Comp. Appl. Math. (2015) 34:1–16
DOI 10.1007/s40314-013-0073-z

Laplace transform-homotopy perturbation method as a powerful tool to solve nonlinear problems with boundary conditions defined on finite intervals

U. Filobello-Nino · H. Vazquez-Leal · Y. Khan · A. Perez-Sesma · A. Diaz-Sanchez · V. M. Jimenez-Fernandez · A. Herrera-May · D. Pereyra-Diaz · J. M. Mendez-Perez · J. Sanchez-Orea

Received: 25 June 2013 / Accepted: 8 August 2013 / Published online: 29 August 2013

Abstract This article proposes Laplace transform-homotopy perturbation method (LT-HPM) to solve nonlinear differential equations with Dirichlet, mixed, and Neumann boundary conditions. After comparing figures between approximate and exact solutions, we will see that the proposed solutions are of high accuracy and, therefore, that LT-HPM is extremely efficient.

Keywords Homotopy perturbation method · Nonlinear differential equation · Approximate solutions · Laplace transform · Laplace transform homotopy perturbation method · Dirichlet · Boundary condition · Neumann boundary condition · Gelfand's differential equation

Mathematics Subject Classification (2000) 34L30

Communicated by Cristina Turner.

U. Filobello-Nino · H. Vazquez-Leal (✉) · A. Perez-Sesma · V. M. Jimenez-Fernandez · D. Pereyra-Diaz · J. M. Mendez-Perez · J. Sanchez-Orea
Electronic Instrumentation and Atmospheric Sciences School, University of Veracruz, Circuito Gonzalo Aguirre Beltrán S/N, 91000 Xalapa, Veracruz, Mexico
e-mail: hvazquez@uv.mx; hvlcom@hotmail.com

Y. Khan
Department of Mathematics, Zhejiang University, Hangzhou 310027, China

A. Diaz-Sanchez
National Institute for Astrophysics, Optics and Electronics, Luis Enrique Erro #1, Sta. María Tonantzintla, 72840 Puebla, Mexico

A. Herrera-May
Micro and Nanotechnology Research Center, University of Veracruz, Calzada Ruiz Cortines 455, 94292 Boca del Rio, Veracruz, Mexico

1 Introduction

Laplace transform (LT) (or operational calculus) has played an important role in mathematics, not only for its theoretical interest, but also because his methods let to solve, in a simpler fashion, many problems in science and engineering, in comparison with other mathematical techniques. In particular, the Laplace transform is useful not only for solving linear ordinary differential equations with constant coefficients, with initial conditions, but also for solving some cases of differential equations with variable coefficients and partial differential equations (Spiegel 1988). On the other hand, applications of LT for nonlinear ordinary differential equations mainly aim to find approximate solutions; thus in reference (Aminikhan and Hemmatnezhad 2012) was reported a combination of homotopy perturbation (HPM) and LT methods (LT-HPM) to obtain highly accurate solutions for these equations. However, just as with LT, LT-HPM method has been used mainly to find solutions to problems with initial conditions (Aminikhan and Hemmatnezhad 2012; Aminikhah 2012), because it is directly related to them. Therefore, this paper presents three application examples of LT-HPM, in the search for approximate solutions for nonlinear problems with Dirichlet, mixed, and Neumann boundary conditions defined on finite intervals.

The case of equations with boundary conditions on infinite intervals has been studied in some articles and correspond often to problems defined on semi-infinite ranges (Hossein 2011; Khan et al. 2011). However the methods of solving these problems are different from what will be presented in this paper.

The importance of research on nonlinear differential equations is that many phenomena, practical or theoretical, are of nonlinear nature. In recent years, several methods focused to find approximate solutions to nonlinear differential equations, as an alternative to classical methods, have been reported, such those based on variational approaches (Assas 2007; He 2007; Kazemnia et al. 2008; Noorzad 2008), tanh method (Evans and Raslan 2005), exp-function (Xu 2007; Mahmoudi et al. 2008), Adomian's decomposition method (Adomian 1988; Babolian and Biazar 2002; Kooch and Abadyan 2012, 2011; Vanani et al. 2011; Chowdhury 2011), parameter expansion (Zhang and Xu 2007), homotopy perturbation method (He 2000, 1999, 2006, 2008; Belendez et al. 2009; El-Shaed 2005; He 2006; Ganji et al. 2009, 2008; Fereidon et al. 2010; Sharma and Methi 2011; Hossein 2011; Vazquez-Leal et al. 2012a, b; Filobello-Nino et al. 2012; Biazar and Aminikhan 2009; Biazar and Ghazvini 2009; Filobello-Nino et al. 2012; Yasir Khan and Qingbiao Wu 2011; Madani et al. 2011; Aminikhan and Hemmatnezhad 2012; Aminikhah 2012; Khan et al. 2011), homotopy analysis method (Patel et al. 2012), and perturbation method (Filobello-Nino et al. 2013) among many others. From all the above methods, the HPM method is one of the most employed because it has been successfully used in many nonlinear problems, and its practical application is simpler than other techniques.

This paper is organized as follows: in Sect. 2, we introduce the basic idea of standard HPM method. Section 3 provides a basic idea of Laplace transform. For Sect. 4 we introduce Laplace transform homotopy perturbation method. Additionally, Sect. 5 presents three case studies. Besides, a discussion on the results is presented in Sect. 6. Finally, a brief conclusion is given in Sect. 7.

2 Standard homotopy perturbation method (HPM)

The standard HPM was proposed by Ji Huan He as a powerful tool to approach various kinds of nonlinear problems. The homotopy perturbation method is considered as a combination of the classical perturbation technique and the homotopy (whose origin is in the topology), but not restricted to small parameters as occur with traditional perturbation methods. For example, HPM method requires neither small parameter nor linearization, but only few iterations to obtain highly accurate solutions (He 2000, 1999).

To figure out how HPM method works, consider a general nonlinear equation in the form

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with the following boundary conditions:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ a known analytical function, and Γ is the domain boundary for Ω. A can be divided into two operators L and N, where L is linear and N nonlinear so that (1) can be rewritten as

$$L(u) + N(u) - f(r) = 0, \tag{3}$$

Generally, a homotopy can be constructed as (He 2000, 1999)

$$H(U, p) = (1-p)\left[L(U) - L(u_0)\right] + p\left[L(U) + N(U) - f(r)\right] = 0, \quad p \in [0, 1], \ r \in \Omega. \tag{4}$$

or

$$H(U, p) = L(U) - L(u_0) + p\left[L(u_0) + N(U) - f(r)\right] = 0, \quad p \in [0, 1], \ r \in \Omega \tag{5}$$

where p is a homotopy parameter, whose values are within range of 0 and 1, and u_0 is the first approximation for the solution of (3) that satisfies the boundary conditions.

Assuming that solution for (4) or (5) can be written as a power series of p

$$U = v_0 + v_1 p + v_2 p^2 + \ldots \tag{6}$$

Substituting (6) into (5) and equating identical powers of p terms, there can be found values for the sequence $v_0, v_1, v_2, \ldots$

When $p \to 1$, it yields in the approximate solution for (1) in the form

$$U = v_0 + v_1 + v_2 + v_3 \ldots \tag{7}$$

3 Basic idea of laplace transform

Laplace transform of $F(t)$ is denoted by $\Im\{F(t)\}$ and is defined by the integral

$$\Im\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)dt \tag{8}$$

Among its most important properties is linearity, that is,

$$\Im\{c_1 F_1(t) + c_2 F_2(T)\} = c_1 f_1(s) + c_2 f_2(s), \tag{9}$$

where c_1 and c_2 are constants and $\Im\{F_1(t)\} = f_1(s)$, $\Im\{F_2(t)\} = f_2(s)$.

Some known properties of LT, employed in this study are

i)

$$\Im\{1\} = \frac{1}{s} \quad (s > 0) \tag{10}$$

ii)

$$\Im\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad (s > 0) \tag{11}$$

iii)

$$\Im\left\{F^{(n)}(t)\right\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0), \tag{12}$$

where $F^{(n)}(t)$ denotes the n-th derivative of $F(t)$ and $\Im\{F(t)\} = f(s)$.

If Laplace transform of $F(t)$ is $f(s)$, then $F(t)$ is called the inverse Laplace transform of $f(s)$ and is expressed by $F(t) = \Im^{-1}\{f(s)\}$, where $\Im^{-1}$ is called the inverse Laplace transform operator.

From equations (10) and (11) it is clear that

$$1 = \Im^{-1}\left(\frac{1}{s}\right), \tag{13}$$

$$t^n = \Im^{-1}\left(\frac{n!}{s^{n+1}}\right), \tag{14}$$

The following result is obtained from (9) and denotes the linearity property of $\Im^{-1}$:

$$\Im^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 F_1(t) + c_2 F_2(T). \tag{15}$$

4 Basic idea of Laplace transform homotopy perturbation method (LT-HPM)

The objective of this section is employ LT-HPM to find analytical approximate solutions of ODEs, as (3).

For this purpose LTHPM follows the same steps of standard HPM until (5); next we apply Laplace transform on both sides of homotopy equation (5) to obtain

$$\Im\{L(U) - L(u_0) + p[L(u_0) + N(U) - f(r)\} = 0, \tag{16}$$

using the differential property of LT, we have

$$\begin{aligned} s^n\Im\{U\} - s^{n-1}U(0) - s^{n-2}U'(0) - \cdots - U^{(n-1)}(0) = \Im\{&L(u_0) - pL(u_0) \\ &+p[-N(U) + f(r)]\}, \end{aligned} \tag{17}$$

or

$$\begin{aligned} \Im(U) = &\left(\frac{1}{s^n}\right)\left\{s^{n-1}U(0) + s^{n-2}U'(0) + \cdots + U^{(n-1)}(0)\right\} + \Im\{L(u_0) - pL(u_0) \\ &+p[-N(U) + f(r)]\} \end{aligned} \tag{18}$$

Applying inverse Laplace transform to both sides of (18), we obtain

$$\begin{aligned} U = \Im^{-1}\Bigg\{ &\left(\frac{1}{s^n}\right)\left\{s^{n-1}U(0) + s^{n-2}U'(0) + \cdots + U^{(n-1)}(0)\right\} \\ &+\Im\left\{L(u_0) - pL(u_0) + p[-N(U) + f(r)]\right\}\Bigg\} \end{aligned} \tag{19}$$

Assuming that the solutions of (3) can be expressed as a power series of p

$$U = \sum_{n=0}^{\infty} p^n v_n, \tag{20}$$

Then substituting (20) into (19), we get

$$\begin{aligned}\sum_{n=0}^{\infty} p^n v^n &= \Im^{-1}\left\{\left(\frac{1}{s^n}\right)\left\{s^{n-1}U(0) + s^{n-2}U'(0) + \ldots + U^{(n-1)}(0)\right\}\right.\\ &\quad \left. + \Im\left\{L(u_0) - pL(u_0) + p\left[-N\left(\sum_{n=0}^{\infty} p^n v^n\right) + f(r)\right]\right\}\right\},\end{aligned} \tag{21}$$

Comparing coefficients of p with the same power leads to

$$\begin{aligned}
&p^0 : v_0 = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)(s^{n-1}U(0) + s^{n-2}U'(0) + \ldots + U^{(n-1)}(0)) + \Im\{L(u_0)\}\right\},\\
&p^1 : v_1 = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)(\Im\{N(v_0) - L(u_0) + f(r)\})\right\},\\
&p^2 : v_2 = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)\Im\{N(v_0, v_1)\}\right\},\\
&p^3 : v_3 = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)\Im\{N(v_0, v_1, v_2)\}\right\},\\
&\ldots\\
&p^j : v_j = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)\Im\left\{N(v_0, v_1, v_2, \ldots, v_j)\right\}\right\},\\
&\ldots
\end{aligned} \tag{22}$$

Assuming that the initial approximation has the form $U(0) = u_0 = \alpha_0$, $U'(0) = \alpha_1, \ldots, U^{n-1}(0) = \alpha_{n-1}$; therefore, the exact solution may be obtained as follows:

$$u = \lim_{p \to 1} U = v_0 + v_1 + v_2 + \ldots \tag{23}$$

5 Case studies

5.1 Dirichlet boundary conditions

The equation to solve is Gelfand's differential equation which governs combustible gas dynamics (He 2006).

$$\frac{d^2 y(x)}{dx^2} + \varepsilon e^{y(x)} = 0, \quad 0 \le x \le 1, \; y(0) = 0, \; y(1) = 0, \tag{24}$$

where ε is a positive parameter.
It is possible to find a handy solution by applying the LT-HPM method.
Identifying terms:

$$L(y) = y''(x), \tag{25}$$

$$N(y) = \varepsilon e^{y(x)}, \tag{26}$$

where prime denotes differentiation with respect to x.

To solve approximately (24), first we expand the exponential term of Gelfand's problem, resulting in

$$y'' + \varepsilon\left(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \cdots\right) = 0, \quad 0 \le x \le 1,\ y(0) = 1,\ y(1) = 0 \tag{27}$$

In order to obtain an approximate analytical solution we construct a homotopy in accordance with (4)

$$(1-p)(y'' - y_0'') + p\left[y'' + \varepsilon\left(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3\right)\right] = 0, \tag{28}$$

or

$$y'' = y_0'' + p\left[-y_0'' - \varepsilon\left(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3\right)\right]. \tag{29}$$

Applying Laplace transform algorithm, we get

$$\Im\left(y''\right) = \Im\left(y_0'' + p\left(-y_0'' - \varepsilon\left(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3\right)\right)\right), \tag{30}$$

and after substituting (12) for $n = 2$, we obtain

$$s^2Y(s) - sy(0) - y'(0) = \Im\left(y_0'' + p\left(y_0'' - \varepsilon\left(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3\right)\right)\right). \tag{31}$$

Solving for $Y(s)$ and applying Laplace inverse transform $\Im^{-1}$

$$y(x) = \Im^{-1}\left\{\frac{1}{s^2}\left(A + \Im\left(y_0'' + p\left(-y_0'' - \varepsilon\left(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3\right)\right)\right)\right)\right\}, \tag{32}$$

where we defined $A = y'(0)$ and employed the condition $y(0) = 0$.

Next, we assume a series solution for $y(x)$, in the form

$$y(x) = \sum_{n=0}^{\infty} p^n v_n, \tag{33}$$

and choosing

$$v_0(x) = Ax, \tag{34}$$

as the first approximation for the solution of (24) that satisfies the condition $y(0) = 0$. Substituting (33) and (34) into (32), we get

$$\sum_{n=0}^{\infty} p^n v_n = \Im^{-1}\left\{\frac{1}{s^2}\left(A - \varepsilon\Im\left(p(1 + v_0 + pv_1 + p^2v_2 + \cdots \frac{1}{2}\left(v_0 + pv_1 + p^2v_2 + \cdots\right)^2 + \frac{1}{6}(v_0 + pv_1 + p^2v_2 + \cdots)^3)\right)\right)\right\} \tag{35}$$

On comparing the coefficients of like powers of p we have

$$p^0 : \nu_0(x) = \Im^{-1}\left\{\frac{A}{s^2}\right\}, \tag{36}$$

$$p^1 : \nu_1(x) = \Im^{-1}\left\{\left(-\frac{\varepsilon}{s^2})\Im(1+v_0+\frac{v_0^2}{2}+\frac{v_0^3}{6}\right)\right\}, \tag{37}$$

$$p^2 : \nu_2(x) = \Im^{-1}\left\{\left(-\tfrac{\varepsilon}{s^2})\Im(v_1+v_0v_1+\tfrac{v_0^2v_1}{2}\right)\right\}, \tag{38}$$
$$\ldots$$

Solving the above equations for $\nu_0(x),\ \nu_1(x),\ \nu_2(x), \ldots$, we obtain

$$\nu_0(x) = Ax, \tag{39}$$

$$\nu_1(x) = -\frac{\varepsilon}{2}x^2 - \frac{\varepsilon A}{6}x^3 - \frac{\varepsilon A^2}{24}x^4 - \frac{\varepsilon A^3}{120}x^5, \tag{40}$$

$$\nu_2(x) = \frac{\varepsilon^2}{24}x^4 + \frac{\varepsilon^2 A}{30}x^5 + \frac{11\varepsilon^2 A^2}{720}x^6 + \frac{\varepsilon^2 A^3}{315}x^7 + \frac{\varepsilon^2 A^4}{1920}x^8 + \frac{\varepsilon^2 A^5}{17280}x^9,$$
$$\ldots \tag{41}$$

and so on.

By substituting solutions (39), (40) and (41) into (23) and calculating the limit when $p \to 1$ results in a second-order approximation

$$y(x) = Ax - \frac{\varepsilon}{2}x^2 - \frac{\varepsilon A}{6}x^3 + \left(\frac{\varepsilon^2}{24} - \frac{\varepsilon A^2}{24}\right)x^4 + \left(\frac{\varepsilon^2 A}{30} - \frac{\varepsilon A^3}{120}\right)x^5$$
$$+\frac{11\varepsilon^2 A^2}{720}x^6 + \frac{\varepsilon^2 A^3}{315}x^7 + \frac{\varepsilon^2 A^4}{1920}x^8 + \frac{\varepsilon^2 A^5}{17280}x^9 \tag{42}$$

In order to calculate the value of A, we require that (42) satisfies the boundary condition $y(1) = 0$, resulting an equation for A. After considering the value of $\varepsilon = 1/2$ as a case study, we obtain the following result:

$$A = 0.2603191187. \tag{43}$$

Substituting (43) into (42), we obtain

$$y(x) = 0.2603191187x - 0.25x^2 - 0.02169326x^3 + 0.0090048741x^4 + 0.002095822669x^5$$
$$+0.0002588286386x^6 + 0.00001400063x^7 + 5.979474818x10^{-7}x^8$$
$$+1.729524016x10^{-8}x^9 \tag{44}$$

5.2 Mixed boundary conditions

Now we will consider the following nonlinear differential equation:

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} - \varepsilon y^4(x) = 0, \quad 0 \le x \le 1,\ y'(0) = 0,\ y(1) = 1 \tag{45}$$

which describes the temperature distribution in a uniformly thick rectangular fin radiation to free space with nonlinearity of high order (Marinca and Herisanu 2011).

We will find a handy solution for (45) by applying the LTHPM method.

Identifying terms:

$$L(y) = y''(x), \tag{46}$$
$$N(y) = -\varepsilon y^4(x), \tag{47}$$

where prime denotes differentiation with respect to x.

In order to obtain an approximate analytical solution we construct a homotopy in accordance with (4)

$$(1-p)(y''-y_0'') + p\left[y'' - \varepsilon y^4\right] = 0, \tag{48}$$

or

$$y'' = y_0'' + p\left[-y_0'' + \varepsilon y^4\right]. \tag{49}$$

Applying Laplace transform algorithm we get

$$\Im(y'') = \Im\left(y_0'' + p\left(-y_0'' + \varepsilon y^4\right)\right), \tag{50}$$

and after substituting (12) for $n = 2$

$$s^2 Y(s) - sy(0) - y'(0) = \Im(y_0'' + p(-y_0'' + \varepsilon y^4)). \tag{51}$$

Solving for $Y(s)$and applying Laplace inverse transform $\Im^{-1}$

$$y(x) = \Im^{-1}\left\{\frac{1}{s^2}\left(A + \Im\left(y_0'' + p\left(-y_0'' + \varepsilon y^4\right)\right)\right)\right\}, \tag{52}$$

where we defined $A = y(0)$, and employed the condition $y'(0) = 0$.

Next, we assume a series solution for $y(x)$, in the form

$$y(x) = \sum_{n=0}^{\infty} p^n v_n, \tag{53}$$

and we choose

$$v_0(x) = A, \tag{54}$$

as the first approximation for the solution of (45) that satisfies the condition $y'(0) = 0$.

Substituting (53) and (54) into (52), we get

$$\sum_{n=0}^{\infty} p^n v_n = \Im^{-1}\left\{\frac{1}{s^2}\left(As + p\varepsilon\Im\left(\left(v_0 + pv_1 + p^2 v_2 + \cdots\right)^4\right)\right)\right\} \tag{55}$$

On comparing the coefficients of like powers of p we have

$$p^0 : \nu_0(x) = \Im^{-1}\left\{\frac{A}{s}\right\}, \tag{56}$$

$$p^1 : \nu_1(x) = \varepsilon \Im^{-1}\left\{\left(\frac{1}{s^2}\right)\Im\{\nu_0^4\}\right\}, \tag{57}$$

$$p^2 : \nu_2(x) = \varepsilon \Im^{-1}\left\{\left(\frac{1}{s^2}\right)\Im(4\nu_0^3\nu_1)\right\}, \tag{58}$$

$$p^3 : \nu_3(x) = \varepsilon \Im^{-1}\left\{\left(\frac{1}{s^2}\right)\Im(6\nu_0^2\nu_1^2 + 4\nu_0^3\nu_2)\right\} \tag{59}$$

$$p^4 : \nu_4(x) = \varepsilon \Im^{-1}\left\{\left(\frac{1}{s^2}\right)\Im(4\nu_1^3\nu_0 + 12\nu_0^2\nu_1\nu_2 + 4\nu_0^3\nu_3)\right\}, \tag{60}$$

$$\ldots$$

Solving the above Laplace transforms for $\nu_0(x),\ \nu_1(x),\ \nu_2(x),\ldots$ we obtain

$$p^0 : \nu_0(x) = A, \tag{61}$$

$$p^1 : \nu_1(x) = \frac{\varepsilon A^4}{2}x^2, \tag{62}$$

$$p^2 : \nu_2(x) = \frac{\varepsilon^2 A^7}{6}x^4, \tag{63}$$

$$p^3 : \nu_3(x) = \frac{13\varepsilon^3 A^{10}}{180}x^6, \tag{64}$$

$$p^4 : \nu_4(x) = \frac{161\varepsilon^4 A^{13}}{5040}x^8,$$
$$\ldots \tag{65}$$

and so on.

By substituting solutions (61)–(65) into (23) and calculating the limit when $p \to 1$ results in a fourth-order approximation

$$y(x) = A + \frac{\varepsilon A^4}{2}x^2 + \frac{\varepsilon^2 A^7}{6}x^4 + \frac{13\varepsilon^3 A^{10}}{180}x^6 + \frac{161\varepsilon^4 A^{13}}{5040}x^8 + \cdots \tag{66}$$

In order to calculate the value of A, we require that (66) satisfies the boundary condition $y(1) = 1$; this gives rise to an equation for A. Considering as case study $\varepsilon = 1$, we obtain the value

$$A = 0.7792914176 \tag{67}$$

Substituting (67) into (66), we obtain

$$\begin{aligned} y(x) &= 0.7792914176 + 0.1844036774x^2 + 0.02909028680x^4 \\ &\quad + 0.005965814985x^6 + 0.001248803057x^8. \end{aligned} \tag{68}$$

5.3 Neumann boundary conditions

Finally, we will find an approximate solution for

$$\frac{d^2 y(x)}{dx^2} + y(x) - y^2(x) = 0, \quad 0 \le x \le 1,\ y'(0) = 0,\ y'(1) = \pi/4, \tag{69}$$

To obtain a precise solution for (69) by applying the LTHPM method, we identify

$$L(y) = y''(x), \tag{70}$$
$$N(y) = y(x) - y^2(x), \tag{71}$$

where prime denotes differentiation with respect to x.

We construct the following homotopy in accordance with (4):

$$(1-p)(y''-y_0'') + p\left[y''+y-y^2\right] = 0, \tag{72}$$

or

$$y'' = y_0'' + p\left[-y_0''-y+y^2\right]. \tag{73}$$

Applying Laplace transform to (73) we get

$$\Im(y'') = \Im(y_0'' + p(-y_0''-y+y^2)), \tag{74}$$

and after substituting (12), for $n=2$

$$s^2Y(s) - sy(0) - y'(0) = \Im\left(y_0'' + p\left(-y_0''-y+y^2\right)\right). \tag{75}$$

Solving for $Y(s)$and applying Laplace inverse transform $\Im^{-1}$

$$y(x) = \Im^{-1}\left\{\frac{A}{s}\right\} + \Im^{-1}\left\{\frac{1}{s^2}\left(\Im\left(y_0''+p\left(-y_0''-y+y^2\right)\right)\right)\right\}, \tag{76}$$

where we defined $A = y(0)$, and employed the condition $y'(0) = 0$.

Next, suppose that the solution for (69) has the form

$$y(x) = \sum_{n=0}^{\infty} p^n v_n, \tag{77}$$

and choosing

$$v_0(x) = A, \tag{78}$$

as the first approximation for the solution of (69) that satisfies the condition $y'(0) = 0$.

Substituting (77) and (78) into (76), we get

$$\sum_{n=0}^{\infty} p^n v_n = \Im^{-1}\left\{\frac{A}{s} + p\left\{\left(\frac{1}{s^2}\right)\Im\left\{-(v_0+pv_1+p^2v_2+\cdots)\right.\right.\right.$$
$$\left.\left.\left.+(v_0+pv_1+p^2v_2+\cdots)^2\right\}\right\}\right\} \tag{79}$$

Equating terms with identical powers of p, we obtain

$$p^0:\ \nu_0(x)=\Im^{-1}\left\{\frac{A}{s}\right\},\tag{80}$$

$$p^1:\ \nu_1(x)=\Im^{-1}\left\{\left(\frac{1}{s^2}\right)\Im(-\nu_0+\nu_0^2)\right\},\tag{81}$$

$$p^2:\ \nu_2(x)=\Im^{-1}\left\{\left(\frac{1}{s^2}\right)\Im(-\nu_1+2\nu_0\nu_1)\right\},\tag{82}$$

$$p^3:\nu_2(x)=\Im^{-1}\left\{\left(\frac{1}{s^2}\right)\Im(-\nu_2+\nu_1^2 2\nu_0\nu_2)\right\},\tag{83}$$

$$p^4:\nu_2(x)=\Im^{-1}\left\{\left(\tfrac{1}{s^2}\right)\Im(-\nu_3+2\nu_0\nu_3+2\nu_1\nu_2)\right\}, \\ \ldots \tag{84}$$

Solving the above equations for $\nu_0(x),\ \nu_1(x),\ \nu_2(x),\ldots$ we obtain

$$p^0:\ \nu_0(x)=A,\tag{85}$$

$$p^1:\nu_1(x)=\frac{(A^2-A)x^2}{2},\tag{86}$$

$$p^2:\nu_2(x)=\frac{(A^2-A)(2A-1)x^2}{4!},\tag{87}$$

$$p^3:\nu_3(x)=\frac{\left[(A^2-A)(2A-1)^2+6(A^2-A)^2\right]x^6}{6!},\tag{88}$$

$$p^4:\nu_4(x)=\frac{(2A-1)(A^2-A)\left[(2A-1)^2+36(A^2-A)\right]x^8}{8!}, \\ \ldots \tag{89}$$

and so on.

By substituting solutions (85)–(89) into (23) results in a fourth-order approximation

$$y(x)=A+\left(\frac{A^2-A}{2}\right)x^2+\frac{(A^2-A)(2A-1)}{4!}x^4+\frac{((A^2-A)(2A-1)^2+6(A^2-A)^2)}{6!}x^6 \\ +\frac{(A^2-A)(2A-1)((2A-1)^2+36(A^2-A))}{8!}x^8.\tag{90}$$

In order to calculate the value of A, we require that (90) satisfies the boundary condition $y'(1)=\pi/4$ so that we obtain

$$A=-0.6793160999.\tag{91}$$

Substituting (91) into (90), we obtain

$$y(x)=-0.6793160999+0.5703932320x^2-0.1121123203x^4 \\ +0.01965933892x^6-0.003111881558x^8.\tag{92}$$

6 Discussion

In this work, LT-HPM was used in the search for handy accurate analytical approximate solutions for nonlinear ordinary differential equations with boundary conditions defined on finite intervals. In order to show the versatility of this method, we chose equations subject to Dirichlet, mixed and Neumann boundary conditions; as a matter of fact, Figs. 1, 2, 3, 4, 5, 6, which compare our approximations with the numerical solution, showed good confirmation for all cases. Since LT-HPM is expressed in terms of initial conditions for a given differential equation [see (22)], our procedure was aimed to express the approximate solutions in terms of an unknown quantity A, related to $y(0)$ or $y'(0)$. We noted that the value of A can be determined requiring that approximate solution satisfies a given second boundary condition, from which one obtain an algebraic equation in terms of A, whose solution concludes the procedure. The above was systematically shown with our examples and proved to be very efficient.

In the first case study, we found an approximate solution for Gelfand's equation (24), which is subject to Dirichlet boundary conditions. Figure 1 shows the comparison between numerical solution and (44) solutions for $\varepsilon = 1/2$. It can be noticed that curves are in good agreement, from which is clear the accuracy of our approximation, as a matter of fact Fig. 2, shows that the biggest absolute error (AE) of (44) is scarcely between 0.0005 and 0.0006, which is remarkably precise, above all taking into account that (44) is just a second order approximate solution for (24).

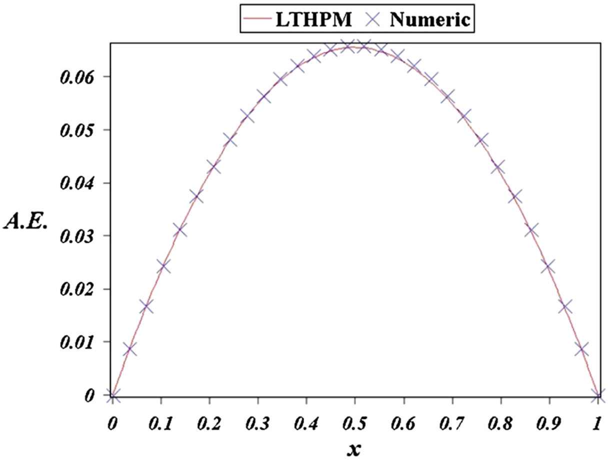

Fig. 1 Comparison numerical solution of (24) and LT-HPM approximation (44)

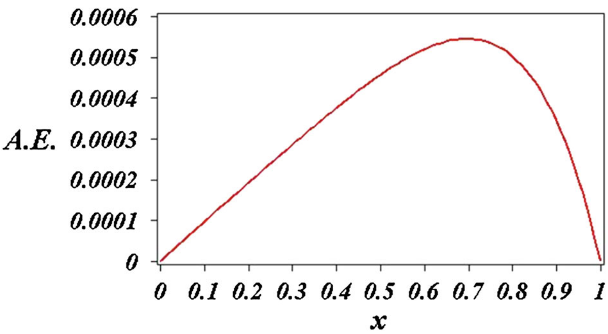

Fig. 2 Absolute error (AE) between numerical solution of (24) and LT-HPM approximation (44)

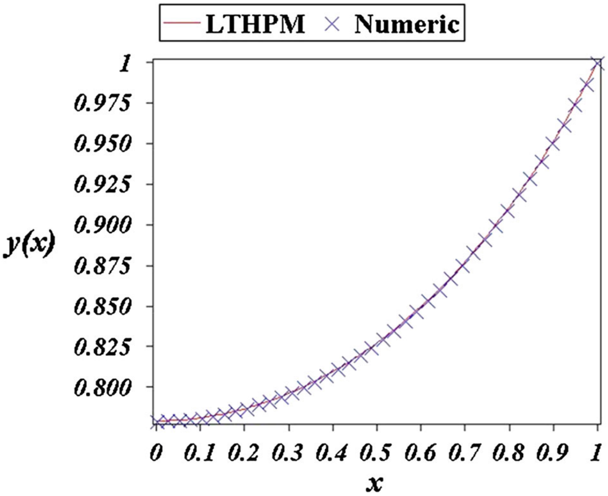

Fig. 3 Comparison numerical solution of (45) and LT-HPM approximation (68)

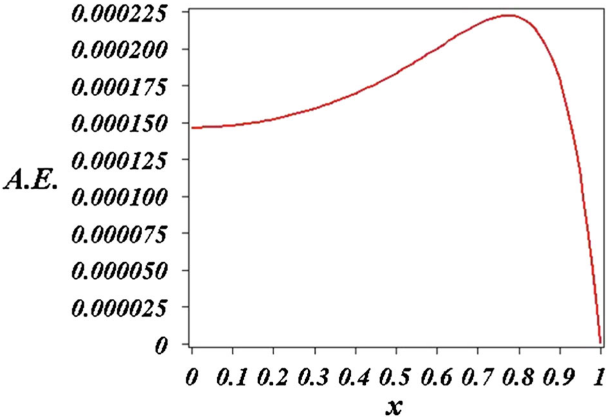

Fig. 4 Absolute error (AE) between numerical solution of (45) and LT-HPM approximation (68)

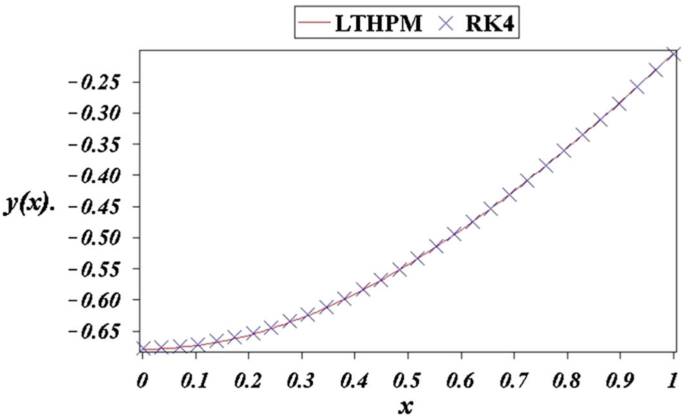

Fig. 5 Comparison numerical solution of (69) and LT-HPM approximation (92)

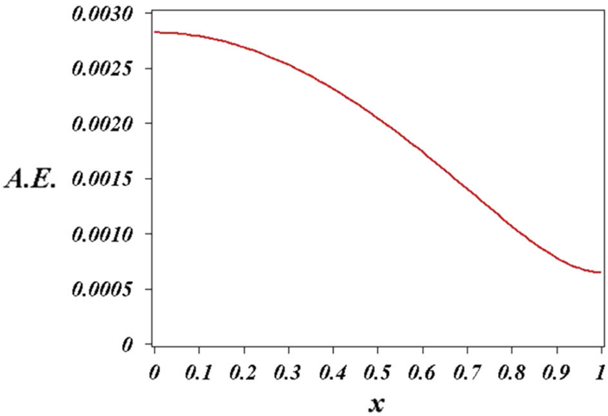

Fig. 6 Absolute error (AE) between numerical solution of (69) and LTPHM approximation (92)

Next, we found an approximate solution for temperature distribution equation (45), which is subject to mixed boundary conditions. Figure 3 shows that (68) is an accurate analytical approximate solution for (45); from Fig. 4 we deduce that the biggest absolute error is only 0.000225, whereby the reliability of LT-HPM method in the search for approximate solutions of equations like (45)is clear .

In the last case study, we employed LT-HPM method to find an approximate solution for (69), subject to Neumann boundary conditions. In the same way, we obtained a very precise result, as shown in Figs. (5) and (6), from where it is clear that our fourth-order approximation (92) has an absolute error whose value is between 0.0005 and 0.0030, which is also accurate. From the examples studied, we conclude that the approximate solutions, obtained by using LTHPM are of high accuracy.

7 Conclusions

In this paper LT-HPM was employed to provide approximate analytical solutions for nonlinear differential equations with Dirichlet, mixed and Neumann boundary conditions. The proposed procedure is to express the problem of solving a nonlinear ordinary differential equation, in terms of solving an algebraic equation for an unknown condition A [see (43), (67), and (91)]. Figs. 1,2,3,4,5,6 show the efficiency of this method for solving boundary value nonlinear problems.

The above is an additional advantage for the method, considering that LT-HPM does not need to solve several recurrence differential equations, by which is a tool extremely efficient, useful and precise in practical applications.

References

Adomian G (1988) A review of decomposition method in applied mathematics. Math Anal Appl 135:501–544
Aminikhah H (2011) Analytical approximation to the solution of nonlinear Blasius viscous flow equation by LTNHPM. International Scholarly Research Network ISRN Mathematical Analysis, vol 2012, Article ID 957473. doi:10.5402/2012/957473
Aminikhah H (2012) The combined Laplace transform and new homotopy perturbation method for stiff systems of ODE's. Appl Math Model 36:3638–3644

Aminikhan H, Hemmatnezhad M (2012) A novel effective approach for solving nonlinear heat transfer equations. Heat Transf Asian Res 41(6):459–466
Assas LMB (2007) Approximate solutions for the generalized K-dV–Burgers' equation by He's variational iteration method. Phys Scr 76:161–164. doi:10.1088/0031-8949/76/2/008
Babolian E, Biazar J (2002) On the order of convergence of Adomian method. Appl Math Comput 130(2):383–387. doi:10.1016/S0096-3003(01)00103-5
Belendez A, Pascual C, Alvarez ML, Méndez DI, Yebra MS, Hernández A (2009) High order analytical approximate solutions to the nonlinear pendulum by He's homotopy method. Physica Scripta 79(1):1–24. doi:10.1088/0031-8949/79/01/015009
Biazar J, Aminikhan H (2009) Study of convergence of homotopy perturbation method for systems of partial differential equations. Comput Math Appl 58(11–12): 2221–2230
Biazar J, Ghazvini H (2009) Convergence of the homotopy perturbation method for partial differential equations. Nonlinear Anal Real World Appl 10(5): 2633–2640
Chowdhury SH (2011) A comparison between the modified homotopy perturbation method and Adomian decomposition method for solving nonlinear heat transfer equations. J Appl Sci 11:1416–1420. doi:10.3923/jas.2011.1416.1420
El-Shaed M (2005) Application of He's homotopy perturbation method to Volterra's integro differential equation. Int J Nonlinear Sci Numer Simul 6:163–168
Evans DJ, Raslan KR (2005) The Tanh function method for solving some important nonlinear partial differential. Int J Comput Math 82:897–905. doi:10.1080/00207160412331336026
Fereidon A, Rostamiyan Y, Akbarzade M, Ganji DD (2010) Application of He's homotopy perturbation method to nonlinear shock damper dynamics. Arch Appl Mech 80(6): 641–649. doi:10.1007/s00419-009-0334-x
Filobello-Nino U, Vazquez-Leal H, Castaneda-Sheissa R, Yildirim A, Hernandez Martinez L, Pereyra Diaz D, Perez Sesma A, Hoyos Reyes C (2012) An approximate solution of Blasius equation by using HPM method. Asian J Math Stat 2012: 10. doi:10.3923/ajms.2012. ISSN 1994-5418
Filobello-Nino U, Vazquez-Leal H, Khan Y, Yildirim A, Jimenez-Fernandez VM, Herrera May AL, Castaneda-Sheissa R, Cervantes-Perez J (2013) Using perturbation methods and Laplace-Padé approximation to solve nonlinear problems. Miskolc Mathematical Notes 14(1):89–101. e-ISSN: 1787–2413
Filobello-Nino U, Vazquez-Leal H, Khan Y, Yildirim A, Pereyra-Diaz D, Perez-Sesma A, Hernandez-Martinez L, Sanchez-Orea J, Castaneda-Sheissa R, Rabago-Bernal F (2012) HPM applied to solve nonlinear circuits: a study case. Appl Math Sci 6(85–88):4331–4344
Ganji DD, Babazadeh H, Noori F, Pirouz MM, Janipour M (2009) An application of homotopy perturbation method for non linear Blasius equation to boundary layer flow over a flat plate, ACADEMIC World Academic Union, ISNN 1749–3889 (print), 1749–3897 (online). Int J Nonlinear Sci 7(4):309–404
Ganji DD, Mirgolbabaei H, Miansari Me, Miansari Mo (2008) Application of homotopy perturbation method to solve linear and non-linear systems of ordinary differential equations and differential equation of order three. J Appl Sci 8:1256–1261. doi:10.3923/jas.2008.1256.1261
He JH (2000) A coupling method of a homotopy technique and a perturbation technique for nonlinear problems. Int J Non-Linear Mech 35(1): 37–43. doi:10.1016/S0020-7462(98)00085-7
He JH (2006) Some asymptotic methods for strongly nonlinear equations. Int J Modern Phys B 20(10):1141–1199
He JH (1999) Homotopy perturbation technique. Comput Methods Appl Mech Eng 178:257–262. doi:10.1016/S0045-7825(99)00018-3
He JH (2006) Homotopy perturbation method for solving boundary value problems. Phys Lett A 350(1–2):87–88
He JH (2006) Some asymptotic methods for strongly nonlinear equations. Int J Modern Phys B 20(10):1141–1199. doi:10.1142/S0217979206033796
He JH (2007) Variational approach for nonlinear oscillators. Chaos Solitons Fractals 34:1430–1439. doi:10.1016/j.chaos.2006.10.026
He JH (2008) Recent development of the homotopy perturbation method. Topol Methods Nonlinear Anal 31(2):205–209
Kazemnia M, Zahedi SA, Vaezi M, Tolou N (2008) Assessment of modified variational iteration method in BVPs high-order differential equations. J Appl Sci 8:4192–4197. doi:10.3923/jas.2008.4192.4197
Khan Y, Qingbiao W (2011) Homotopy perturbation transform method for nonlinear equations using He's polynomials. Comput Math Appl 61(8):1963–1967
Khan M, Gondal MA, Vanani K (2011) A new study between homotopy analysis method and homotopy perturbation transform method on a semi infinite domain. Math Comput Model 55:1143–1150
Kooch A, Abadyan M (2011) Evaluating the ability of modified Adomian decomposition method to simulate the instability of freestanding carbon nanotube: comparison with conventional decomposition method. J Appl Sci 11:3421–3428. doi:10.3923/jas.2011.3421.3428

Kooch A, Abadyan M (2012) Efficiency of modified Adomian decomposition for simulating the instability of nano-electromechanical switches: comparison with the conventional decomposition method. Trends Appl Sci Res 7:57–67. doi:10.3923/tasr.2012.57.67
Madani M, Fathizadeh M, Khan Y, Yildirim A (2011) On the coupling of the homotopy perturbation method and Laplace transformation. Math Comput Model 53(9–10):1937–1945
Mahmoudi J, Tolou N, Khatami I, Barari A, Ganji DD (2008) Explicit solution of nonlinear ZK-BBM wave equation using exp-function method. J Appl Sci 8:358–363. doi:10.3923/jas.2008.358.363
Murray RS (1988) Teoría y Problemas de Transformadas de Laplace, primera edición. Serie de compendios Schaum
Noorzad R, Tahmasebi Poor A, Omidvar M (2008) Variational iteration method and homotopy-perturbation method for solving Burgers equation in fluid dynamics. J Appl Sci 8:369–373. doi:10.3923/jas.2008.369.373
Patel T, Mehta MN, Pradhan VH (2012) The numerical solution of Burger's equation arising into the irradiation of tumour tissue in biological diffusing system by homotopy analysis method. Asian J Appl Sci 5:60–66. doi:10.3923/ajaps.2012.60.66
Sharma PR, Methi G (2011) Applications of homotopy perturbation method to partial differential equations. Asian J Math Stat 4:140–150. doi:10.3923/ajms.2011.140.150
Vanani SK, Heidari S, Avaji M (2011) A low-cost numerical algorithm for the solution of nonlinear delay boundary integral equations. J Appl Sci 11:3504–3509. doi:10.3923/jas.2011.3504.3509
Vasile M, Nicolae H (2011) Nonlinear dynamical systems in engineering, 1 st edn. Springer, Berlin
Vazquez-Leal H, Castañeda-Sheissa R, Filobello-Nino U, Sarmiento-Reyes A, Sánchez-Orea J (2012) High accurate simple approximation of normal distribution related integrals. Mathematical Problems in Engineering, vol. 2012, Article ID 124029. doi:10.1155/2012/124029
Vazquez-Leal H, Filobello-Niño U, Castañeda-Sheissa R, Hernandez Martinez L, Sarmiento-Reyes A (2012a) Modified HPMs inspired by homotopy continuation methods. Mathematical Problems in Engineering, vol. 2012, Article ID 309123. doi:10.155/2012/309123
Xu F (2007) A generalized soliton solution of the Konopelchenko–Dubrovsky equation using exp-function method. Zeitschrift Naturforschung: Sect A J Phys Sci 62(12):685–688
Zhang L-N, Xu L (2007) Determination of the limit cycle by He's parameter expansion for oscillators in a $u^3/1+u^2$ potential. Zeitschrift für Naturforschung: Sect A J Phys Sci 62(7–8):396–398