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Complex-valued Thiran allpole filters

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ABSTRACT

This paper describes the generalization of the real-valued Thiran allpole filter to the complex case. The design specifications for the complex case are the phase value ϕ_α, the group delay τ, and the degree of flatness K at $\omega = 0$. The complex filter coefficients are obtained by solving a set of closed form linear equations derived from the design specification. Depending on the parity of the degree of flatness K, we proposed three classes of complex-valued allpole filters. We also establish the stability conditions, which depend on τ and ϕ_α. Finally, the application of the proposed filters for the design of a casual complex-valued cardinal orthogonal scaling function is described.

1. Introduction

The design of real-valued allpole filter with a maximally flat group delay at $\omega = 0$ was first proposed by Thiran in [1]. The degree of flatness is even since the group delay is an even function. The coefficients of the filter are obtained by solving closed form linear equations. In order to ensure stability, it was shown that the group delay at $\omega = 0$ should be larger than zero.

This filter has been used in many applications such as fractional delay filters, IIR filters with the specified degrees of flatness and constant group delay, FIR Hilbert transform pairs, among others. The design of maximally flat fractional delay filters is addressed in [2,3]. The method applies Thiran allpole filter to design a desired allpass filter since the corresponding group delays are linearly related. In [4], the author proposed the design of IIR wavelet filters based on maximally flat allpole filters. In a similar way, the design of IIR wavelet filter with desired degree of flatness is described in [5,6]. The Thiran allpole filter is also applied to the design of IIR filters with the specified degrees of flatness and constant group delay characteristics in [7]. The design of FIR Hilbert transform pairs of wavelet bases based on Thiran allpole filter is formulated in [8,9].

There are also some extended results of the maximally flat group delay filter, e.g., [10,11]. Selesnick proposed the design of maximally flat group delay allpole filter with the desired degrees of flatness at $\omega = 0$ and $\omega = \pi$ [10]. The closed form equations to calculate the filter coefficients are also given. Additionally, the stability of the filter is considered for the particular cases. In [11] the authors proposed the design of allpole filters with the prescribed degrees of flatness at $\omega = 0$ and $\omega = \pi$, and the desired phase at the frequency point ω_p, $0 < \omega_p < \pi$. The resulting allpole filters are used in the design of maximally flat IIR filters.

The complex-valued digital filters have some desired properties useful for many applications [8,9,12–17]. Complex signal processing provides magnitude and phase information needed, for example, in digital communications and image processing [9,12,15,17]. This was our principal motivation to consider an extension of the Thiran filters to the complex case.

The rest of the paper is organized as follows. Section 2 focuses on the design of complex-valued Thiran allpole filters. Depending on the parity of the degree of flatness, we present three classes of complex filters. Section 3 deals with the stability of the proposed filters and presents the conditions in terms of group delay at $\omega = 0$ and a phase parameter. Finally, the interesting application in the design of a complex-valued cardinal orthogonal scaling function is described in Section 4.

2. Complex-valued maximally flat group delay filters

This section focuses on the design of complex-valued allpole filters with the desired characteristics. We consider the following system function:

$$D(z) = \frac{e^{j\phi_\alpha}}{1 + \sum_{n=1}^{N} d_n z^{-n}}, \tag{1}$$

where ϕ_α is a phase constant, N is the order of the allpole filter $D(z)$, and d_n, $n = 1, \ldots, N$, are the filter coefficients. The real and imaginary parts of d_n are expressed as $d_{R,n}$ and $d_{I,n}$, respectively. The group delay function of $D(z)$ is

$$G(\omega) = -\frac{d\phi_D(\omega)}{d\omega}, \tag{2}$$

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where $\phi_D(\omega)$ is the phase response. We now establish the following maximally flat conditions on the allpole filter:

i. The phase response at $\omega = 0$ equals zero, i.e., $\phi_D(0) = 0$,
ii. The group delay evaluated at $\omega = 0$ is τ,
iii. In order to have an allpole filter with maximally flat group delay response, the relationships

$$\left.\frac{d^k G(\omega)}{d\omega^k}\right|_{\omega=0} = 0, \quad k = 1, \dots, K,$$

should be satisfied, where K is the degree of flatness.

In [18], the authors proposed the design of digital filters with the desired phase and degree of flatness at any set of frequency points. Based on the results [18] and conditions i–iii, the following set of linear equations is obtained:

$$\sum_{n=1}^{N}(n+\tau)^{\ell} d_{R,n} - \frac{1}{\tan\phi_\alpha}\sum_{n=1}^{N}(n+\tau)^{\ell} d_{I,n} = -\tau^{\ell}, \quad \ell \text{ even}, \tag{3}$$

$$\sum_{n=1}^{N}(n+\tau)^{\ell} d_{R,n} + \tan\phi_\alpha\sum_{n=1}^{N}(n+\tau)^{\ell} d_{I,n} = -\tau^{\ell}, \quad \ell \text{ odd}, \tag{4}$$

where $\ell = 0, \dots, K+1$. Observe that $\tan\phi_\alpha$ is a periodic function with the period of π. Therefore, the phase value ϕ_α is restricted to be in $[-\pi/2, \pi/2]$. Furthermore, there are $K+2$ equations, N real coefficients, and N imaginary coefficients. In particular, we propose three classes of complex-valued allpole filters, denoted as Types I–III. In Type I, we consider that $K = 2(N-1)$ with $N \geqslant 1$, while Type II results in $K = 2N-3$ with $N \geqslant 2$. Similar to Type II, Type III results in $K = 2N-3$. However, in Type III, the order N satisfies $N \geqslant 1$. We also define the filter coefficients $d_n^{(I)}$, $d_n^{(II)}$, and $d_n^{(III)}$ for Types I, II, and III, respectively. In the following subsections, we solve the set of linear equations (3) and (4) using Type I–III allpole filters.

2.1. Complex Thiran filter Type I

We first consider that the degree of flatness K is even and N is a minimum order. To obtain a consistent set of linear equations, the relation between K and N must be $K = 2(N-1)$, that is, the order of the allpole filter N is $K/2+1$. Moreover, the resulting system matrix is composed of two Vandermonde-like matrices. As the Vandermonde matrices are invertible, the solution for $d_n^{(I)}$ always exists and it is unique. Furthermore, the closed form equation for the filter coefficient is

$$d_n^{(I)} = (-1)^n \binom{N}{n} \frac{2(2\tau+1)_{n-1}}{(2\tau+N+1)_n}\left(\tau + n e^{j(\phi_\alpha - \pi/2)}\sin\phi_\alpha\right), \tag{5}$$

where τ is the group delay at $\omega = 0$, i.e., $G(0) = \tau$ and the binomial coefficient is

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}. \tag{6}$$

The Pochhammer symbol $(x)_n$ can be expressed by

$$(x)_n = \begin{cases} \prod_{k=0}^{n-1}(x+k), & n > 0; \\ 1, & n = 0; \\ \prod_{k=1}^{|n|}(x-k)^{-1}, & n < 0. \end{cases} \tag{7}$$

We note that for $\tau = -(N+k)/2$, $k = 1, \dots, n$, no solution exists since the denominator in (5) vanishes. Additionally, from the numerator in (5), the values $\tau = -k/2$, $k = 1, \dots, n-1$, make filter coefficient $d_n^{(I)}$ be zero. Finally, evaluating τ at $-n/2$, $d_n^{(I)}$ equals $e^{2j\phi_\alpha}$. In summary, we have

$$d_n^{(I)} = \begin{cases} 0, & \text{if } \tau = 0 \text{ and } \phi_\alpha = 0, \\ 0, & \text{if } \tau = -n \text{ and } \phi_\alpha = \pm\pi/2, \\ 0, & \text{if } \tau = -k/2 \text{ for } k = 1, \dots, n-1, \\ e^{2j\phi}, & \text{if } \tau = -n/2, \\ \infty, & \text{if } \tau = -(N+k)/2 \text{ for } k = 1, \dots, n, \\ (-1)^n\binom{N}{n}, & \text{if } \tau \to \pm\infty. \end{cases} \tag{8}$$

It should also be noted that if one sets $\phi_\alpha = 0$ in (5), then $D(z)$ reduces to the Thiran allpole filter for the real case, that is, the filter coefficients $d_n^{(I)}$, for $n = 0, \dots, N$, are given by [1],

$$d_n^{(I)} = (-1)^n \binom{N}{n} \frac{(2\tau)_n}{(2\tau+N+1)_n}. \tag{9}$$

Example 1. We calculate the filter coefficients for the first and second order allpole filters. The filter coefficients of the first order filter are

$$d_0^{(I)} = 1, \tag{10}$$

$$d_1^{(I)} = -\frac{\tau + \sin\phi_\alpha e^{j(\phi_\alpha - \pi/2)}}{\tau+1}. \tag{11}$$

The resulting filter $D(z)$ satisfies conditions i and ii.

Similarly, for the second order, we have

$$d_0^{(I)} = 1, \tag{12}$$

$$d_1^{(I)} = -\frac{4}{2\tau+3}\left(\tau + e^{j(\phi_\alpha - \pi/2)}\sin\phi_\alpha\right), \tag{13}$$

$$d_2^{(I)} = \frac{2\tau+1}{(2\tau+3)(\tau+2)}\left(\tau + 2e^{j(\phi_\alpha - \pi/2)}\sin\phi_\alpha\right). \tag{14}$$

In this case, conditions i–iii are satisfied using $K = 2$.

We now show that the corresponding allpole filter for $\phi_\alpha = \pm\pi/2$ has one pole at $z = 1$. Placing $\phi_\alpha = \pm\pi/2$ into (5), the filter coefficient $d_n^{(I)}$ becomes

$$d_n^{(I)} = (-1)^n \binom{N}{n} \frac{(2\tau+1)_{n-1}}{(2\tau+N+1)_n}(2\tau+2n), \tag{15}$$

which can be rewritten as (see Appendix A for details)

$$d_n(I) = (-1)^n \binom{N-1}{n} \frac{(2\tau+1)_n}{(2\tau+N+1)_n} - (-1)^{n-1}\binom{N-1}{n-1}\frac{(2\tau+1)_{n-1}}{(2\tau+N+1)_{n-1}}. \tag{16}$$

From z-transform of (16), we have one zero at $z = 1$. Consequently, the allpole filter $D(z)$ has one pole at $z = 1$. The stability issues are discussed in Section 3.

2.2. Complex Thiran filter Type II

This class of filter considers that the degree of flatness K is odd and the order N has a minimum value. Therefore, we have $K = 2N-3$. Additionally, in order to satisfy K odd, we impose that $d_{I,N}^{(II)} = 0$. Solving the set of linear equations (3) and (4), we arrive at

$$d_n^{(II)} = (-1)^n \binom{N}{n} \frac{2(2\tau+1)_{n-1}}{(2\tau+N+1)_n} \times \left(\tau + n - \frac{n(N-n)e^{j\phi_\alpha}\cos\phi_\alpha}{2\tau+N}\right). \tag{17}$$

Similarly to Type I coefficients, the filter coefficient $d_n^{(II)}$ satisfies

$$d_n^{(II)}=\begin{cases}0, & \text{if } \tau=-k/2 \text{ for } k=1,\dots,n-1,\\ 0, & \text{if } \tau=-n \text{ and } \phi_\alpha=\pm\pi/2,\\ e^{2j\phi}, & \text{if } \tau=-n/2,\\ 1, & \text{if } \tau=-N/2 \text{ and } n=N,\\ -1, & \text{if } n=N=1,\\ \infty, & \text{if } \tau=-(N+k)/2 \text{ for } k=0,\dots,n, n\neq N,\\ (-1)^n\binom{N}{n}, & \text{if } \tau\to\pm\infty.\end{cases} \tag{18}$$

It is straightforward to show that for $\phi_\alpha=\pm\pi/2$, the z-transform of (17) results in one zero at $z=1$.

Example 2. The corresponding filter coefficients of the first order allpole filter are

$$d_0^{(II)}=1, \tag{19}$$
$$d_1^{(II)}=-1. \tag{20}$$

Observe that the resulting allpole filter has the pole at $z=1$ and consequently does not fulfill conditions i–iii. On the other hand, second order filter satisfies conditions i–iii with $K=1$. The corresponding filter coefficients are given by,

$$d_0^{(II)}=1, \tag{21}$$
$$d_1^{(II)}=-\frac{4}{2\tau+3}\left(\tau+1-\frac{e^{j\phi_\alpha}\cos\phi_\alpha}{2\tau+2}\right), \tag{22}$$
$$d_2^{(II)}=\frac{2\tau+1}{2\tau+3}. \tag{23}$$

2.3. Complex Thiran filter Type III

Finally, consider the case where the degree of flatness K is odd, N is minimum, $K=2N-3$ and $d_{R,N}^{(III)}=0$. Consequently, from (3) and (4), the filter coefficients $d_n^{(III)}$ are now given by,

$$d_n^{(III)}=(-1)^n\binom{N}{n}\frac{2(2\tau+1)_{n-1}}{(2\tau+N+1)_n}\left(\tau+n-\frac{n(1+j\tan\phi_\alpha)}{N}\right.$$
$$\left.\times\left(\tau+n+\frac{(N-n)(\tau+N\cos^2\phi_\alpha)}{2\tau+N}\right)\right). \tag{24}$$

The filter coefficients defined in (24) have the following properties:

$$d_n^{(III)}=\begin{cases}0, & \text{if } \tau=-k/2 \text{ for } k=1,\dots,n-1,\\ e^{2j\phi}, & \text{if } \tau=-n/2,\\ j\tan\phi_\alpha, & \text{if } \tau=-N/2 \text{ and } n=N,\\ \infty, & \text{if } \tau=-(N+k)/2 \text{ for } k=0,\dots,n, n\neq N,\\ (-1)^n\binom{N}{n}\left(\frac{N-n}{N}+j\frac{n}{N}\tan\phi_\alpha\right), & \text{if } \tau\to\pm\infty.\end{cases} \tag{25}$$

We show that the Type III allpole filter has one pole at $z=\pm j$ if $\phi_\alpha=\pm\pi/4$. Thus evaluating $d_n^{(III)}$ at $\phi_\alpha=\pm\pi/4$, the filter coefficient becomes

$$d_n^{(III)}=(-1)^n\binom{N}{n}\frac{2(2\tau+1)_{n-1}}{(2\tau+N+1)_n}\left(\tau+n-\frac{n(1\pm j)}{N}\right.$$
$$\left.\times\left(\tau+n+\frac{(N-n)(\tau+N/2)}{2\tau+N}\right)\right), \tag{26}$$

which can be rewritten as,

$$d_n^{(III)}=(-1)^n\binom{N-1}{n}\frac{(2\tau+1)_n}{(2\tau+N+1)_n}$$
$$\mp j(-1)^{n-1}\binom{N-1}{n-1}\frac{(2\tau+1)_{n-1}}{(2\tau+N+1)_{n-1}}. \tag{27}$$

Consequently, from the z-transform of (27), it follows that the corresponding filter $D(z)$ has one pole at $z=\pm j$.

Example 3. The filter coefficients for the first order allpole filter are

$$d_0^{(III)}=1, \tag{28}$$
$$d_1^{(III)}=-j\tan\phi_\alpha. \tag{29}$$

Observe that in this case only condition i is satisfied. However, a second order filter can fulfill conditions i–iii with $K=1$. Using (24), the values of $d_n^{(III)}$ for $N=2$ are given by,

$$d_0^{(III)}=1, \tag{30}$$
$$d_1^{(III)}=-\frac{4}{2\tau+3}\left(\tau+1-\frac{e^{j\phi_\alpha}}{2\cos\phi_\alpha}\left(\tau+1+\frac{\tau+2\cos^2\phi_\alpha}{2(\tau+1)}\right)\right), \tag{31}$$
$$d_2^{(III)}=-j\tan\phi_\alpha\frac{2\tau+1}{2\tau+3}. \tag{32}$$

Some comments on the results for Type II and Type III allpole filters are in order. Obverse that the degree of flatness at $\omega=0$ for both Type II and Type III is the same, i.e., $K=2N-3$. However, according to Examples 2 and 3, by setting $N=1$, the condition $\phi_D(0)=0$ is satisfied only for Type III allpole filter. This justifies the proposed Type III allpole filter. On the other hand, Type III has unstable poles if $\phi_\alpha=\pm\pi/4$, while Type II is unstable if $\phi_\alpha=\pm\pi/2$. This implies, as we shall see in Section 3, that Type II allpole filter offers better stability conditions than Type III allpole filter.

2.4. Recursive relations

This section proposes the recursive relations to compute the coefficients $d_n^{(I)}$, $d_n^{(II)}$, and $d_n^{(III)}$, as it was suggested in [10]. From (5), (17), and (24), we arrive at

$$\frac{d_{n+1}^{(I)}}{d_n^{(I)}}=-\frac{(N-n)(2\tau+n)}{(n+1)(2\tau+N+n+1)}$$
$$\times\left(1+\frac{1}{\tau(1+j\cot\phi_\alpha)+n}\right), \tag{33}$$
$$\frac{d_{n+1}^{(II)}}{d_n^{(II)}}=-\frac{(N-n)(2\tau+n)}{(n+1)(2\tau+N+n+1)}$$
$$\times\left(1+\frac{1-j\tan\phi_\alpha-\frac{N-2n-1}{2\tau+N}}{(\tau+n)(1-j\tan\phi_\alpha)-\frac{n(N-n)}{2\tau+N}}\right), \tag{34}$$
$$\frac{d_{n+1}^{(III)}}{d_n^{(III)}}=-\frac{(N-n)(2\tau+n)}{(n+1)(2\tau+N+n+1)}$$
$$\times\left(1+\frac{1-\frac{(1+j\tan\phi_\alpha)}{N}\left(\tau+2n+1+\frac{(N-2n-1)(\tau+N\cos^2\phi_\alpha)}{2\tau+N}\right)}{\tau+n-\frac{n(1+j\tan\phi_\alpha)}{N}\left(\tau+n+\frac{(N-n)(\tau+N\cos^2\phi_\alpha)}{2\tau+N}\right)}\right). \tag{35}$$

The initial value in the recursive equation (33) is $d_0^{(I)}=1$. Accordingly, the filter coefficient $d_{n+1}^{(I)}$ is computed from $d_n^{(I)}$ by just multiplying $d_n^{(I)}$ with this ratio. In a similar way, we can compute $d_{n+1}^{(II)}$ and $d_{n+1}^{(III)}$. The corresponding MATLAB program is given in Appendix B.

We illustrate the design of complex-valued allpole filters in the following example.

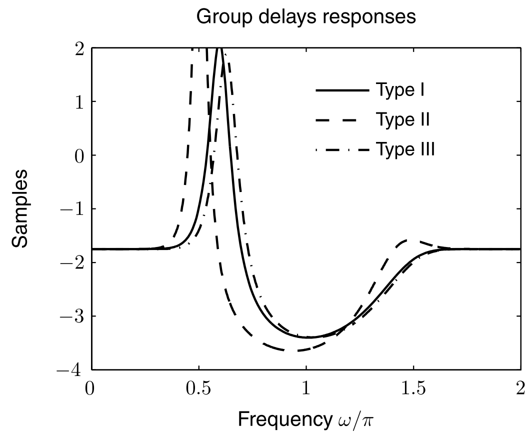

Fig. 1. Group delay responses of the designed complex-valued allpole filters in Example 4.

Example 4. We consider that the filter design parameter are the phase value ϕ_α is $-\pi/8$, the filter order $N = 7$, and group delay $\tau = 1/4$. The resulting group delays responses of the designed allpole filters are shown in Fig. 1. Observe that the degrees of flatness for Types I, II, and III are 12, 11, and 11, respectively,

3. Stability issues

We turn our attention to the stability of the proposed complex-valued allpole filters. At first, the algebraic stability test procedure is reviewed [19,20]. Consider a polynomial $F(z)$ given by

$$F(z) = \sum_{n=0}^{N} f_n z^{-n}, \tag{36}$$

where f_n, $n = 0, \dots, N$ are complex coefficients. According to [19, 20], the zeros of the polynomial $F(z)$ are strictly inside the unit circle if and only if

$$|f_{N-m,N-m}|^2 < 1, \quad m = 0, \dots, N-1, \tag{37}$$

where

$$f_{N-m-1,n} = \frac{f_{N-m,n} - f^*_{N-m,N-m-n} f_{N-m,N-m}}{1 - |f_{N-m,N-m}|^2}, \tag{38}$$

for $N \geqslant 2$, $m = 0, \dots, N-2$, and $n = 0, \dots, N-m-1$. The initial value for the recursive equation is $f_{N,n} = f_n$. In the following, we first illustrate the stability of the first and second order allpole filters.

3.1. First order allpole filters

3.1.1. Type I

The stability condition for $N = 1$ is

$$\left|d_1^{(I)}\right|^2 < 1. \tag{39}$$

From (11), the condition (39) can be rewritten as

$$0 < (1 + 2\tau)\cos^2\phi_\alpha. \tag{40}$$

In order to satisfy the inequality (40), the group delay τ should be larger than $-1/2$, and the phase value ϕ_α cannot be equal to $\pm\pi/2$.

3.1.2. Type II

In this case, as it is pointed out in Example 2, the allpole filter has a pole at $z = 1$.

3.1.3. Type III

Using (11) and the condition

$$\left|d_1^{(II)}\right|^2 < 1, \tag{41}$$

we found that the allpole filter is stable if $|\phi_\alpha| < \pi/4$.

3.2. Second order allpole filters

3.2.1. Type I

For the second order allpole filter, we have two stability conditions, i.e.,

$$\left|d_2^{(I)}\right|^2 < 1, \tag{42}$$

$$\left|\frac{d_1^{(I)} - (d_1^{(I)})^* d_2^{(I)}}{1 - |d_2^{(I)}|^2}\right|^2 < 1. \tag{43}$$

From (13) and (14), the conditions become

$$0 < \left(1 + 2\cos^2\phi_\alpha\right)(\tau + 1) \times \left(\left(\tau + \frac{2 + \cos^2\phi_\alpha}{1 + 2\cos^2\phi_\alpha}\right)^2 + \frac{9\cos^2\phi_\alpha}{2(1 + 2\cos^2\phi_\alpha)^2}\right), \tag{44}$$

$$0 < (2\tau + 1)(2\tau + 3)\cos^4\phi_\alpha. \tag{45}$$

Observe that (44) is satisfied for $\tau > -1$. Additionally, the condition (45) establishes that $\tau > -1/2$ and $\phi_\alpha \neq \pm\pi/2$. As a consequence, the allpole filter is stable if the following conditions are satisfied: $\tau > -1/2$ and $|\phi_\alpha| < \pi/2$.

3.2.2. Type II

In a similar way, using the stability test procedure, the conditions are

$$\left|d_2^{(II)}\right|^2 < 1, \tag{46}$$

$$\left|\frac{d_1^{(II)} - (d_1^{(II)})^* d_2^{(II)}}{1 - |d_2^{(II)}|^2}\right|^2 < 1, \tag{47}$$

which are equivalent to

$$0 < \tau + 1, \tag{48}$$

$$0 < (2\tau + 1)(2\tau + 3)\cos^4\phi_\alpha. \tag{49}$$

Therefore, the stability conditions for the second order allpole filter are $\tau > -1/2$ and $|\phi_\alpha| < \pi/2$. Note that the same conditions stand for the Type I allpole filter.

3.2.3. Type III

Finally, the conditions for the stability of the Type III allpole filters are equivalent to

$$0 < \left(1 - \tan^2\phi_\alpha\right)\left(\tau + \frac{1}{2}\frac{\tan\phi_\alpha + 3}{\tan\phi_\alpha + 1}\right)\left(\tau + \frac{1}{2}\frac{\tan\phi_\alpha - 3}{\tan\phi_\alpha - 1}\right), \tag{50}$$

$$0 < (2\tau + 3)(2\tau + 1)\left(2\cos^2\phi_\alpha - 1\right)\left(1 + 2\cos^2\phi_\alpha + 2\tau\right)^3. \tag{51}$$

To obtain the stability condition in terms of τ and ϕ_α, we use a geometric approach. The area where the values τ and ϕ_α satisfy the inequality (50) is the shaded area in Fig. 2(a). Similarly the shaded area in Fig. 2(b) illustrates the values of τ and ϕ_α, which satisfy (51). Therefore, an stable allpole filter has the values of τ and ϕ_α in the shaded area shown in Fig. 2(c).

It is worth to highlight that the stability regions for $-1 < \tau < -1/2$ are critical regions since a small variation on the phase of ϕ_α would make the allpole filter unstable. Accordingly, for practical purposes, the allpole filter is stable if $\tau > -1/2$ and $|\phi_\alpha| < \pi/4$.

The following subsection gives a general method to obtain the stability conditions of the complex-valued allpole filters.

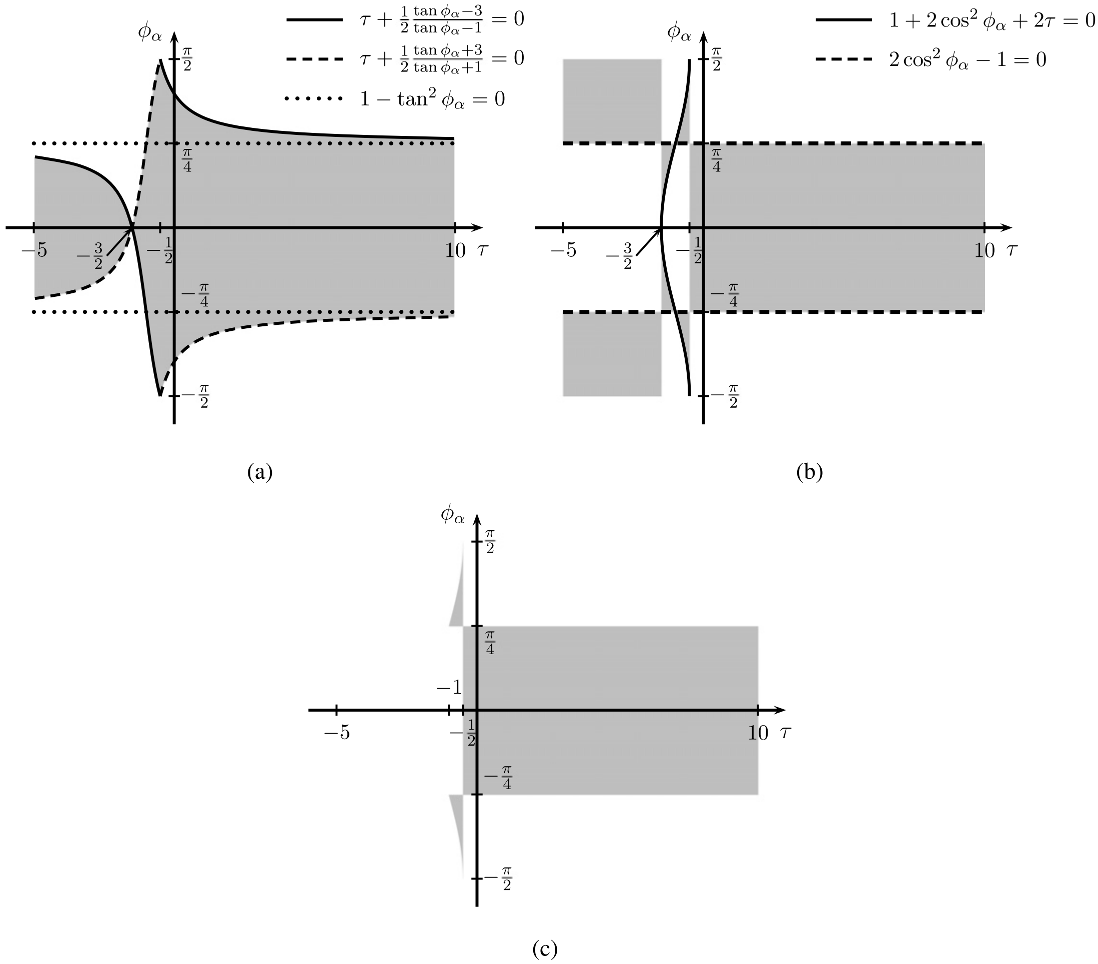

Fig. 2. Stability regions for second order Type III allpole filter. (a) Stability region for (50), (b) stability region for (51) and (c) stability region for the complex-valued allpole filter.

3.3. General stability conditions

3.3.1. Type I

At first, we rewrite (38) in terms of the filter coefficients $d_n^{(I)}$, i.e.,

$$d_{N-m-1,n}^{(I)}=\frac{d_{N-m,n}^{(I)}-(d_{N-m,N-m-n}^{(I)})^{*}d_{N-m,N-m}^{(I)}}{1-|d_{N-m,N-m}^{(I)}|^{2}},\tag{52}$$

for $N\geqslant 2$, $m=0,\dots,N-2$, $n=0,\dots,N-m-1$ and $d_{N,n}^{(I)}=d_n^{(I)}$. Next we illustrate the interesting behavior of $d_{N-m-1,n}^{(I)}$ at the points $\tau=-k/2$ for $k=1,\dots,N-m-1$ and $m=0,\dots,N-2$. Additionally, we consider that $|\phi_\alpha|<\pi/2$. This implies that there is no zero in $d_n^{(I)}$ at $\tau=-n$.

Considering the case $m=0$, Eq. (52) becomes

$$d_{N-1,n}^{(I)}=\frac{d_{N,n}^{(I)}-(d_{N,N-n}^{(I)})^{*}d_{N,N}^{(I)}}{1-|d_{N,N}^{(I)}|^{2}}.\tag{53}$$

We are interested in the values of $d_{N-1,n}^{(I)}$ at $\tau=-k/2$, for $k=1,\dots,N-1$. Thus using (8), the values of $d_{N,N}^{(I)}$ vanish at those points and consequently, we have $d_{N-1,n}^{(I)}=d_{N,n}^{(I)}$.

Now, consider the case when τ approaches ∞. In this case, $d_{N-1,n}^{(I)}$ has an indeterminate form. However, in order to overcome this problem, we use $d_{N,n}^{(I)}=(-\varepsilon)^n\binom{N}{n}$ and ε approaches to 1. Consequently, we obtain

$$\lim_{\tau\to\infty}d_{N-1,n}^{(I)}=\lim_{\varepsilon\to 1}(-1)^n\binom{N}{n}\frac{(-\varepsilon)^n-(-\varepsilon)^{2N-n}}{1-\varepsilon^{2N}}.\tag{54}$$

Applying L'Hopital's rule, we finally arrive at

$$\lim_{\tau\to\infty}d_{N-1,n}^{(I)}=(-1)^n\binom{N-1}{n}.\tag{55}$$

Similarly, for $m=1$, we have

$$d_{N-2,n}^{(I)}=\frac{d_{N-1,n}^{(I)}-(d_{N-1,N-1-n}^{(I)})^{*}d_{N-1,N-1}^{(I)}}{1-|d_{N-1,N-1}^{(I)}|^{2}}.\tag{56}$$

Consider now the points $\tau=-k/2$, for $k=1,\dots,N-2$. Since $d_{N-1,N-1}^{(I)}=d_{N,N-1}=0$ at those points, we obtain $d_{N-2,n}^{(I)}=d_{N-1,n}^{(I)}=d_n^{(I)}$ and the following relation holds:

$$\lim_{\tau\to\infty}d_{N-2,n}^{(I)}=(-1)^n\binom{N-2}{n}.\tag{57}$$

In general, we have

$$d_{N-m-1,n}^{(I)}=\begin{cases}d_n^{(I)}, & \text{if } \tau=-k/2 \text{ for } k=1,\dots,N-m-1,\\ (-1)^n\binom{N-m-1}{n}, & \tau\to\infty.\end{cases}\tag{58}$$

In the following, we present general stability conditions in terms of τ and ϕ_α for the Type I allpole filters. From (37), the first condition that should be satisfied is

$$\left|d_N^{(I)}\right|^2<1.\tag{59}$$

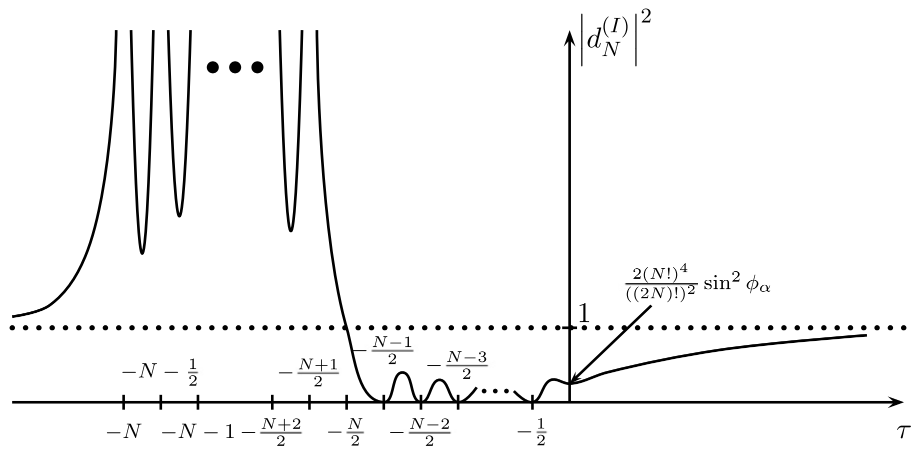

Fig. 3. Square magnitude of $d_N^{(I)}$ as a function of τ.

Our goal is to find values of τ and ϕ_α such that (59) is fulfilled. Recalling Section 2, we know that the allpole filter has one pole at $z=1$ if $\phi_\alpha=\pm\pi/2$. Consequently, we use $|\phi_\alpha|<\pi/2$. Thus, using (8), we arrive at

$$d_N^{(I)}=\begin{cases}0, & \text{if } \tau=0 \text{ and } \phi_\alpha=0,\\ 0, & \text{if } \tau=-k/2 \text{ for } k=1,\dots,N-1,\\ e^{2j\phi_\alpha}, & \text{if } \tau=-N/2,\\ \infty, & \text{if } \tau=-(N+k)/2 \text{ for } k=1,\dots,N,\\ (-1)^N, & \text{if } \tau\to\pm\infty.\end{cases} \tag{60}$$

Fig. 3 shows the plot of $|d_N^{(I)}|^2$ versus τ. Observe that $|d_N^{(I)}|^2>1$ for $\tau<-N/2$ because the points $\tau=-(N+k)/2$, for $k=1,\dots,N$, make $|d_N^{(I)}|$ unbounded and $|d_N^{(I)}|^2$ approaches to 1 when $\tau\to-\infty$. Furthermore, because of the zeros of $d_N^{(I)}$, the square magnitude of $d_N^{(I)}$ is less that 1 in $-N/2<\tau<0$. Finally, $|d_N^{(I)}|^2$ is an increasing function for $\tau>0$ and bounded by 1. Consequently, (59) is satisfied if $\tau>-N/2$.

The second stability condition is

$$\left|d_{N-1,N-1}^{(I)}\right|^2<1. \tag{61}$$

From (58), we observe that $|d_{N-1,N-1}^{(I)}|^2=|d_{N-1}^{(I)}|^2=1$ at $\tau=-(N-1)/2$ and $|d_{N-1,N-1}^{(I)}|^2<1$ for $-(N-1)/2<\tau<0$ due to the zeros of $d_{N-1}^{(I)}$. Furthermore, using (55), $|d_{N-1}^{(I)}|^2$ is an increasing function of τ and bounded by 1 for $\tau>0$. Consequently, Eq. (61) is satisfied when $-(N-1)/2<\tau$.

In a similar way, we found that $|d_{N-m,N-m}^{(I)}|^2<1$ if $-(N-m)/2<\tau$, for $m=0,\dots,N-1$, which consequently means that the Type I allpole filter is stable, if $\tau>-1/2$ and $|\phi_\alpha|<\pi/2$.

This interesting result shows that the traditional real-valued Thiran allpole filter is stable when $\tau>-1/2$. Thiran showed that the allpole filter is stable if $\tau>0$ [1]. Therefore, the approach presented in this paper extends the results proposed in [1]. To justify this result, in [2] the author applies the Thiran allpole filter with $-1/2<\tau<0$ for designing fractional delay allpass filters. The author pointed out that the designed filters were stable without a proof of this statement. Here we have proved the validation of this assumption.

3.3.2. Type II

Similar to the Type I allpole filter, we found that,

$$d_{N-m-1,n}^{(II)}=\begin{cases}d_n^{(II)}, & \text{if } \tau=-k/2 \text{ for } k=1,\dots,N-m-1,\\ (-1)^n\binom{N-m-1}{n}, & \tau\to\infty,\end{cases} \tag{62}$$

for $N\geqslant 2$. Therefore, with $N\geqslant 2$, the Type II allpole filter is stable if $\tau>-1/2$ and $|\phi_\alpha|<\pi/2$.

Table 1
Stability conditions for the proposed complex-valued allpole filters.

Type I $N\geqslant 1$	Type II $N\geqslant 2$	Type III $N\geqslant 1$
$\tau>-1/2$ $\lvert\phi_\alpha\rvert<\pi/2$	$\tau>-1/2$ $\lvert\phi_\alpha\rvert<\pi/2$	$\tau>-1/2$ $\lvert\phi_\alpha\rvert<\pi/4$

3.3.3. Type III

Finally, we derive the stability for the Type III allpole filters. From (37), the first condition that should be satisfied is

$$\left|d_N^{(III)}\right|^2<1. \tag{63}$$

Assuming $|\phi_\alpha|<\pi/4$ and using (25), we arrive at:

$$d_N^{(III)}=\begin{cases}0, & \text{if } \tau=-k/2 \text{ for } k=1,\dots,N-1,\\ j\tan\phi_\alpha, & \text{if } \tau=-N/2,\\ \infty, & \text{if } \tau=-(N+k)/2 \text{ for } k=0,\dots,N,\\ (-1)^n j\tan\phi_\alpha, & \text{if } \tau\to\pm\infty.\end{cases} \tag{64}$$

Consequently, (63) is satisfied if $\tau>-N/2$ and $|\phi_\alpha|<\pi/4$.

In general for $m=0,\dots,N-2$ and $N>2$, we have:

$$d_{N-m-1,n}^{(III)}=\begin{cases}d_n^{(III)}, & \text{if } \tau=-k/2 \quad \text{for } k=1,\dots,N-m-1,\\ (-1)^n\binom{N-m-1}{n}, & \tau\to\infty.\end{cases} \tag{65}$$

Therefore, the Type III allpole filter is stable if $\tau>-1/2$ and $|\phi_\alpha|<\pi/4$. The summary of this section is given in Table 1.

4. Application: design of casual complex-valued IIR filter for cardinal orthogonal scaling functions

This section deals with the design of casual complex-valued cardinal orthogonal scaling function. In particular, we extend the results proposed in [21] to include higher order approximation and complex-valued coefficients, as well.

The Fourier transform of the cardinal orthogonal scaling function $\phi(t)$ is expressed in terms of an infinite product of an IIR filter $H(e^{j\omega})$ as [21]:

$$\Phi(\omega)=\prod_{k=1}^{\infty}H\left(e^{j\frac{\omega}{2^k}}\right), \tag{66}$$

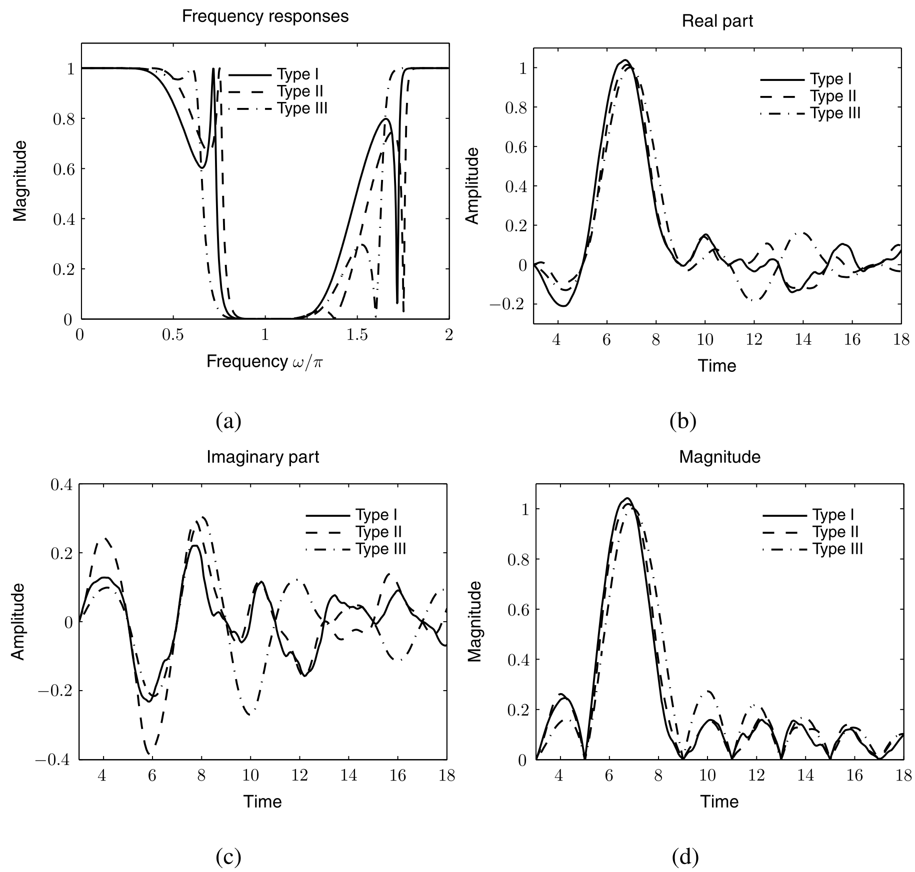

Fig. 4. Cardinal orthogonal scaling functions in Example 5. (a) Magnitudes response of $H(z)$, (b) real parts of $\phi(t)$, (c) imaginary parts of $\phi(t)$, (d) magnitudes of $\phi(t)$.

where $H(e^{j\omega})$ is a lowpass filter and is given by [22],

$$H\left(e^{j\omega}\right) = \frac{1}{2} + \frac{1}{2} e^{-jM\omega} A\left(e^{j2\omega}\right), \tag{67}$$

where $A(e^{j\omega})$ is an allpass filter of order N and M is an odd integer. Therefore, the problem of the IIR filter design is reduced to the design of an allpass filter $A(z)$ with the desired characteristics.

In order to obtain a proper scaling function, the magnitude response of the lowpass filter should be maximally flat at $\omega = \pi$. This implies that the group delay response $G_A(\omega)$ of the allpass filter is maximally flat at $\omega = 0$ [10]. Furthermore, using (67), it is straightforward to show that $G_A(0) = M/2$ [10]. In the following, the design of $A(z)$ based on complex-valued maximally flat allpole filter is described.

We now relate the allpass filter of order N with the allpole filter as,

$$\begin{aligned} A(z) &= z^{-N} \frac{D(z)}{\widetilde{D}(z)} \\ &= e^{j2\phi_\alpha} \frac{d_N^* + d_{N-1}^* z^{-1} d_{N-2}^* z^{-2} + \cdots + 1}{1 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_N z^{-N}}, \end{aligned} \tag{68}$$

where $\widetilde{D}(z)$ is the paraconjugate of $D(z)$, i.e., it is generated by conjugating the coefficients of $D(z)$ and by replacing z by z^{-1}.

From (68), the group delay of the complex-valued allpass filter $A(z)$, is given by

$$G_A(\omega) = N + 2G(\omega), \tag{69}$$

where $G(\omega)$ is the group delay of $D(z)$. Consequently, the desired group delay τ at $\omega = 0$ can be rewritten as,

$$\tau = \frac{M - 2N}{4}. \tag{70}$$

Using the results of Section 3, the stability of $A(z)$ is fulfilled if

$$M > 2(N - 1). \tag{71}$$

We illustrate the design of $H(z)$, which generates the complex-value cardinal orthogonal scaling functions, in the following example.

Example 5. We discuss the design of $H(z)$ using the following parameters: $N = 4$, and $M = 7$. At first, we design $A(z)$ based on the Type I allpole filter. In this case we use $\phi_\alpha = 3\pi/16$. Similarly, for Types II and III, the phase values for ϕ_α are $3\pi/16$ and $3\pi/32$, respectively. From (70), we have $\tau = -1/4$. Fig. 4 shows the resulting magnitude responses of the designed lowpass filters and the corresponding cardinal orthogonal scaling functions.

In [21], the authors proposed the design of $A(z)$ of order N for $N \leqslant 3$. Here, we have not only extended the method [21] to include higher order allpass filters ($N \geqslant 1$) but also to include complex coefficients. Additionally, the conditions for the stability of $A(z)$, as well as $H(z)$, are provided.

5. Conclusions

This paper extends the traditional Thiran allpole filter to the complex case. In the complex case, the degree of flatness can be either even or odd. Particularly, we propose three classes of filters, namely Types I–III. The design parameters are the phase value ϕ_α, the group delay τ and the degree of flatness K at $\omega = 0$. The complex filter coefficients are obtained from a set of linear equations. The MATLAB script for computing the filter coefficients is also provided. The stability conditions are defined for the proposed filters.

Table 2
Matlab script to compute the complex filter coefficients.

```
function d = CompThiran(N,tau,phi_a,Type)


% CompThiran computes the complex filter coefficients of Thiran allpole
% filter. The resulting complex-valued allpole filter is maximally flat
% group delay at w = 0.
%
% Input:
% N      is the order of the complex-valued allpole filter
% tau    is the value of the group delay at w = 0
% phi_a  is the phase value of the constant phi_alpha
% Type   is the Thiran allpole filter Type ('one','two','three')
%
% Output:
% d      is a row vector with the allpole filter coefficients
%
n = 0:N-1;
switch lower(Type)
    case 'one'
        d = cumprod([1,-(N-n)./(n+1).*(2*tau+n)./(2*tau+N+n+1).*...
            (1+1./(tau*(1+1j*cot(phi_a))+n))]);
    case 'two'
        d = cumprod([1,-(N-n)./(n+1).*(2*tau+n)./(2*tau+N+n+1).*...
            (1+((1-1j*tan(phi_a))-(N-2*n-1)/(2*tau+N))./((tau+n)*...
            (1-1j*tan(phi_a))-n.*(N-n)/(2*tau+N)))]);
    case 'three'
        d = cumprod([1,-(N-n)./(n+1).*(2*tau+n)./(2*tau+N+n+1).*...
            (1+(1-(1+1j*tan(phi_a))/N.*(tau+2*n+1+(N-2*n-1)*...
            (tau+N*cos(phi_a)^2)/(2*tau+N)))./(tau+n-n*(1+1j*tan(phi_a))/N.*...
            (tau+n+(N-n)*(tau+N*cos(phi_a)^2)/(2*tau+N))))]);
    otherwise
            disp('Unknown type')
end
```

The extension of the stability conditions of a real Thiran allpole filter is also presented. Finally, the application of the proposed filters in the design of complex-valued cardinal orthogonal scaling functions is described.

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Appendix A. Derivation of (16)

This section addresses the detail derivation of (16). We first expand the term $(2\tau + 2n)$ as follows,

$$2\tau + 2n = 2\tau + n + n. \tag{A.1}$$

Substituting (A.1) into (16) and regrouping terms, we arrive at

$$\begin{aligned} d_n = {} & (-1)^n \binom{N}{n} \frac{(2\tau+1)_n}{(2\tau+N+1)_n} \\ & + n(-1)^n \binom{N}{n} \frac{(2\tau+1)_{n-1}}{(2\tau+N+1)_n}. \end{aligned} \tag{A.2}$$

In this way, we expand the first binomial coefficient and the last binomial coefficient is multiplied by N/N, i.e.,

$$\begin{aligned} d_n = {} & (-1)^n \left(\binom{N-1}{n} + \binom{N-1}{n-1} \right) \frac{(2\tau+1)_n}{(2\tau+N+1)_n} \\ & + (-1)^n \binom{N-1}{n-1} \frac{(2\tau+1)_{n-1}}{(2\tau+N+1)_{n-1}} \frac{N}{2\tau+N+n}. \end{aligned} \tag{A.3}$$

Accordingly, Eq. (A.3) reduces to

$$d_n = (-1)^n \left(\binom{N-1}{n} + \binom{N-1}{n-1} \right) \frac{(2\tau+1)_n}{(2\tau+N+1)_n} + (-1)^n \binom{N-1}{n-1} \frac{(2\tau+1)_{n-1}}{(2\tau+N+1)_{n-1}} \left(1 - \frac{2\tau+n}{2\tau+N+n} \right). \tag{A.4}$$

From (A.4), Eq. (16) holds.

Appendix B. MATLAB program

This appendix shows the MATLAB script to compute the filter coefficients of the proposed complex-valued allpole filter. The script `CompThiran.m` is shown in Table 2.

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