Proposal for realization of the Majorana equation in a tabletop experiment

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We introduce the term Majoranon to describe particles that obey the Majorana equation, which are different from the Majorana fermions widely studied in various physical systems. A general procedure to simulate the corresponding Majoranon dynamics, based on a decomposition of the Majorana equation into two Dirac equations, is described in detail. It allows the simulation of the two-component chiral spinors, the building blocks of modern gauge theories, in the laboratory with current technology. Specifically, a Majoranon in one spatial dimension can be simulated with a single qubit plus a continuous degree of freedom, for example, a single trapped ion. Interestingly, the dynamics of a Majoranon deviates most clearly from that of a Dirac particle in the rest frame, in which the continuous variable is redundant, making a possible laboratory implementation feasible with existing setups.

Introduction. Quantum simulation was originally envisioned as an approach to study complex quantum systems that are difficult to understand using conventional methods [1,2]. However, it has recently been realized that the concept can also be used to engineer quantum dynamics not readily accessible in naturally occurring physical systems; e.g., elementary particles in lower space-time dimensions. A notable example is an experimental demonstration of quantum simulation of the Dirac equation [3,4] in $1+1$-dimensional space-time using trapped ions [5]. There are many other proposals and experimental demonstrations regarding the quantum simulation of relativistic equations and phenomena [6–18], including a recent proposal to simulate the Majorana equation [19].

The Majorana equation is a Lorentz covariant equation similar to the Dirac equation, discovered by Majorana [20,21]:

$$\iota \gamma^\mu \partial_\mu \psi = m \psi_c, \qquad (1)$$

where we have set $\hbar = c = 1$ for convenience. The gamma matrices γ^μ obey $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$; the symbol ψ_c stands for charge conjugation of the spinor ψ, $\psi_c \equiv C(\gamma^0)^T \psi^*$, where C obeys $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$, $C^\dagger = C^{-1}$, and $C^T = -C$; $g^{\mu\nu}$ is the standard metric in the vacuum [22]. To avoid confusion, let us discuss here the nomenclature found in the literature. A field obeying the Majorana condition, $\psi_c = \psi$, is called a Majorana field (fermion, particle), in which case Eq. (1) reduces to the Dirac equation. The (Dirac) equation obeyed by a Majorana field is sometimes called the Majorana equation [23], but we will reserve the term Majorana equation for Eq. (1) when the condition does not hold and call the hypothetical particles that obey it Majoranons. We would like to emphasize that Majoranons are thus different from Majorana fermions that are widely studied in the literature.

The Majorana equation has received a renewed interest since the discovery of finite masses of neutrinos. A neutrino is a neutral fermion; therefore it could be either a Majorana

fermion, in which case it is its own antiparticle, or a Dirac particle, in which case a neutrino will be different from an antineutrino. For this reason, most studies on Eq. (1) are limited to Majorana fermions, and the Majorana equation by itself seems to be of academic interest only. However, Casanova *et al.* have recently proposed a general procedure to implement nonphysical operations such as charge conjugation and time reversal, allowing for an experimental study of the equation [19].

Closely related to the Majorana equation are the so-called two-component Majorana equations (TCMEs). In $1+1$ and $2+1$ dimensions, it turns out that one of the two TCMEs is equivalent to the Majorana equation, whereas in $3+1$ dimensions the two equations have to be combined together to form the Majorana equation [24]. The importance of the TCMEs is that they are obeyed by the building blocks of the modern gauge theory, chiral spinors [24,25]. In this work, we show that it is possible to decompose one of the equations, say for a right-chiral field, into two Hamiltonian equations that can be simulated. Our method differs from the earlier proposal [19], which depends on enlarging the Hilbert space of the system to turn the Majorana equation into a higher-dimensional Dirac equation. An advantage is that the size of the Hilbert space required is smaller, but a trade-off exists, in that a complete reconstruction of the spinor, including the continuous mode, is required. These points are discussed further below.

This Rapid Communication is organized as follows. We start by giving a short summary on quantum simulation of the Majorana equation based on Hilbert space expansion mentioned above, followed by a brief introduction to TCMEs for nonspecialists. We show how the equations can be motivated by trying to introduce Lorentz invariant mass terms in the Weyl equations [24], and describe how relativistic equations for higher-dimensional spinors can be constructed from these equations. We then prove that a general fermion obeying what we will call a Dirac-Majorana equation can be decomposed into two Majorana particles [22,24,26], which implies that such an equation can be physically realized. It is shown that this decomposition allows a general simulation procedure.

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Quantum simulation by Hilbert space expansion. As shown in [19], it is possible to implement unphysical operations such as charge conjugation, complex conjugation, and time reversal, via mapping to an enlarged Hilbert space. It is thus possible to simulate the Majorana equation which in $1 + 1$ dimensions reads

$$\imath \partial_t \psi = \sigma_x p_x \psi - \imath m \sigma_y \psi^*, \tag{2}$$

where $\psi = (\psi^{(1)}, \psi^{(2)})$, $\psi^{(i)} \in \mathbb{C}$ so $\psi \in \mathbb{C}^2$. To see how, first note that this equation can be rewritten as

$$\imath \partial_t (\psi + \psi^*) = \sigma_x p_x (\psi + \psi^*) + \imath m \sigma_y (\psi - \psi^*), \tag{3}$$

$$\imath \partial_t (\psi - \psi^*) = \sigma_x p_x (\psi - \psi^*) - \imath m \sigma_y (\psi + \psi^*). \tag{4}$$

Now, by mapping ψ to an extended Hilbert space such that $\Psi = [\mathrm{Re}(\psi^{(1)}), \mathrm{Re}(\psi^{(2)}), \mathrm{Im}(\psi^{(1)}), \mathrm{Im}(\psi^{(2)})]$, $\psi = M\Psi$, $M = (1_2, \imath 1_2)$, the Majorana equation can be written as the Schrödinger equation, $\imath \partial_t \Psi = H_M \Psi$, where

$$H_M = (1_2 \otimes \sigma_x) \, p_x - m(\sigma_x \otimes \sigma_y). \tag{5}$$

This Hamiltonian, describing the Dirac equation, can be implemented with two trapped ions.

Two-component formalism. Here we give a brief introduction to the two-component formalism of chiral spinors and show how they are related to the usual four-component Dirac equation and the Majorana equation [22,24,25]. Being the smallest irreducible representations of the Lorentz group, the chiral (or Weyl) spinors are the basic building blocks of modern gauge theories [22]. Moreover, chirality plays an important role in the standard model due to the chiral nature of electroweak interaction, i.e., the electroweak interaction distinguishes between left- and right-chiral fields.

Let us start with the description of massless fermions by the Weyl equations:

$$(\partial_t - \vec{\sigma} \cdot \vec{\nabla})\psi_L = 0, \tag{6}$$

$$(\partial_t + \vec{\sigma} \cdot \vec{\nabla})\psi_R = 0, \tag{7}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. The Weyl spinors ψ_R and ψ_L transform according to the two-dimensional irreducible representation of the Lorentz group and its complex conjugate, respectively, and are called two-component chiral spinors. In the massless case, the chirality coincides with the helicity defined as the projection of a particle's spin along its momentum.

The left-chiral Weyl spinor was believed to describe the massless neutrinos and had been incorporated in the old standard model before the discovery of nonzero neutrino masses [22]. To introduce mass terms in the Weyl equations, one must be careful about the Lorentz covariance of the equation. In order to keep the equations covariant, one should note that $m\epsilon^{-1}\psi^*$, where ψ is either ψ_L or ψ_R and $\epsilon = i\sigma_y$, transforms like the differential part of the corresponding Weyl equation [24]. Therefore it is possible to write the Lorentz invariant equations for massive fermions as

$$i(\partial_t - \vec{\sigma} \cdot \vec{\nabla})\psi_L(x) - m\epsilon \psi_L^*(x) = 0, \tag{8a}$$

$$i(\partial_t + \vec{\sigma} \cdot \vec{\nabla})\psi_R(x) + m\epsilon \psi_R^*(x) = 0, \tag{8b}$$

called the left-chiral and right-chiral two-component Majorana equations. These spinors form the building blocks of higher-dimensional spinors such as the Dirac or Majorana fields. They do not correspond to any massive particles by themselves, however, and a direct observation of their dynamics is ordinarily impossible. In fact, Eqs. (8) are equivalent to the Majorana equation, which can be easily seen by constructing the four-component spinor (ψ_R, ψ_L). In $1 + 1$ or $2 + 1$ dimensions, the situation simplifies because a Majoranon can be represented by a two-component spinor and it can be shown, by explicit construction of γ^μ in terms of Pauli matrices, that one of the two TCMEs is equivalent to the Majorana equation.

It is also possible to build a Majorana field from either of the chiral fields. To see this, for ψ_L say, note that $\epsilon \psi_L^*$ obeys the equation

$$i(\partial_t + \vec{\sigma} \cdot \vec{\nabla})\epsilon \psi_L^*(x) + m\epsilon \epsilon \psi_L(x) = 0. \tag{9}$$

That is, $\epsilon \psi_L^*$ behaves like a right-chiral two-component spinor. Thus noticing that Eq. (8a) mixes the left- and right-chiral spinors, let us define a four-component spinor

$$\begin{pmatrix} \epsilon \psi_L^* \\ \psi_L \end{pmatrix},$$

which obeys the Dirac equation

$$i\gamma^\mu \partial_\mu \Psi_M - m\Psi_M = 0, \tag{10}$$

where γ^μ are the Dirac matrices in the chiral representation.

We have thus shown that a four-component spinor obeying the Dirac equation can be constructed from a two-component left-chiral spinor. Using the properties of the Dirac matrices it is possible to show that Ψ_M has a special property

$$(\Psi_M)_c = \tilde{C}(\gamma^0)^T \Psi_M^* = \Psi_M, \tag{11}$$

where

$$\tilde{C} = \begin{pmatrix} i\sigma_y & 0 \\ 0 & -i\sigma_y \end{pmatrix}$$

in the chiral representation. The subscript c denotes charge conjugation and the (Majorana) condition states that the spinor is invariant under charge conjugation. A Majorana particle is therefore necessarily charge neutral, which is why such a particle can be its own antiparticle.

To build up a Dirac field from two-component spinors we need two independent left-chiral spinors, say ψ_L and χ_L. Then $(\epsilon \psi_L^*, \chi_L)^T$ obeys the Dirac equation, but not the Majorana condition.

The fact that the TCME can be converted into the four-component Dirac equation suggests that a two-component Majoranon is equivalent to a four-component Majorana fermion. The resulting equation can then be simulated with a single trapped ion using the method proposed in [4]. Furthermore, if we restrict the spatial dimension to 1 it is possible to write down the equation of motion in the form of interacting qubits coupled to a phonon mode, where only one qubit is coupled to the phonons. This fact has already been noticed in [19] through different argument and allows quantum simulation of the Majorana equation using trapped two-level ions. Our argument here can thus be seen as an alternative derivation of the result, starting from the two-component formalism. The salient point is that in $1 + 1$ dimensions a

Majoranon is described by a two-component spinor obeying either of Eqs. (8) with one spatial degree of freedom and therefore it should be possible to convert this to a Hamiltonian dynamics by constructing a Majorana fermion as shown earlier. To see this explicitly, consider the Dirac equation for a Majoranon confined to move in the z direction. In the Majorana representation $\gamma_M^0 = \sigma_y \otimes \sigma_x$, $\gamma_M^3 = i\sigma_y \otimes \sigma_y$, and $\Psi_M^* = \Psi_M$, and Eq. (10) becomes

$$i\partial_t \Psi_M = 1 \otimes \sigma_z (i\partial_z)\Psi_M + m(\sigma_y \otimes \sigma_x)\Psi_M. \qquad (12)$$

The Hamiltonian for this Schrödinger equation is equivalent to (5) up to a unitary transformation and thus can be readily simulated with trapped ions as shown explicitly in the reference.

Decomposition of a Majoranon into two Majorana fields. Here, it is useful to go back to Eq. (5) briefly and discuss an interesting fact. Using the unitary operation $U = ie^{-\iota\pi\sigma_y/4} \otimes e^{-\iota\pi\sigma_x/4}$ with the basis $\Psi = U\Phi \in \mathbb{R}^4$, such that $\psi = MU\Phi$, the "Majorana Hamiltonian" H_M in Eq. (5) becomes

$$H = U^\dagger H_M U, \qquad (13)$$

$$= (1_2 \otimes \sigma_x) p_x + m(\sigma_z \otimes \sigma_z). \qquad (14)$$

This Hamiltonian leads to two uncoupled Dirac equations:

$$\iota\partial_t \phi_\pm = H_\pm \phi_\pm, \qquad (15)$$

$$H_\pm = \sigma_x p_x \pm m\sigma_z, \qquad (16)$$

with $\phi_+ = (\Phi^{(1)}, \Phi^{(2)})$ and $\phi_- = (\Phi^{(3)}, \Phi^{(4)})$ such that $\Phi = (\phi_+, \phi_-)$. It is interesting that $\phi_\pm$ are invariant under charge conjugation, which in $1 + 1$ dimensions is defined as $-i\sigma_z\sigma_y\phi_\pm^*$. The uncoupling thus suggests that a Majoranon in $1 + 1$ dimensions can be decomposed into two Majorana fermions. In fact it is well known that a field (which in $1 + 1$ dimensions has two components) obeying the Dirac-Majorana equation, a general relativistic field equation that has both the Dirac and the Majorana mass terms, can be broken down into two Majorana fermions with different masses [24,26]. We provide a simple proof here.

To construct a general proof for the Dirac-Majorana equation, we first write it in the Majorana representation (in which charge conjugation is equal to complex conjugation denoted by the superscript $*$):

$$\iota\gamma^\mu \partial_\mu \Psi = m_M \Psi^* + m_D \Psi, \qquad (17)$$

where m_M and m_D refer to the Majorana and the Dirac mass, respectively. Decomposing the four-component spinor as $\Psi = (\Psi_+ + \iota\Psi_-)/\sqrt{2}$, where $\Psi_\pm$ are the Majorana fields, i.e., $\Psi_\pm = \Psi_\pm^*$, it is readily seen that

$$\iota\gamma^\mu \partial_\mu \Psi_\pm = (m_D \pm m_M)\Psi_\pm. \qquad (18)$$

Therefore a Dirac-Majoranon obeying Eq. (17) can be decomposed into two Majorana fermions obeying their respective Dirac equations. The following proposal for quantum simulation thus works not only for a Majoranon but for a Dirac-Majoranon too.

A proposal for quantum simulation. The decomposition described above has an interesting consequence for quantum simulation of a Majoranon: It is possible to simulate the Majorana equation by simulating two Dirac equations with the opposite mass signs. An advantage is that only a single qubit

is required, so the single-qubit addressability and qubit-qubit interaction is not an issue anymore. The price to pay is that the method requires complete state reconstruction, including the continuous degree of freedom. This is because the full information on spinors ϕ_+ and ϕ_- is required in order to calculate expectation values. Despite this caveat, the method provides a good alternative as far as observing exotic physics goes, especially for the case that only requires qubit state reconstruction, which we describe below. Note that the exotic physics of the Majorana equation does not manifest itself in the relativistic regime, in which the mass term becomes negligible and the Majorana equation reduces to the Dirac equation. Instead it is most prominent in the stationary case as discussed in [27].

It is also possible to simulate the TCME in two spatial dimensions, i.e., simulate a two-component chiral field in $2 + 1$ dimensions, using the fact that a Majoranon can be decomposed into two Majorana fermions. To see this, first note that the (right-chiral) two-component Majorana equation and the Majorana equation are equal and can be written as

$$i\partial_t \psi = (\sigma_x p_x + \sigma_y p_y)\psi - im\sigma_y \psi^*. \qquad (19)$$

By decomposing $\psi = (\psi_+ + i\psi_-)/\sqrt{2}$ and assuming $\psi_\pm = -i\sigma_z\sigma_y\psi_\pm^*$, it is easily seen that the two Dirac equations

$$i\partial_t \psi_\pm = -(\sigma_x p_x + \sigma_y p_y)\psi_\pm \pm m\sigma_z \psi_\pm \qquad (20)$$

are equivalent to Eq. (19). The same decomposition does not work in $3 + 1$ dimensions because the charge conjugation operator cannot be constructed for two-component spinors. This seems to corroborate the fact that chiral spinors are the irreducible representations of the Lorentz group, i.e., they cannot be decomposed into smaller representations. One is forced to use the four-component spinors in this case, requiring a quantum simulator in $3 + 1$ dimensions proposed in [4] for example. An interesting open question is whether two qubits instead of a single four-level system can be used to simulate the Dirac equation in $3 + 1$ dimensions.

Simulation procedure. Thus far, we have shown that in $1 + 1$ or $2 + 1$ dimensions a (two-component) chiral spinor, obeying the TCME or equivalently the Majorana equation, can be either turned into a four-component Majorana field or decomposed into two two-component Majorana fields. The latter allows quantum simulation of the Majorana equation in $1 + 1$ ($2 + 1$) dimensions with a single qubit and one (two) continuous degree(s) of freedom, given that one can reconstruct the states completely. The simulation procedure can be summarized into four steps: (i) Choose a system that simulates the Dirac equation; e.g., trapped ions [4,5], stationary light polaritons [8], or optical lattice [17] schemes. (ii) Prepare the initial state that you want to simulate; i.e., write down the spinor corresponding to a particular initial state and decompose it into two spinors obeying the Majorana conditions: $\psi = \psi_+ + i\psi_-$, where $(\psi_\pm)_c = \psi_\pm$. We will show specific examples below. (iii) Time evolve the states $\psi_\pm$ according to the Dirac equation with $\pm m$, respectively. (iv) Evaluate an observable in terms of ψ_+ and ψ_-.

To successfully verify the simulation of the Majorana equation, one has to choose an observable that clearly distinguishes between the Majorana and Dirac dynamics. One interesting observable that accomplishes this goal is

the pseudohelicity operator $\Sigma = \sigma_x p_x$ ($\sigma_x p_x + \sigma_y p_y$ in $2+1$ dimensions), which is conserved by the Majorana dynamics but not by the Dirac one [19]. In the limit of $m \to 0$, this quantity is conserved by both types of dynamics, but in general a massive Dirac particle is known to violate helicity conservation. In the rest-frame limit, a revealing observable would be $\langle \sigma_z \rangle$, which is conserved by the Dirac equation but not by the Majorana equation [27]. In our decomposition method, this quantity is preserved for the two fields ψ_+ and ψ_- separately as required by the conservation property in the Dirac equation, but the coherence between them prohibits the conservation in the total Majorana dynamics.

Lastly, we discuss two possible initial conditions that can be employed to show the Majorana dynamics. In $1+1$ or $2+1$ dimensions, the Majorana condition $\chi_\pm = -i\sigma_z\sigma_y\chi_\pm^*$ is satisfied by the basis vectors $\chi_+ = (1,-1)/\sqrt{2}$ and $\chi_- = (i,i)/\sqrt{2}$, and one can expand any initial state in terms of these basis spinors. A given state $\psi(0)$ can be expanded as $[\psi_+(0) + i\psi_-(0)]/\sqrt{2}$, in terms of the Majorana fields $\psi_\pm$ using the following relations:

$$\psi_+(0) = \frac{1}{2}[\psi(0) - i\sigma_z\sigma_y\psi^*(0)], \tag{21}$$

$$\psi_-(0) = \frac{-i}{2}[\psi(0) + i\sigma_z\sigma_y\psi^*(0)]. \tag{22}$$

For example, $\psi(0) = (1,0)$ is equivalent to $\psi_+(0) = (1,-1)/\sqrt{2}$ and $\psi_-(0) = -(i,i)/\sqrt{2}$. This example corresponds to a particle at rest discussed in [27]. In this case, there is no momentum dependence and the simulation only requires qubit state reconstruction, which can be carried out easily in various systems [28]. Therefore, one can simulate the Majorana equation in the rest frame with a single qubit, a trapped ion for example, instead of two. As explained earlier, this corresponds to the limiting case where the differences between the Majorana and the Dirac dynamics are most prominent.

As another example, consider a general state that distinguishes between the Majorana and the Dirac equation, i.e., a state that is not charge conjugate invariant,

$$\psi(0) = e^{ip_0 x} e^{-(x^2/4\Delta^2)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{23}$$

This state can be created in the trapped ion setup [13], for example. It is decomposed into

$$\psi_+(0) = \sqrt{2}\sin(p_0 x)e^{-(x^2/4\Delta^2)}\chi_-, \tag{24}$$

$$\psi_-(0) = -\sqrt{2}\cos(p_0 x)e^{-(x^2/4\Delta^2)}\chi_-. \tag{25}$$

In this case, evaluation of an observable requires complete information on $\psi_\pm$, including the spatial dependence. This means the full quantum state tomography of the qubit plus the phonon mode for trapped ions, whereas in other setups, such as stationary light polaritons or an optical lattice, it would mean a spatially resolved detection scheme.

Conclusion. A Majoranon, i.e., a particle that obeys the Majorana equation but not the Majorana condition, can be decomposed into two Majorana fermions, i.e., particles that answer to both the Majorana equation and condition. It is therefore possible to simulate a chiral spinor with previously proposed quantum simulators of the Dirac equation. Quantum simulation in $1+1$ or $2+1$ dimensions requires a single qubit plus one or two continuous degrees of freedom, instead of two qubits as proposed before, with the trade-off that complete state reconstruction is required. An interesting exception is for a particle initially at rest, for which the differences between the dynamics of the Majorana and the Dirac equations are most obvious. In the latter case, only single qubit state reconstruction is needed, greatly simplifying the complexity of a possible experimental implementation of our approach.

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