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On the Estimation of the Minimum Number of Distinct Multipliers in FIR Hilbert Transformers based on Frequency Transformation

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Abstract

Frequency Transformation (FT) is a technique to design Finite Impulse Response (FIR) filters with repeated simple identical subfilters, which allows reducing the overall number of distinct multipliers, the most costly elements in a digital filter. This paper presents a new expression to estimate the length of HTs, straightforwardly derived from the formula of I. Koichi et al. The proposed formula avoids the high error for cases with wide ripple, which is found in a previous formula used in the recently proposed approach to design FT-based FIR Hilbert Transformers (HTs) where the estimated number of distinct multipliers is minimized. It is shown that, with the proposed formula, the FT-based design of HTs can be expressed under the same optimization approach for any of the already existing FT-based architectures.

Keywords: Finite Impulse Response (FIR) filters, Hilbert transformers, Identical Subfilters, Frequency Transformation, Pipelining-Interleaving

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1. Introduction

A Hilbert Transformer (HT) is a special class of filter that introduces a $\pi/2$ phase shift in the input signal and whose ideal frequency response is [1]:

$$H(e^{j\omega}) = H(\omega)e^{j\theta(\omega)}, \tag{1a}$$

$$H(\omega) = 1, \quad \theta(\omega) = \begin{cases} -\pi/2 & \text{for } 0 < \omega < \pi \\ \pi/2 & \text{for } -\pi < \omega < 0. \end{cases} \tag{1b}$$

HTs are usually used in telecommunications, speech processing and medical imaging, among others [2]. HTs can be designed either as Finite Impulse Response (FIR) filters or Infinite Impulse Response (IIR) filters. However, FIR filters can have exact linear phase and their stability is guaranteed. Additionally, they are less sensitive to the coefficients rounding and their phase response is not affected by this rounding. As a result, FIR HTs are often preferred. Nevertheless, FIR filters have a high complexity when very sharp transition bands and small ripples are required. This complexity is dominated by multipliers, the most expensive elements in a digital filter. A linear-phase HT, which has an anti-symmetrical impulse response, can be designed with either an odd (Type III) or an even (Type IV) length. The number of multipliers m is estimated in terms of the filter length L as

$$m \approx C \cdot L, \tag{2}$$

where $C = 0.25$ if the filter is Type III or $C = 0.5$ if it is Type IV.

In order to decrease the complexity of direct designs based on the well-known Parks-McClellan algorithm, different techniques have been developed for efficient design of HTs. One of the most important techniques to reduce the overall number of multipliers is the design based on Frequency Transformation (FT). The FT method, introduced by T. Saramaki [3], was modified to design FIR HTs in the work of Y. L. Tai and T. P. Lin [4]. In that proposal HTs are designed as a tapped cascaded interconnection of repeated simple basic building blocks constituted by two identical subfilters. Taking advantage of the repetitive use of these basic blocks, the recent proposal of M. G. C. Jimenez et al., [5] gives a simple and efficient method to design multiplierless HTs, where a combination of the FT method with the Pipelining Interleaving (PI) technique of Z. Jiang and A. N. Willson [9] allows getting a time-multiplexed implementation with three subfilters. The FT-PI-based structure proposed by M. G. C. Jimenez et al. was modified in the work of D. E. T. Romero et al. [6] to use two subfilters and an optimization approach was developed to minimize the overall number of filter coefficients in that architecture. This method is hereafter referred to as *optimization-based FT method*. The same authors proposed a new PI-based structure in [7], which requires only one time-multiplexed subfilter, to reduce further the number of multiplier coefficients in the overall design, such that the total number of coefficients is the same as the number of distinct multipliers obtained with the FT method. Due to the effectiveness of the FT method, a brief introduction to the multi-level FT method has been recently reported by J. C. Liu and Y. L. Tai [8], where the basic idea consists on simplifying further the subfilters by using again the FT in every subfilter.

In this paper we extend our previous optimization-based FT method [6]. It is shown that the optimization approach, even though proper, can not be applied to other FT-based structures because the HT length estimation formula does not provide exact results for cases with wide ripple. To solve this problem, we straightforwardly derive a new formula to estimate the length of HTs from the proposal of I. Koichi et al. [10]. The obtained formula is less sensitive to the variation of the ripple specifications. As a consequence, with this improvement the FT-based method can be presented under a generalized optimization-based approach where the estimated number of multiplier coefficients can be minimized.

The organization of this paper is as follows. Section 2 gives a brief introduction of the FT method to design HTs. The analysis of the optimization-based FT method is detailed in Section 3. Then, Section 4 introduces the derivation of the expression to estimate the length of HTs from the formula of I. Koichi et al. Comparison of this expression with the one used in the optimization-based FT method are discussed in Section 5 and Section 6 presents a design example and the discussion of results. Finally, Section 7 presents concluding remarks and future work.

2. Frequency Transformation method for FIR Hilbert transformers

Consider the magnitude response $|H(\omega)|$ of a desired FIR HT given as

$$(1-\delta) \le |H(\omega)| \le (1+\delta), \quad \text{for } \omega_L \le \omega \le \omega_H , \tag{3}$$

where δ is the allowed passband ripple, ω_L is the low passband edge and ω_H is given as $\omega_H = \pi - \omega_L$ if the desired HT is a Type III filter or $\omega_H = \pi$ if it is Type IV. The FT method consists of designing two simple filters, a prototype filter and a subfilter, both of them being HTs. The number of times that the subfilter is used and the tapped coefficients among subfilters depend on the prototype filter. The subfilter can be a Type III or Type IV filter according to the type of the desired HT whereas the prototype filter is always Type IV. To accomplish the desired specification for $|H(\omega)|$, the magnitude response of the prototype filter, $|P(\Omega)|$, must satisfy the following condition,

$$(1-\delta) \le |P(\Omega)| \le (1+\delta), \quad \text{for } \Omega_L \le \Omega \le \pi , \tag{4}$$

where Ω denotes the frequency domain of the prototype filter and Ω_L is its low passband edge frequency. The magnitude response of the subfilter, $|G(\omega)|$, must fulfill simultaneously

$$v_d - \delta_G \le |G(\omega)| \le 1, \quad \text{for } \omega_L \le \omega \le \omega_H , \tag{5a}$$

$$v_d = \tfrac{1}{2} + \tfrac{1}{2}\sin\left(\tfrac{\Omega_L}{2}\right), \tag{5b}$$

$$\delta_G = \tfrac{1}{2} - \tfrac{1}{2}\sin\left(\tfrac{\Omega_L}{2}\right). \tag{5c}$$

The overall transfer function for a FT-based HT, $H(z)$, is

$$H(z) = G(z) \sum_{n=0}^{L_P/2-1} z^{-(L_G-1)(L_P/2-1-n)} \alpha(n) \left[H_1(z)\right]^n , \tag{6a}$$

$$H_1(z) = z^{-(L_G-1)} + 2G^2(z) , \tag{6b}$$

with L_P and L_G being the respective filter lengths of the prototype filter and the subfilter, whereas $G(z)$ is the transfer function of the subfilter. The coefficients $\alpha(n)$ can be derived from the prototype filter coefficients through the Chebyshev polynomials (more details in [4], [5] or [6]).

3. Analysis of the method to minimize the overall number of multipliers

The minimization of the number of multipliers presented in the optimization-based FT method is a special

case given only for a PI-based structure which requires two subfilters, where the cost function (the estimated overall number of multipliers) is $m \approx 2C\mathcal{L}_G + \mathcal{L}_P/2$. The values $\mathcal{L}_G$ and $\mathcal{L}_P$ are the respective approximations to the lengths L_G and L_P through a function $\varphi(x, y)$, where x denotes the ripple of the HT and y denotes its low passband edge frequency. These approximations are given in accordance to the respective ripples and low passband edge frequencies of the subfilter and the prototype filter as follows,

$$L_G \approx \mathcal{L}_G = \varphi(\delta_G / v_d,\ \omega_L)\,, \tag{7}$$

$$L_P \approx \mathcal{L}_P = \varphi(\delta,\ \Omega_L)\,. \tag{8}$$

The formula $\varphi(x, y)$ utilized in that proposal, derived from the one introduced by O. Hermann et al. [11], is

$$\varphi(x,\ y) = \left[0.002655\left(\log_{10}(x)\right)^3 + 0.031843\left(\log_{10}(x)\right)^2 - 0.554993\log_{10}(x) - 0.049788\right]/(\tfrac{y}{2\pi}) + 1. \tag{9}$$

With the use of the normalized ripple argument of $\mathcal{L}_G$ as in (7) the objective function has the problem of estimating the number of required multiplier coefficients as zero or negative values. As an example, for a Type III HT with the following specification, $\omega_L = 0.0001\pi$ and $\delta = 0.0001$, we get an estimated overall number of multipliers, m, about –220. Using (9), the estimated lengths for the prototype filter and the subfilter are, respectively, $L_P \approx 276$ and $L_G \approx -719$. While the estimated length L_P can be considered proper, the value for L_G is completely pointless.

The above mentioned problem was partially solved in the optimization-based FT method with the use of δ_G as the subfilter ripple specification in (7), instead of using the normalized ripple δ_G/v_d. However, this solution has a second problem, namely, a highly inaccurate estimation of the number of required multipliers. For instance, consider the Example 1 presented in that work, where the estimated value for L_G was 38.37 whereas the real value was 23. The absolute error of this estimation is about 66% and, as the desired value ω_L decreases, the error becomes higher.

Now, consider the single-rate structure to implement the transfer function (6), whose number of multiplier coefficients can be estimated as $m \approx C(\mathcal{L}_P - 1)\mathcal{L}_G + \mathcal{L}_P/2$. Another problem is the following. By using (7) and (8) to substitute in $\mathcal{L}_G$ and $\mathcal{L}_P$, the method [6] would give an optimum value $\Omega_L = \Omega^*_L$ to be used in (4), for which the overall number of multipliers is supposed to be minimum. However, if the optimal value Ω^*_L is obtained with the cost function $m \approx 2C\mathcal{L}_G + \mathcal{L}_P/2$, which corresponds to a different structure, a less number of coefficients can be obtained. That means that the optimization-based FT method can not be applied for the single-rate structure to implement the transfer function (6). The inexactitude of the current function $\varphi(x, y)$ makes the optimization-based FT method inappropriate when it is applied to other FT-based structures.

4. Derivation of an expression to estimate the length of HTs from the Koichi's et al formula

The main problems highlighted in the previous subsection are due to the function $\varphi(x, y)$ used to estimate the filter lengths, which is given in (9). To our knowledge, a most exact formula to estimate the length of a FIR filter optimized under the minimax criterion is given by I. Koichi et al. [10]. We use the following straightforward approach to adjust this formula for the case of HTs:

1. Consider that L_{III} and L_{IV} are the lengths of a Type III and a Type IV HT, respectively. From the relation between a half-band filter and a Type III HT we have the passband and stopband edge frequencies of the former, ω_p and ω_s, as well as its passband and stopband ripples, δ_p and δ_s, given in terms of the low passband edge frequency y and the ripple x of the latter as follows:

$$\omega_p = \pi/2 - y, \qquad \omega_s = \pi/2 + y, \qquad \delta_p = \delta_s = x/2. \tag{10}$$

Thus, we substitute the half-band filter specification in the general formula given by I. Koichi et al. using the expressions for ω_p, ω_s, δ_p and δ_s in terms of x and y and with this we can estimate the length L_{III}.

2. The Type III HT is derived from the Type IV HT by substituting each delay with two delays. With this, the lengths are related as $L_{\mathrm{III}} = 2L_{\mathrm{IV}} - 1$ and the respective low passband edge frequencies of the Type III HT and the Type IV HT are $y < \pi/2$ and $\tilde{y} = 2y$. Let us express $L_{\mathrm{IV}} = (L_{\mathrm{III}} + 1)/2$. Substituting L_{III} from step 1, L_{IV} becomes an estimation of the length of a Type IV HT with ripple x and low passband edge $2y$. In order to be able to use y in the range $0 < y < \pi$, the transition band considered in the estimation of L_{III} of step 1 must be divided by 2.

Following the two aforementioned steps, we get

$$\varphi(x, y) = \frac{1}{2} + \left[\frac{1.101\left[-\log_{10}(x)\right]^{1.1}}{(\frac{y}{2\pi})} + 1 \right] \cdot \left[\frac{2}{3\pi} \arctan\left\{ \left[2.325\left[0.30103 - \log_{10}(x)\right]^{-0.445} \cdot (\tfrac{y}{2\pi})^{-1.39} \right] \times \left[\frac{1}{0.5 - (\frac{y}{2\pi})} \right] \right\} + \frac{1}{6} \right]. \tag{11}$$

5. Comparison

In the following we will compare the previous formula used in the optimization-based FT method with the proposed formula. To this end, consider (4) and (5a) to (5c). Note that the prototype filter and the subfilter have the ripple specification and the transition band specification of the desired filter, respectively. According to the optimization approach introduced in [6], the optimum value Ω^*_L balances the narrowing of the transition band of the prototype filter and the shrinking of the ripple of the subfilter in such a way that the overall number of multiplier coefficients required in a given structure is minimized. Therefore, it can be assumed that the prototype filter will have a non-stringent transition band specification, whereas the subfilter will have a non-stringent ripple specification.

With this in mind, let us analyze (9) and (11) for two cases: the first one corresponds to prototype filters, i.e., the design of Type IV HTs using 20 different low passband edge values ω_L equally spaced in the range from 0.1π to 0.9π, considering the ripple values $\delta = 0.0001$ and $\delta = 0.005$. The second one corresponds to subfilters, i.e., the design of Type III HTs using 20 different ripple values δ equally spaced in the range from 0.1 to 0.9, considering the low passband edge values $\omega_L = 0.0001\pi$ and $\omega_L = 0.005\pi$. In this second case the choice of Type III HTs is arbitrary and the results for Type IV subfilters are quite similar. Figure 1a shows the percentage of absolute error for the estimation of the length corresponding to prototype filters. Figure 1b shows the percentage of absolute error for the estimation of the length of the subfilters.

From Fig. 1a we have that (9) and (11) exhibit almost the same percentage of error for the length estimation of prototype filters. For any low passband edge less than $\omega_L = 0.6\pi$, the maximum absolute error is no more than 13% for the cases with ripple $\delta = 0.005$ and no more than 17% for the cases with ripple $\delta = 0.0001$. For the other passband edge values some absolute errors are between 20% and 30% and for the specification $\omega_L = 0.9\pi$ and $\delta = 0.0001$ the absolute error is about 65%. However, note that in these cases the filter length is small because the transition band is considerably wide. Therefore, even though the percentage

of error is high, the difference between the estimated and the real length is small. For the highest difference, the estimated and the real lengths are 6.6 and 4, respectively, which corresponds to the error equal to 65%. In such case the estimated number of multipliers is 3.3, whereas the real is 2.

Fig. 1b shows that (11) has a much less percentage of error than (9) for the length estimation of subfilters. Using (9), for any ripple less than $\delta = 0.3$ the maximum absolute error is less than 3% in both cases, with low passband edge $\omega_L = 0.0001\pi$ and with $\omega_L = 0.005\pi$. Using (11), the maximum absolute error is less than 5% in both cases, with low passband edges $\omega_L = 0.0001\pi$ and $\omega_L = 0.005\pi$. As the ripple value δ increases, eq. (9) shows a growing error which is close to or more than 100% when $\delta \geq 0.8$. On the other hand, eq. (11) exhibits a good length estimation over all the cases proved, having a worst absolute error less than 24%. It is obvious that (11) does provide reliable results for cases with wide ripple and it can be used with the optimization-based FT method to minimize the estimated number of multiplier coefficients in any FT-based architecture.

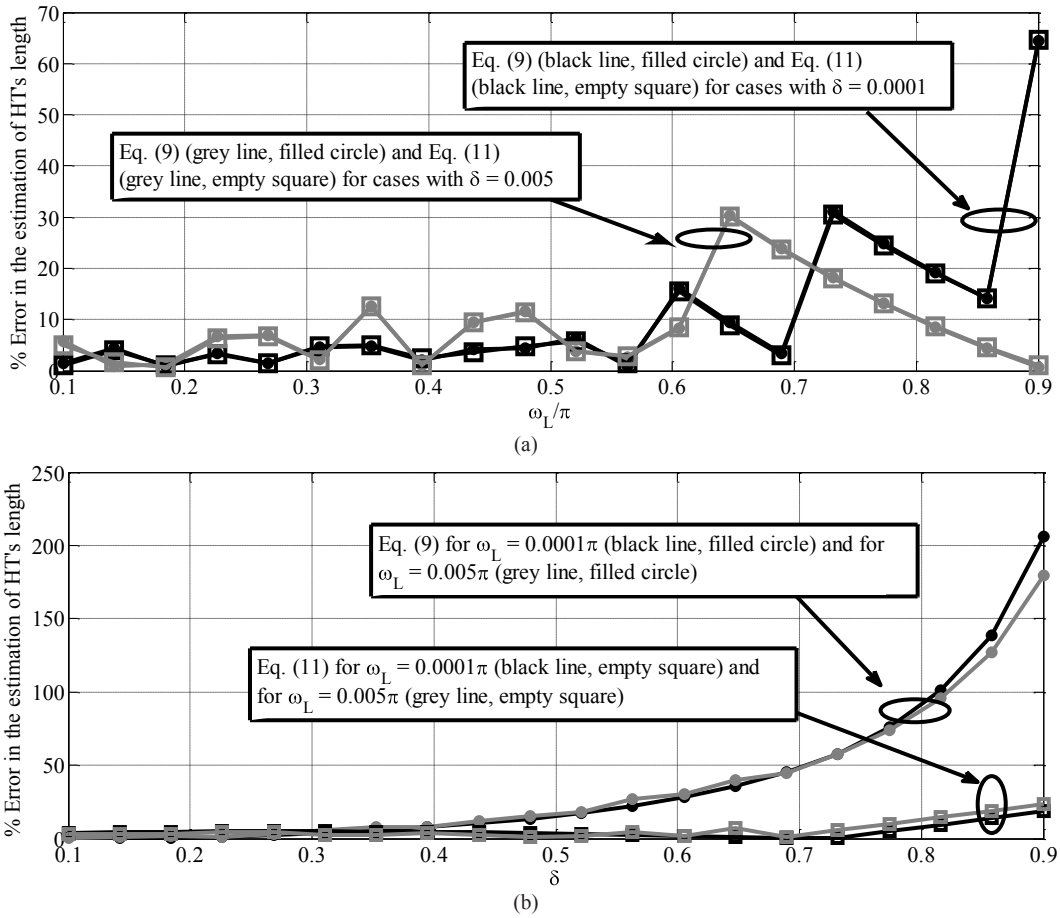

Fig. 1. Percentage of absolute error in the estimation of lengths of HTs using equations (9) and (11); (a) cases with ripples $\delta = 0.0001$ and $\delta = 0.005$, with ω_L in the range from 0.1π to 0.9π, (b) cases with relative frequencies $\omega_L = 0.0001\pi$ and $\omega_L = 0.005\pi$, with δ in the range from 0.1 to 0.9.

6. Design example and discussion of results

In this section we detail a design example to show the usefulness of the proposed formula. It is worth

highlighting that the multiplierless HTs can be obtained by coefficients rounding or by optimizing the filter in a discrete-coefficients space. Therefore, in this example we consider filters with multiplier coefficients, taking in account that the resulting design can be made multiplierless in a subsequent design stage.

Example 1 [6]: Design a Type III HT to satisfy (3) with $\delta = 0.004$ and $\omega_L = 0.01\pi$. Consider that the filter is restricted to operate at a clock rate not higher than 9 times the data rate.

In the original optimization-based FT method the system uses the PI-based structure with three subfilters. For this structure, the estimated overall number of multiplier coefficients is given by $m \approx 3C\mathcal{L}_G + \mathcal{L}_P/2$, with $C = 0.25$, $\mathcal{L}_G = \varphi(\delta_G/v_d, \omega_L)$, $\mathcal{L}_P = \varphi(\delta, \Omega_L)$, δ_G and v_d respectively given in (5b) and (5c) and $\varphi(x, y)$ introduced in (11). With the proper substitutions we get a function $m(\Omega_L)$ and the optimization consists on finding the optimal value for Ω_L, Ω^*_L, that minimizes $m(\Omega_L)$. Moreover, the lower bound for Ω_L, which is a consequence of the aforementioned clock rate constraint, is given as $\Omega_{low} = 0.1503\pi$. After the optimization we obtain $\Omega^*_L = 0.1503\pi$.

Note that the result is the same as the one reported in the original optimization-based FT method, but an important observation follows: in that work, the estimated subfilter and prototype filter lengths are, respectively, $\mathcal{L}_P = 20$ and $\mathcal{L}_G = 38.37$. With the proposed formula, the estimated subfilter and prototype filter lengths are, respectively, $\mathcal{L}_P = 20.099$ and $\mathcal{L}_G = 21.475$. The filter meets the specifications with $L_P = 20$ and $L_G = 23$. This means that a long trial and error search for the correct length L_G is avoided with the proposed formula. The resulting overall number of multiplier coefficients is 28.

Now let us consider that the overall number of multipliers is the same as the number of distinct multiplier coefficients in the FT method. A technique to solve cases like this has been recently proposed by J. C. Liu and Y. L. Tai [8]. The method of J. C. Liu and Y. L. Tai, even though being a time-consuming heuristic, provides near-optimal solutions when the minimum number of distinct coefficients is searched. However, it does not consider clock rate constraints in the structure and it completely relies on assuming that the numbers of multipliers of both, the prototype filter and the subfilter, are nearly equal. In fact, such method does not provide information about what structure should be used. Therefore, if that method is employed, we obtain $L_P = 16$ and $L_G = 27$, with a number of distinct multipliers equal to 15. Note that this solution needs to use the PI-based structure with three subfilters, thus the overall number of multipliers is 29. Other PI-based structures can not be employed because they do not meet the clock rate constraint.

On the other hand, the proposed formula (11) gives the possibility of using a PI-based structure with only a single subfilter. Following the same criteria introduced in the original optimization-based FT method to find the lower bound for Ω_L we get $\Omega_{low} = 0.3173\pi$, which preserves the clock rate limitation. The objective function is $m \approx C\mathcal{L}_G + \mathcal{L}_P/2$. After the optimization we obtain $\Omega^*_L = 0.3173\pi$, $L_P = 10$ and $L_G = 47$, and the overall number of multipliers, which is also the number of distinct coefficients in the FT method, is equal to 17. Finally, Table 1 summarizes the aforementioned results.

Table 1. Comparison of results for the design of a Type III HT designed with FT method.

Design method	Type of structure employed	Number of multipliers
D. E. T. Romero et al. 2012 [6]	PI-based with three subfilters	28
J. C. Liu and Y. L. Tai 2012 [8]	PI-based with three subfilters	29
D. E. T. Romero et al. 2012 with proposed formula	PI-based with one subfilter	17

7. Concluding remarks and future work

In this paper we have analyzed the recently proposed optimization-based method to design FIR Hilbert

transformers based on Frequency Transformation (FT), where the estimated overall number of multiplier coefficients is minimized. It was shown that the formula employed to estimate the length of the subfilters involved in the FT-based design does not provide accurate results for filters with wide ripple. Therefore, even though the optimization insight is proper, the method could not be able to provide optimal solutions for FT-based structures proposed in literature other than the one presented in the analyzed work. To solve this problem, we have straightforwardly derived a new formula to estimate the length of FIR Hilbert transformers. It has been shown that the new formula provides better results in comparison with previous FT-based schemes.

It is worth highlighting that, with the proposed formula, the optimization method can be generalized to minimize any objective functions that properly estimate a given cost of the overall HT in terms of the lengths of the involved subfilters. Thus, the analysis provided in this paper paves the way to develop a general optimization scheme where not only the minimization of the number of multiplier coefficients, but also any other cost function can be performed for any given architecture. As a future work we will develop a generalized optimization scheme which involves also the multi-level FT scheme.

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