Stability of real-valued maximally flat Thiran allpole filters

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Proposed is the necessary and sufficient condition, which depends on the group delay evaluated at $\omega = 0$, for the stability of maximally flat group delay Thiran filters. This interesting result complements the condition originally proposed by Thiran.

Introduction: Thiran proposed the design of the allpole filter with maximally flat group delay response at the frequency $\omega = 0$ in [1]. Solving the derived set of linear equations, closed form equations for the computation of the filter coefficients were proposed. Thiran also showed that the allpole filter is stable if the group delay evaluated at $\omega = 0$ is larger than zero. To be specific, the Thiran allpole filter $D(z)$ is given by [1]

$$D(z) = \frac{\sum_{n=0}^{N} d_n}{\sum_{n=0}^{N} d_n z^{-n}} \tag{1}$$

where N is the filter order and

$$d_n = (-1)^n \binom{N}{n} \frac{(2\tau)_n}{(2\tau + N + 1)_n} \tag{2}$$

where τ represents the group delay evaluated at $\omega = 0$. The binomial coefficient is defined by

$$\binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{3}$$

while the Pochhammer symbol $(x)_n$ can be expressed by

$$(x)_n = \begin{cases} \prod_{k=0}^{n-1} (x+k), & n > 0 \\ 1, & n = 0 \end{cases} \tag{4}$$

The allpole filter satisfies

$$G(\omega)|_{\omega=0} = \tau, \quad \left.\frac{d^k G(\omega)}{d\omega^k}\right|_{\omega=0} = 0, \quad k = 1, \ldots, K \tag{5}$$

where $G(\omega)$ stands for the group delay response and K is the degree of flatness and is related to the filter order N as $K = 2(N-1)$.

The Thiran allpole filter has been used in many applications. The design of maximally flat fractional delay filters is addressed in [2, 3]. The method applies the Thiran allpole filter to design a desired allpass filter since their corresponding group delays are linearly related. Additionally, in this approach, the value of τ ranges from $-1/2$ to 0. In [4], the design of IIR wavelet filters based on maximally flat allpole filters was proposed. In a similar way, the design of an IIR wavelet filter with the desired degree of flatness is described in [5, 6]. The Thiran allpole filter is successfully applied to design IIR filters with specified degrees of flatness and constant group delay characteristics in [7]. The design of FIR Hilbert transform pairs of wavelet bases based on the Thiran allpole filter is formulated in [8, 9].

Thiran expresses the stability condition of the allpole filter as $\tau > 0$ [1]. On the other hand, using numerical results in [2] it was pointed out that Thiran filters were also stable if $-1/2 < \tau < 0$. These results give us the motivation to reformulate the stability condition of the allpole filter. Therefore, this Letter gives the necessary and sufficient condition to solve the stability problem.

Necessary and sufficient condition: This Section reformulates the stability of the Thiran filter. At first, we give some interesting properties of the filter coefficients. These properties play a key role to determine the necessary and sufficient condition for the stability of the Thiran filter. Thus, we establish the algebraic stability test presented in [10, 11].

We introduce some properties of the coefficients d_n, $n = 1, \ldots, N$. Observe that for $\tau = -(N+k)/2$, $k = 1, \ldots, n$ no solution exists since the denominator in (2) vanishes. Additionally, from the numerator in (2), the values $\tau = -k/2$, $k = 0, \ldots, n-1$ make the filter coefficient d_n zero. Additionally, evaluating τ at $-n/2$, d_n equals 1. Finally, if τ approaches to infinity d_n tends to $(-1)^n \binom{N}{n}$. In summary, it follows that

$$d_n = \begin{cases} 0, & \text{if } \tau = -k/2 \text{ for } k = 0, \ldots, n-1 \\ 1, & \text{if } \tau = -n/2 \\ \infty, & \text{if } \tau = -(N+k)/2 \text{ for } k = 1, \ldots, n, \\ (-1)^n \binom{N}{n}, & \text{if } \tau \to \pm\infty \end{cases} \tag{6}$$

As a next step, we define the following recursive equation, which helps us to derive the new stability condition,

$$d_{N-m-1,n} = \frac{d_{N-m,n} - d_{N-m,N-m-n} d_{N-m,N-m}}{1 - d_{N-m,N-m}^2} \tag{7}$$

where $N \geq 2$, $m = 0, \ldots, N-2$, and $n = 0, \ldots, N-m-1$. The initial value in (7) is $d_{N,n} = d_n$. We illustrate an interesting behaviour of $d_{N-m-1,n}$ when $\tau = -k/2$ for $k = 0, \ldots, N-m-1$ and $m = 0, \ldots, N-2$.

Accordingly, for $m = 0$, (7) becomes

$$d_{N-1,n} = \frac{d_n - d_{N-n} d_N}{1 - d_N^2} \tag{8}$$

Here we are interested in the values of $d_{N-1,n}$ at $\tau = -k/2$, for $k = 0, \ldots, N-1$. From (6), the value d_N vanishes at those points and, therefore, $d_{N-1,n}$ equals d_n. Now consider the case where τ approaches to infinity. In this case, $d_{N-1,n}$ has an indeterminate form. However, in order to overcome this problem, we use $d_n = (-\varepsilon)^n \binom{N}{n}$ and ε approaches 1. Consequently, we obtain

$$\lim_{\tau\to\infty} d_{N-1,n} = \lim_{\varepsilon\to 1} (-1)^n \binom{N}{n} \frac{(-\varepsilon)^n - (-\varepsilon)^{2N-n}}{1 - \varepsilon^{2N}} \tag{9}$$

Applying the L'Hopital rule, we finally arrive at

$$\lim_{\tau\to\infty} d_{N-1,n} = (-1)^n \binom{N-1}{n} \tag{10}$$

In a similar fashion, for $m = 1$, it follows that

$$d_{N-2,n} = \frac{d_{N-1,n} - d_{N-1,N-1-n} d_{N-1,N-1}}{1 - d_{N-1,N-1}^2} \tag{11}$$

Here we are interested in the values of $d_{N-2,n}$ at the points $\tau = -k/2$, for $k = 0, \ldots, N-2$. Considering $d_{N-1,N-1} = d_{N-1} = 0$ at those points [see (6)], we obtain $d_{N-2,n} = d_{N-1,n} = d_n$ and

$$\lim_{\tau\to\infty} d_{N-2,n} = (-1)^n \binom{N-2}{n} \tag{12}$$

Generally, the coefficients defined in (7) satisfy

$$d_{N-m-1,n} = \begin{cases} d_n & \text{if } \tau = -k/2 \text{ for } k = 0, \ldots, N-m-1 \\ (-1)^n \binom{N-m-1}{n} & \tau \to \infty \end{cases} \tag{13}$$

To introduce the necessary and sufficient stability condition, we recall the algebraic stability test proposed in [10, 11], i.e. the poles of $D(z)$ are strictly inside the unit circle if and only if

$$d_{N-i,N-i}^2 < 1, \quad i = 0, \ldots, N-1 \tag{14}$$

Note that the stability problem involves N conditions, which will be solved in the following.

The first condition that should be satisfied is

$$d_N^2 < 1 \tag{15}$$

Our goal is to find the values τ such that (15) is fulfilled. Accordingly, using (6), we arrive at

$$d_N = \begin{cases} 0, & \text{if } \tau = -k/2 \text{ for } k = 0, \ldots, N-1 \\ 1, & \text{if } \tau = -N/2 \\ \infty, & \text{if } \tau = -(N+k)/2 \text{ for } k = 1, \ldots, N \\ (-1)^N & \text{if } \tau \to \pm\infty \end{cases} \tag{16}$$

Fig. 1 illustrates the plot of d_N^2 as a function of τ. Observe that $d_N^2 > 1$ for $\tau < -N/2$ because the points $\tau = -(N+k)/2$, for $k = 1, \ldots, N$, make d_N^2 unbounded and d_N^2 approaches 1 when $\tau \to -\infty$. Furthermore, because of the zeros of d_N, the square magnitude of d_N is less that 1 in $-N/2 < \tau < 0$. Finally, note that d_N^2 is an increasing function for $\tau > 0$ and bounded by 1. Consequently, condition (15) is satisfied if $\tau > -N/2$.

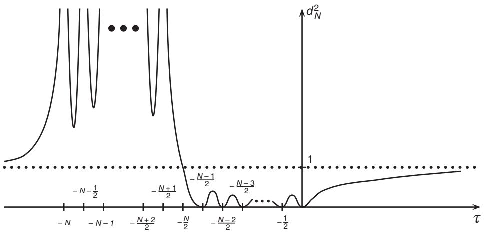

Fig. 1 *Square value of d_N against τ*

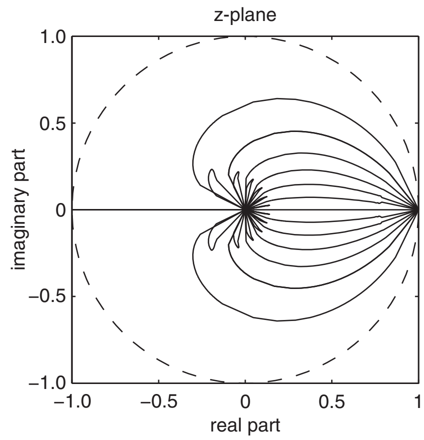

Fig. 2 *Root locus of 13th-order Thiran filter for $-1/2 \leq \tau$*

The second stability condition is

$$d_{N-1,N-1}^2 < 1 \qquad (17)$$

From (13), we observe that $d_{N-1,N-1}^2 = d_{N-1}^2 = 1$ at $\tau = -(N-1)/2$. Similarly, the coefficient $d_{N-1,N-1}^2$ is less than 1 for $-(N-1)/2 < \tau < 0$ because of the zeros of d_{N-1}. Finally, from (10), we note that $d_{N-1,N-1}^2$ is an increasing function and bounded by 1 for $0 < \tau$. Consequently, condition (17) is satisfied when $-(N-1)/2 < \tau$. In a similar way, we found that $d_{N-i,,N-i}^2 < 1$ if $-(N-i)/2 < \tau$, for $i = 0, \ldots, N-1$.

This interesting result verifies the simulation results in [2] and gives the necessary and sufficient condition for the stability of the Thiran filter. The above results are summarised in the following theorem.

Theorem: Let $D(z)$ be an N-order maximally flat group delay allpole filter defined in (1). The filter $D(z)$ is stable if and only if $\tau > -1/2$.

As an illustrative example, Fig. 2 shows the root locus of the 13th-order Thiran filter. Observe that at $\tau = -1/2$ the allpole filter reduces to a first-order filter with the pole at $z = -1$. Similarly, when τ approaches to infinity, the poles tend to $z = 1$. It is worth mentioning that the root locus for $-1/2 < \tau < 0$ is the small lobes around the origin and the line from $z = -1$ to $z = 0$.

Conclusion: This Letter introduces the necessary and sufficient condition for the Thiran filters to be stable and complements the condition originally proposed by Thiran. The proposed result can be useful for other interesting applications, e.g. the design of casual cardinal orthogonal scaling functions [12].

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