ_{NL}. Let us do it here.

Equation (5) for $I^+(\tau_{NL}, \mu_J)$ and (41) for $I^-(\tau_{NL}, \mu_J)$ allow us to compute via eq. (4) the coefficients of a linear relation like

$$J(\tau_{NL}) = \mathcal{LR}\{S(\tau_{NL}), S'(\tau_{NL}), S''(\tau_{NL}), S'''(\tau_{NL})\}. \tag{42}$$

Now thanks to eq. (13) we can compute also the coefficients of the linear relation

$$J'(\tau_{NL}) = \mathcal{LR}\{S(\tau_{NL}), S'(\tau_{NL}), S''(\tau_{NL}), S'''(\tau_{NL})\}. \tag{43}$$

By means of eqs (2) and (12) we can eventually derive the explicit coefficients of the linear relations

$$S(\tau_{NL}) = \mathcal{LR}\{S(\tau_{NL}), S'(\tau_{NL}), S''(\tau_{NL}), S'''(\tau_{NL})\} \tag{44}$$

and

$$S'(\tau_{NL}) = \mathcal{LR}\{S(\tau_{NL}), S'(\tau_{NL}), S''(\tau_{NL}), S'''(\tau_{NL})\}. \tag{45}$$

The two latter relations are the independent conditions to close both the radiative transfer and the spline chain.

According to the cubic approximation for $S(\tau)$, $S'(\tau_{NL})$ is a linear function of $S(\tau_{NL-1})$, $S(\tau_{NL})$, $S''(\tau_{NL-1})$ and $S''(\tau_{NL})$, as shown by eq. (9). The "physical" equation (45) and the spline equation (9) lead to a new linear relation among $S(\tau_{NL-1})$, $S''(\tau_{NL-1})$, $S(\tau_{NL})$ and $S''(\tau_{NL})$ that, together with eqs (39), (40) and (44) lead easily to the explicit values of the latter four variables. Consequently we easily obtain also the values of $S'(\tau_{NL})$ and $S'''(\tau_{NL})$. The explicit values of these variables at τ_{NL} allow us to compute those of the set $\{I^+(\tau_{NL}, \mu_J)\}$ through eq. (5).

Once the explicit values of $S(\tau_{NL-1})$, $S(\tau_{NL})$, $S''(\tau_{NL-1})$ and $S''(\tau_{NL})$ as well as those of the set $\{I^+(\tau_{NL}, \mu_J), J = 1, ND\}$ are known, it is straightforward to compute those of the set $\{I^+(\tau_{NL-1}, \mu_J), J = 1, ND\}$ via eq. (15). Then eqs (27) and (28) will yield the explicit values of $S(\tau_{NL-2})$ and $S''(\tau_{NL-2})$, hence those of the set $\{I^+(\tau_{NL-2}, \mu_J), J = 1, ND\}$. And so on along the back-substitution.

4. *Conclusions*. Our Implicit Integral Method is based on the progressive treatment of the different layers that consitute a model of the stellar atmosphere physical system, from the outermost layer (the surface) to the deepest one (the bottom). The protagonist variables of the method are the upgoing and downgoing specific intensities $I^{\pm}(\tau, \mu)$ as well as the corresponding source functions that besides the thermal sources include a scattering-like integral into which there enter the foregoing specific intensities. Precisely, the study (and the elimination) of each single layer (τ_L, τ_{L+1}) leads to a relation that links linearly the value $S(\tau_L)$ of the source function at τ_L with $S(\tau_{L+1})$, the value at τ_{L+1}. Once obtained via the study of the last layer the relation between the values of the source function at the two last optical depth points, the boundary condition at τ_{NL} given by eq. (5) makes it possible to compute the explicit values of $S(\tau_{NL})$ and $S(\tau_{NL-1})$, hence all the others.

In order to design the required elimination scheme it is necessary to employ a mathematical model for $S(\tau)$. In principle the simplest and easiest model would be a piece-wise linear one, but the discontinuity of the first derivative $S'(\tau)$ at each knot τ_L can imply severe errors and possible numerical instabilities because the above discontinuity is incompatible with the radiative transfer (RT) process itself, where both $I^{\pm}(\tau,\mu)$ and their first derivatives must be continuous, and therefore also the mean intensity $J(\tau)$ and its first derivative. Thus the foregoing model cannot be correct, but for extreme cases of the thermal sources.

A piece-wise parabolic model warrants the continuity of $S'(\tau)$ at all depth points. Such a model shall include also $S'(\tau)$ as a protagonist variable in the process of progressive elimination of the atmospheric layers. Hence $S'(\tau)$ must be put into relation with the foregoing protagonist variables, which can be done either mathematically or physically.

From the mathematical standpoint we can introduce $S'(\tau)$ by means of the formula

$$S'(\tau_L) = 2\frac{S(\tau_{L+1}) - S(\tau_L)}{\tau_{L+1} - \tau_L} - S'(\tau_{L+1}), \tag{46}$$

which could however introduce numerical instabilities because of the difference between the two terms in the right-hand side, above all in the back-substitution process that works with explicit values. On the other hand, from the physical standpoint $S'(\tau)$ could be included by taking into account at all the optical depth points the equations (12) and (13) for the derivatives of the source function. However, in case that $\varepsilon(\tau)$ and $B(\tau)$ show large variations (as it is the case of the formation of Lyman α in cool stars), severe instabilities may appear, too.

These drawbacks can be avoided by introducing a piece-wise cubic approximation, where a further protagonis variable has to be included, namely the second derivative $S''(\tau)$. That is, by means of a cubic spline model that automatically warrant the continuity of $S(\tau)$ and its first two derivatives. To circumvent the explicit calculation of the derivatives makes the above difficulties vanish.

The source function $S(\tau_L)$ at each depth point τ_L will be expressed as a linear function of $S(\tau_{L+1})$ and $S''(\tau_{L+1})$. Therefore we shall transmit from any optical depth to the next one also the (implicit) value of $S''(\tau_{L+1})$. This is achieved thanks to the fundamental relation that assures the continuity properties imposed by the cubic spline condition (cf. eq. (8)). However the propagation of $S(\tau)$ and $S''(\tau)$ via a cubic spline model constitutes a twopoint boundary problem. Nevertheless this is perfectly compatible with the treatment of the transmission of the specific intensities, as it is performed in the scheme for the solution of the two-point boundary value RT problem. Both propagation processes can be treated simultaneously.

Under these conditions we can warrant the elimination of many of the causes of instability that can spoil the algorithm for the solution of the system of specific

RT equations coupled through a scattering-like term in the source function, whose initial conditions are assigned at different points of the physical system.

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ИСПРАВЛЕННЫЙ ВАРИАНТ НЕЯВНОГО ИНТЕГРАЛЬНОГО МЕТОДА РЕШЕНИЯ ЗАДАЧ ПЕРЕНОСА ИЗЛУЧЕНИЯ

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Проблемы переноса излучения (RT), в которых функция источника содержит интеграл рассеяния, являются типичными двуточечными граничными задачами. При решении соответствуюших дифференциальных уравнений приходится делать предположения относительно решения, а именно, относительно удельной интенсивности поля излучения $I(\tau; n)$. В противоположность этому при использовании интегральных методов, предположения относятся к функции источника $S(\tau)$. Последнее кажется оправданным, поскольку можно ожидать, что по сравнению с интенсивностью $S(\tau)$ меняется с глубиной более медленно. В предыдущих работах для функции источника мы использовали кусочно-параболическое приближение, что гарантирует непрерывность $S(\tau)$ и ее первой производной в каждой точке среды. Здесь мы требуем непрерывность второй производной $S''(\tau)$. Другими словами, мы пользуемся представлением функции источника посредством кубических сплайнов, что чрезвычайно стабилизирует численные процессы.

Ключевые слова: *численные методы:перенос излучения - звезды:атмосферы*

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