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Homotopy-Continuation Picard Method

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Abstract

A coupling between Homotopy Continuation methods and Picard's Method (HCPM) is proposed to provide an approximate solution to nonlinear differential equations (NDE). By means of a simple case study, the solution calculated by HCPM will show lower relative error compared to Picard's and HPM. Also, the solution calculated by HCPM has lower number of terms than Picard's method and requires significantly fewer iterations than the employed by the HPM method.

Mathematics Subject Classification: 34L30

Keywords: Picard Method, Homotopy Continuation, Nonlinear Differential Equations.

1 Introduction

The solutions for nonlinear differential equations is an important subject of research, because many physical phenomena can be modelled by such equations. One of the most known and ancient method to approximately solve nonlinear differential equations is Picard's iterative method [1–6]. This article proposes a coupling between the homotopy continuation methods [7–11] and Picard's method; it is called Homotopy-Continuation Picard Method (HCPM). Besides, we will compare the proposed method to the homotopy perturbation method (HPM) [12–26]. Therefore, using a simple case study, we will show a comparison between results obtained by applying HCPM, Picard's, and HPM methods. The comparison shows how the HCPM method generates an useful approximate solution having wider domain of accuracy than the other two methods.

This paper is organized as follows. In Section 2, we introduce the basic concept about HCPM method. Section 3 presents the approximated solution of a case study calculated by HCPM, Picard and HPM methods. In Section 4, we perform numerical simulations and discuss the results. Finally, a brief conclusion is given in Section 5.

2 Basic concept of HCPM

Given a nonlinear n -th order differential equation, which can be expressed as

$$y^{(n)}(x) = L + N + f(x), \quad (1)$$

with initial conditions

$$y(x^*) = C_1, \quad y^{(1)}(x^*) = C_2, \dots, \quad y^{(n-1)}(x^*) = C_n, \quad (2)$$

where L is a linear operator, N is a nonlinear operator and $f(x)$ is a function of the independent variable x .

On one hand, the basic formulation of Picard's method is

$$y_{i+1}(x) = y(x^*) + y^{(1)}(x^*)x + \dots + y^{(n-1)}(x^*)x^{n-1} + \int_{x^*}^x \dots \left[\int_{x^*}^x \left[\int_{x^*}^x y_i(t) dt \right] dt \right] \dots dt, \quad (3)$$

where the number of integrals is n , y_i is the right hand of equation (1), and x^* is the expansion point.

On the other hand, a possible homotopy-continuation formulation is

$$H(z, p) = (1 - p)G(z) + p(F(z)) = 0, \quad (4)$$

where p is the homotopy parameter, $F(z)$ is the nonlinear algebraic equation to solve, $G(z)$ is an operator used as the initial homotopy deformation, and z is the independent variable. When $p = 0$, $H(z, 0) = G(z) = 0$ has a trivial solution.

The proposed coupled equation of Homotopy and Picard's is

$$y_{i+1}(x) = y(x^*) + y^{(1)}(x^*)x + \dots + y^{(n-1)}(x^*)x^{n-1} + \int_{x^*}^x \dots \left[\int_{x^*}^x \left[\int_{x^*}^x ((1 - p_i)G(t) + p_i y_i(t)) \right] dt \right] dt \dots dt, \quad (5)$$

where the number of integrals is n , p_i is a discrete version of the homotopy parameter and G is an arbitrary operator which can be linear or nonlinear.

Also, p_i can be calculated by

$$p_i = \frac{i - 1}{N}, \quad i = [1, 2, \dots, N + 1], \quad N > 1 \quad (6)$$

where N represents the number of required iterations for the approximation.

3 Case Study

In this work a Riccati nonlinear differential equation [23, 27] will be used as a case study, then

$$y'(x) - y(x)^2 + 1 = 0, \quad y(0) = 0, \quad (7)$$

which has an exact solution

$$y(x) = -\tanh(x). \quad (8)$$

3.1 Solution by HCPM

By using HCPM, the following iterative equation is established

$$y_{i+1}(x) = y(0) + \int_0^x [(1 - p_i)a + p_i(y_i(t)^2 - 1)]dt, \quad (9)$$

where, for simplicity, we choose the trial operator as a constant $a = -0.44223818$ and $y_0(x) = 0.79x$.

Now, applying five iterations the result is the following

$$\begin{aligned} p_1 = 0, \quad y_1(x) &= -0.44223818x, \\ p_2 = \frac{1}{4}, \quad y_2(x) &= -0.58167862x + 0.016297882x^3, \\ p_3 = \frac{1}{2}, \quad y_3(x) &= -0.72111913x + 0.18972926E-4x^7 - 0.001896026x^5 + 0.05639167x^3, \\ p_4 = \frac{3}{4}, \quad y_4(x) &= -0.86055957x + 1.7998597E-11x^{15} - 4.1507495E-9x^{13} \\ &\quad + 3.9100531E-7x^{11} - 0.20100303E-4x^9 + 0.63370088E-3x^7 \\ &\quad - 0.12199535E-1x^5 + 0.13000320x^3, \\ p_5 = 1, \quad y_5(x) &= 1.0449984E-23x^{31} - 5.1522529E-21x^{29} \\ &\quad + 1.1594006E-18x^{27} - 1.5877939E-16x^{25} \\ &\quad + 1.4893881E-14x^{23} - 1.0199270E-12x^{21} \\ &\quad + 5.2923010E-11x^{19} - 2.1250309E-9x^{17} \\ &\quad + 6.6720888E-8x^{15} - 0.16431450E-5x^{13} \\ &\quad + 0.31653633E-4x^{11} - 0.47362575E-3x^9 \\ &\quad + 0.0054139546x^7 - 0.0447502x^5 + 0.24685427x^3 - x, \end{aligned} \quad (10)$$

where the last iteration $y_5(x)$ represents the desired approximate solution.

3.2 Solution by Picard's Method

By using Picard's Method, the following equation is established

$$y_{i+1}(x) = y(0) + \int_0^x [y_i(t)^2 - 1]dt, \quad (11)$$

where, as trial function, we select $y_0(x) = 0.79x$.

Now, applying five iterations the results are

$$\begin{aligned}
y_1(x) &= 0.20803335x^3 - x, \\
y_2(x) &= 0.0061825533x^7 - 0.083213338x^5 + 0.33333333x^3 - x, \\
y_3(x) &= 0.25482643E-5x^{15} - 0.79149369E-4x^{13} + 0.10041965E-2x^{11} \\
&\quad - 0.75378517E-2x^9 + 0.039648255x^7 - 0.13333333x^5 + 0.33333333x^3 - x, \\
y_4(x) &= 2.0947262E-13x^{31} - 1.3909898E-11x^{29} \\
&\quad + 4.2157552E-10x^{27} - 7.8951969E-9x^{25} + 1.0450920E-7x^{23} \\
&\quad - 0.10521321E-5x^{21} + 0.83817806E-5x^{19} - 0.54316109E-4x^{17} \\
&\quad + 0.29398940E-3x^{15} - 0.0013543458x^{13} + 0.0053896047x^{11} \\
&\quad - 0.18687267E-1x^9 + 0.053968254x^7 - 0.13333333x^5 + .33333333x^3 - x, \\
y_5(x) &= -x + .33333333x^3 + 0.053968254x^7 - 0.13333333x^5 \\
&\quad + 0.000946507x^{15} - 0.0028945324x^{13} + 0.0082846497x^{11} \\
&\quad - 0.021869489x^9 - 0.13837907E-5x^{25} + 0.58075863E-5x^{23} \\
&\quad - 0.22808990E-4x^{21} + 0.84038583E-4x^{19} - 0.00029089x^{17} \\
&\quad + 1.2158771E-8x^{31} - 1.8877460E-19x^{53} + 3.5927996E-18x^{51} \\
&\quad - 5.7005768E-17x^{49} + 7.7100189E-16x^{47} - \\
&\quad 9.0402088E-15x^{45} + 9.3125912E-14x^{43} - 8.5201075E-13x^{41} \\
&\quad + 6.9864431E-12x^{39} - 5.1747660E-11x^{37} + \\
&\quad 3.4855454E-10x^{35} - 2.1473275E-9)x^{33} + 6.9648854E-28x^{63} \\
&\quad - 9.5532547E-26x^{61} + 6.2729205E-24x^{59} - 2.6378596E-22x^{57} \\
&\quad + 8.0209511E-21x^{55} - 6.3538472E-8x^{29} + 3.0755545E-7x^{27}
\end{aligned} \tag{12}$$

where the last iteration $y_5(x)$ represents the required approximate solution.

3.3 Solution by HPM Method

We establish the following homotopy equation

$$(1 - p)(v'(x) - v'_0(x)) + p(v'(x) - v^2(x) + 1) = 0, \tag{13}$$

where p is the homotopy parameter and $v_0(x) = 0.79x$ is the trial function.

We suppose that solution for (13) has the form

$$v(x) = v_0(x) + pv_1(x) + p^2v_2(x) + \dots \tag{14}$$

Adjusting $p = 1$, we can obtain an approximate solution

$$y(x) = \lim_{p \rightarrow 1} v(x) = v_0(x) + v_1(x) + v_2(x) + \dots \tag{15}$$

Substituting (14) into (13) and equating terms having the same p -powers, it can be solved for $v_0(x)$, $v_1(x)$, $v_2(x)$, and so on (in order to fulfil initial conditions from $v(0) = y(0) = 0$, it follows that $v_0(0) = 0, v_1(0) = 0, v_2(0) = 0$ and so on). Also, to achieve an acceptable accuracy we apply 18th order HPM, giving as result

$$\begin{aligned}
 y(x) = & -x + 0.33333330x^3 - 0.000144722x^{31} - 0.13333331x^5 \\
 & + 0.05396826x^7 + 0.51596259E-5x^{33} + 8.0055189E-10x^{37} \\
 & + 0.0023761396x^{29} - 0.02186949x^9 + 0.00886324x^{11} \\
 & - 0.00359211x^{13} + 0.00145583x^{15} - 0.00059x^{17} - 0.08051462x^{19} \\
 & + 0.40168369x^{21} - 0.35812693x^{23} - 9.9476252E-8x^{35} + \\
 & 0.12725004x^{25} - 0.023073724x^{27}.
 \end{aligned} \tag{16}$$

4 Numerical Simulation and Discussion

Figure 1 and Table 1 shows the comparison between the exact solution (8) for the nonlinear differential equation (7) and the approximate solutions (10), (12), and (16). Taking advantage of the odd symmetry of exact solution and approximations, Table 1 presents the relative error just for the range $x \in [0, 4]$. The result is that the proposed solution (10) possess a relative error $< .097$ within the range of interest, while (12) and (16) achieve a maximum relative error -72547.658 and $1.22777E10$, respectively. Therefore, solution (10) calculated by HCPM has, in general, lower relative error with respect to the solutions obtained by Picard's and HPM.

The proposed approximation (10) has 16 terms, while solution (12) obtained by Picard's method has 32 terms, this is despite that both solutions were calculated after five iterations. Solution (16) has a number of terms similar to (10), 19 in total; nevertheless, HPM required 19 iterations to generate (16), while HCPM required just five iterations to obtain (10). Also, it should be noticed that performing 19 iterations on the HPM method required to establish a system of 19 simultaneous differential equations, while for HCPM and Picard's methods only required to solve five times, iteratively, one integral.

This work presented, by a simple case study, the HCPM method as a powerful tool and with high potential to solve nonlinear differential equations. Finally, in a future work the HCPM method will be tested using partial nonlinear differential equations and systems of nonlinear differential equations.

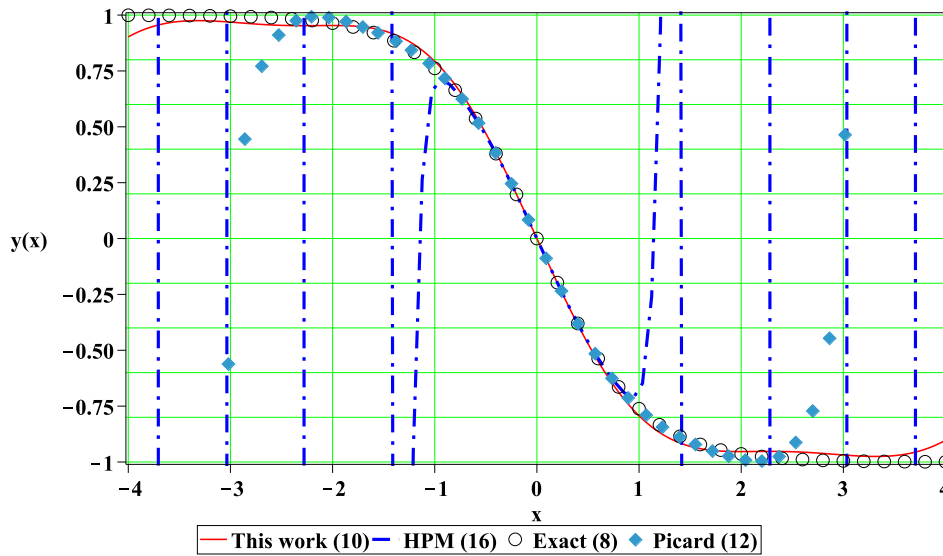


Figure 1: Exact solution (8) for (7) (circles) and its approximate solutions (10) (solid line), (12) (diamonds), and (16) (dash dot).

x	Exact (8)	R.E.		R.E.	
		HCPM (10)	Picard (12)	HPM (16)	
0.0*	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.50	-0.46211716	-0.018140673	-4.5701831E-07	-1.06120e-08	
1.00	-0.76159416	-0.041139194	-0.00025661665	0.090973361	
1.50	-0.90514825	-0.027399138	-0.0058014968	-29.587332	
2.00	-0.96402758	0.011041849	-0.023568177	-26192.739	
2.50	-0.98661430	0.033010551	0.059732685	1345023.3	
3.00	-0.99505475	0.026601002	1.333851	10111232	
3.50	-0.99817790	0.026742207	-245.03831	-8.23202e+08	
4.00	-0.99932930	0.09698956	-72547.658	1.22777e+10	

Table 1: Relative error (R.E.) comparison between exact solution (8) for (7) and approximation results (10), (12), and (16). *For $x = 0.0$ the error presented is the absolute error.

5 Conclusions

This work introduced the Homotopy-Continuation Picard's method (HCPM) as a novel tool with high potential to solve ordinary nonlinear differential equations. A simple case study showed that the solution calculated by HCPM had lower relative error than the solutions found by Picard's and HPM. Also, the HCPM solution has less terms than Picard's; requiring lower number of iterations than HPM method.

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