Applied Mathematics and Computation 219 (2013) 8163–8168

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Optimizing the positive Lyapunov exponent in multi-scroll chaotic oscillators with differential evolution algorithm

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ARTICLE INFO

Keywords:
Chaotic oscillator
Lyapunov exponent
Multi-scrolls
Differential evolution
Bifurcation diagram

ABSTRACT

We introduce the application of the differential evolution algorithm (DE) to optimize the positive Lyapunov exponent in a multi-scroll chaotic oscillator based on saturated nonlinear function series. The positive Lyapunov exponent is optimized from two to nine scrolls by sweeping the coefficients of the chaotic oscillator. In this article, the case of study has four coefficients, so that the feasible solutions for a, b, c, and d_1, are used to generate the bifurcation diagrams for the cases from two to nine scrolls taking c as the bifurcation parameter to demonstrate that high values of the positive Lyapunov exponent can be guaranteed when a, b, d_1 take values higher than 0.7, while c takes values lower than 0.3.

1. Introduction

During the last decade, chaotic oscillators have been investigated to generate multi-scroll attractors [1–5]. The majority of them have been realized with electronic devices [6], and some have been used to design secure communication systems [7–9]. Besides, we appeal that those designs can be enhanced if the chaotic oscillator possesses higher positive Lyapunov exponents, because it determines the unpredictability grade of the chaotic oscillator.

Computing a higher value of the positive Lyapunov exponent requieres varying the coefficients of the chaotic oscillator, leading to a huge number of combinations. For example: the chaotic oscillator described by (1) has four coefficients a, b, c, d_1, so that the search space is calculated from the size of the variables [10], i.e. each coefficient has one significant digit that can be 0 or 1 and if four decimal places are considered, they can have values in $\{0; 9\}$ leading to $2 \times 10 \times 10 \times 10 \times 10 = 2 \times 10^4$ combinations. For the whole problem having four coefficients, the search space will be $(2 \times 10^4)^4 = 16 \times 10^{16}$. This huge search space justifies the use of evolutionary algorithms. That way, we show the application of the differential evolution (DE) algorithm to optimize the positive Lyapunov exponent of the multi-scroll chaotic oscillator described in the following section. In the last section, we show the bifurcation diagrams for the optimized positive Lyapunov exponents from two to nine scrolls. Finally, from those diagrams it is highlighted that higher values of the positive Lyapunov exponent can be guaranteed when a, b, d_1 take values higher than 0.7, while c takes values lower than 0.3.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -ax - by - cz + d_1 f(x; k, h, p, q). \end{aligned} \tag{1}$$

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.01.072

2. Multi-scroll chaotic oscillator

Among different kinds of multi-scroll chaotic oscillators [1,11], this investigation is focused on the one described by (1) that is based on a piecewise-linear (PWL) function approached by (2) [1–3]. It is a series of a saturated function, where: $k \geqslant 2$ is the slope of the saturated function and a multiplier factor to saturated plateaus; $plateau = \pm nk$, with n = odd integer to generate even number of scrolls and n = even integer to generate odd-scrolls; h = saturated delay of the center of the slopes, which must agree with $h_i = \pm mk$, where $i = 1, \ldots, [(scrolls - 2)/2]$ and $m = 2, 4, \ldots, (scrolls - 2)$ to generate even-scrolls, and $i = 1, \ldots, [(scrolls - 1)/2]$ and $m = 1, 3, \ldots, (scrolls - 2)$ to odd-scrolls; p and q are positive integers. Eq. (3) defines $f(x1; k, h, p, q)$.

$$f(x; k, h, p, q) = \sum_{i=-p}^{q} f_i(x; h, k), \tag{2}$$

$$f(x; k, h, p, q) = \begin{cases} (2q+1)k, & x > qh + 1, \\ k(x - ih) + 2ik, & |x - ih| \leqslant 1, \ -p \leqslant i \leqslant q, \\ (2i+1)k, & ih + 1 < x < (i+1)h - 1, \ -p \leqslant i \leqslant q - 1, \\ -(2p+1)k, & x < -ph - 1. \end{cases} \tag{3}$$

3. Lyapunov exponent

Lyapunov exponents are asymptotic measures characterizing the average rate of growth (or shrinking) of small perturbations to the solutions of a dynamical system, e.g. chaotic oscillators. They provide quantitative measures of response sensitivity of a dynamical system to small changes in initial conditions [12]. Therefore, the presence of positive Lyapunov exponents is taken as a signature of chaotic motion [12,13].

In continuous-time chaotic oscillators, the number of Lyapunov exponents equals the number of state variables. Further, if at least one is positive, it is an indication of chaos. Lets us consider an n-dimensional dynamical system:

$$\dot{x} = f(x), \quad t > 0, \quad x(0) = x_0 \in \mathbf{R}^n, \tag{4}$$

where x and f are n-dimensional vector fields. To determine the n-Lyapunov exponents, one should find the long term evolution of small perturbations to a trajectory, which are determined by the variational equation of (4),

$$\dot{y} = \frac{\partial f}{\partial x}(x(t))y = J(x(t))y, \tag{5}$$

where J is the $n \times n$ Jacobian matrix of f. A solution of (1) with a given initial perturbation $y(0)$ can be written as

$$y(t) = Y(t)y(0), \tag{6}$$

with $Y(t)$ as the fundamental solution satisfying

$$\dot{Y} = J(x(t))Y, \quad Y(0) = I_n. \tag{7}$$

In (7), I_n is the $n \times n$ identity matrix. By considering the evolution of an infinitesimal n-parallelepiped $[p_1(t), \ldots, p_n(t)]$ with the axis $p_i(t) = Y(t)p_i(0)$ for $i = 1, \ldots, n$, where $p_i(0)$ denotes an orthogonal basis of R^n, then the i_{th} Lyapunov exponent, which measures the long-time sensitivity of the flow $x(t)$ with respect to the initial data $x(0)$ at the direction $p_i(t)$, is defined by the expansion rate of the length of the ith axis $p_i(t)$ and is given by

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \|p_i(t)\|. \tag{8}$$

Table 1
Positive Lyapunov exponent (LE) and coefficients values applying DE.

Scrolls	LE with $(a, b, c, d_1 = 0.7)$	LE and (a, b, c, d_1) coef. values for DE
2	0.1257	0.3743 (0.279, 0.852, 0.072, 0.492)
3	0.1565	0.4088 (0.839,0.954,0.192,0.914)
4	0.1763	0.4338 (1,1,0.143,0.991)
5	0.1773	0.4416 (1,0.991,0.105,1)
6	0.1795	0.4418 (0.991,0.912,0.105,0.971)
7	0.1763	0.4455 (0.987,0.732,0.111,0.994)
8	0.1972	0.4503 (1,0.755,0.109,1)
9	0.1950	0.4469 (1,0.844,0.074,1)

The Lyapunov exponents can be computed by applying the methods given in [10–14], where a variational system is used to measure the changes in the original dynamical system with respect to the different directions. The method can be summarized as follows:

1. Initial conditions and the variational system are set to $\mathbf{X_0}$ and $\mathbf{I_{nxn}}$, respectively.
2. The systems are integrated until an orthonormalization period (TO), is reached. The integration of the variational system $\mathbf{Y} = [\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}]$ depends on the specific Jacobian that the original system $\mathbf{X}$ is using in the current time-step.
3. The variational system is orthonormalized by using the standard Gram–Schmidt method [15], the logarithm of the norm of each Lyapunov vector contained in $\mathbf{Y}$ is obtained and accumulated in time.
4. The next integration is carried out by using the new orthonormalized vectors as initial conditions. This process is repeated until the full integration period T is reached.
5. The Lyapunov exponents are obtained by evaluating:

$$\lambda_{\mathbf{i}} \approx \frac{\mathbf{1}}{\mathbf{T}} \sum_{\mathbf{j}=TO}^{\mathbf{T}} \ln \|y_i\|,$$

where TO is greater than the time-step used to solve (1).

4. Differential evolution algorithm

DE is an evolutionary algorithm working with a population of tentative solutions to the problem. New solutions are generated by combining the old ones and by surviving the ones with better fitness.

This section describes DE using the version rand/1/bin. It has the ability of a child competing one to one with his father, resulting in a faster rate of convergence. The new vectors of parameters are generated by adding the weighted difference between two vectors of the population to a third vector. If the resultant vector allocates an objective function value lower than a member of the population, the newly generated vector replaces the vector with which it was compared, otherwise the previous vector survives. The pseudocode is shown in Algorithm 1, where the population is represented by $P_O = [\mathbf{x_1}, \ldots, \mathbf{x_n}]$ and each individual is represented by a vector $\mathbf{x} = [x_1, \ldots, x_D]$ $x \in \mathbb{R}^D$.

Algorithm 1 Differential evolution algorithm

```
Population size=N_P
Generations = G
Procedure DE (N_P, G)
for i = 1 : N_P do
  for d = 1 : D do
    x_i[d] = LimitInf + (LimitSup − LimitInf) · rand()      ▷ initialize the population
  end for
  x_i.fit = evaluate(x_i)      ▷ evaluate population
end for
minfit = min(x_i.fit)      ▷ best individual fitness
for i = 1 : G do
  Let j_1, j_2 and j_3 be three random numbers in {1, N_P}
  without replacement and also different to i
  jrand ← [rand() · D] + 1
  for d = 1 : D do
    if rand() < R   OR   d = jrand then
      y[d] = x_i2[d] + F(x_i0[d] − x_i1[d])
      if y[d] < LimitInf   OR   y[d] > LimitSup then
        y[d] = LimitInf + (LimitSup − LimitInf) · rand()
      end if
    else
      y[d] = x_i[d]
    end if
  end for
  y.fit = evaluate(y)      ▷ evaluate new individual
  if y.fit < minfit then
    minfit = y.fit      ▷ best fitness
    x_i = y
  end if
end for
```

5. Optimizing the positive Lyapunov exponent

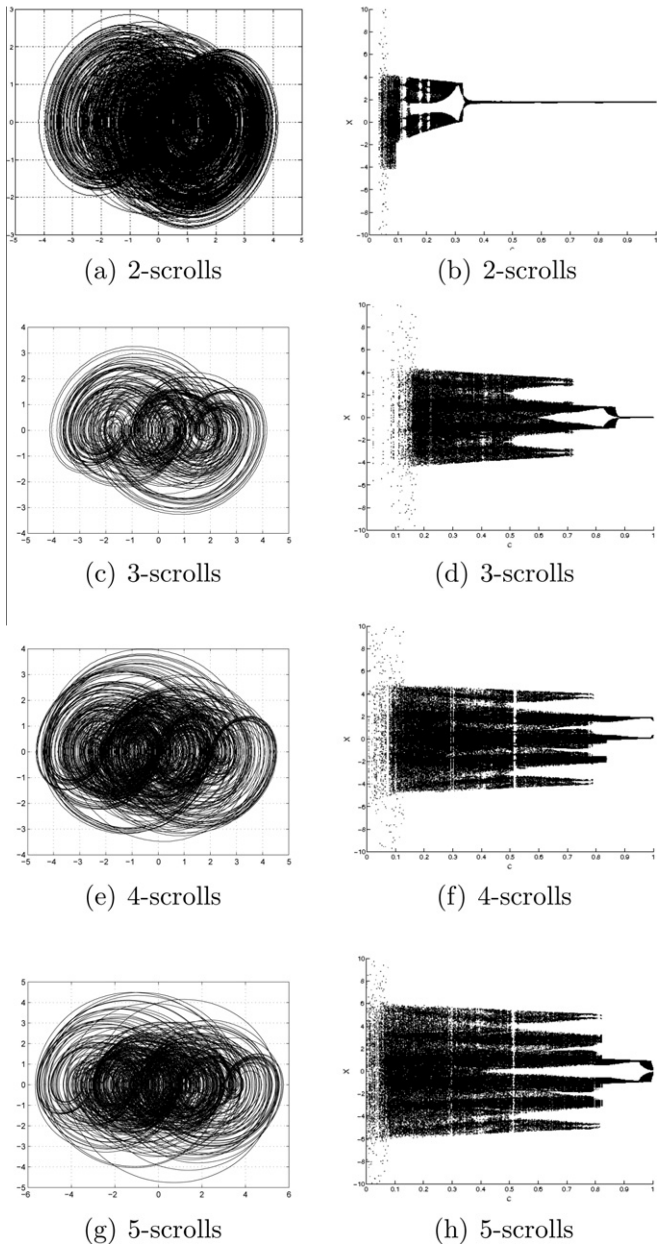
A global optimization problem can be formulated as

$$f : \mathbb{R}^D \to \mathbb{R},$$

$$f(x),\ s.t.\ x_j \in [l_j, u_j], \quad j = 1, \ldots, D,$$

where f is the objective function, and x is a continuous variable vector of D dimensions. The feasible domain of variable x is defined by specifying upper (u_j) and lower (l_j) limits of each component j.

The computation of the Lyapunov exponents for the chaotic oscillator described by (1), is performed by setting: $a = b = c = d_1 = 0.7, k = 10, h = 2$; and by varying the four coefficients using three decimals within the DE given in Algorithm 1. p and q are adjusted to generate from two to nine scrolls. The results are summarized in Table 1, where one can



(a) 2-scrolls (b) 2-scrolls

(c) 3-scrolls (d) 3-scrolls

(e) 4-scrolls (f) 4-scrolls

(g) 5-scrolls (h) 5-scrolls

Fig. 1. Bifurcation diagrams from Table 1.

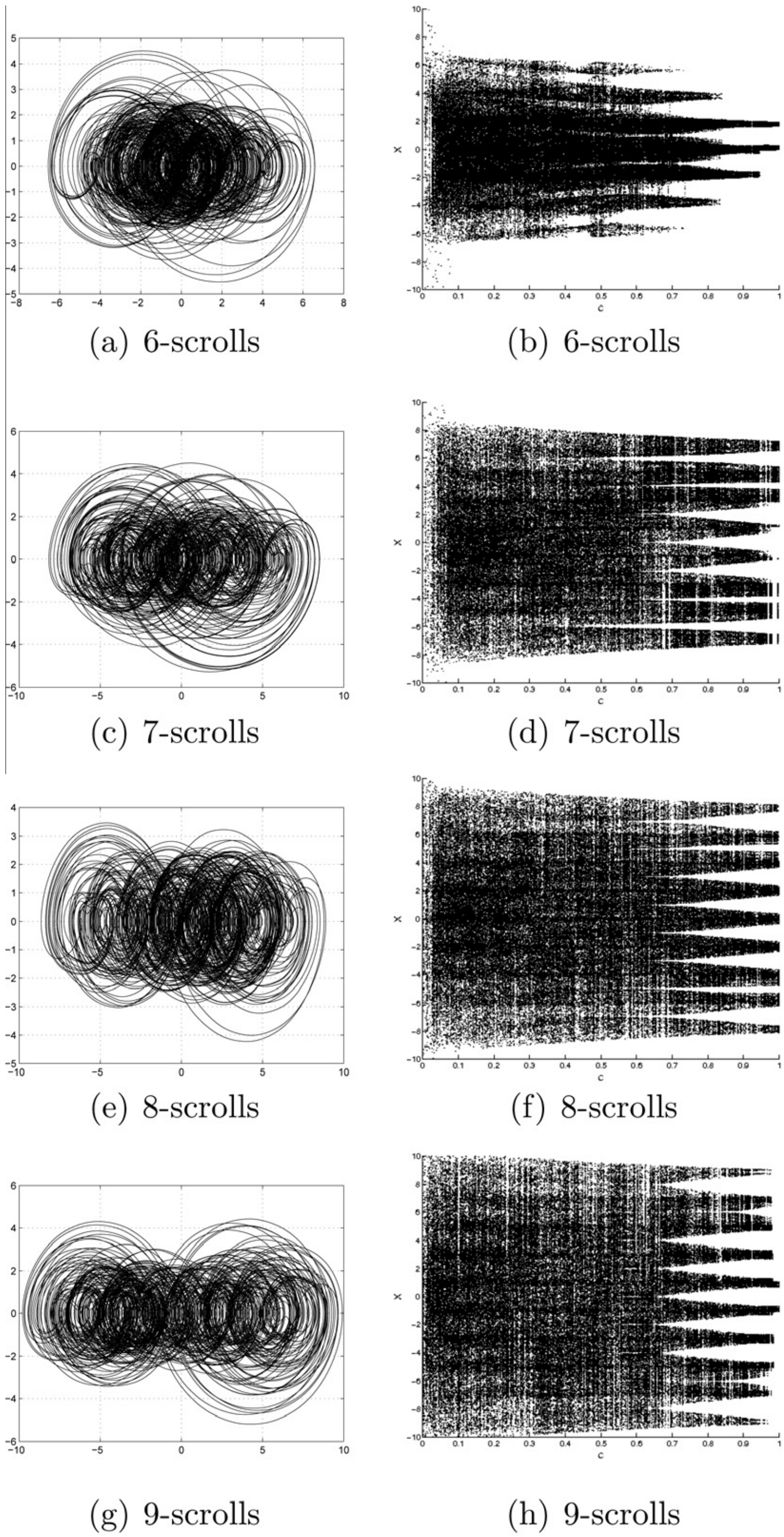


Fig. 2. Bifurcation diagrams from Table 1 (continuation).

see that the positive Lyapunov exponent with all coefficients = 0.7 is relatively small compared with the optimized one applying DE.

DE was implemented using MATLAB and the ODE45 integration package. It was executed with an initial population of 40 individuals and 80 generations. The time-step was selected by using the minimum absolute value among the eigenvalues of the system λ_{min} [16], ψ was chosen well above the sample theorem as 50, and TO was chosen as $50t_{step}$, where $t_{step} = \frac{1}{\lambda_{min}\psi}$.

The integration was carried with a full period T of 4,000. Initial conditions $\mathbf{X_0}$ were chosen in the basin of attraction of the chaotic oscillator from a previous simulation with T = 800 and initial conditions IC=[0.1, 0, 0].

Figs. 1 and 2 show the phase and bifurcation diagrams for the cases listed in Table 1. The bifurcation diagrams were generated by setting a, b, d_1 from (1) according to the values from Table 1 when applying DE, and c was varied being the bifurcation parameter.

The bifurcation diagrams are shown as a projection of the orbit onto the $X - Y$ plane, where the orbits intersect the projection onto the plane of the line passing through the equilibrium points. An algorithm calculates the intersection of two lines projected onto the $X - Y$ plane: the line between the equilibrium points and the line between the two consecutive points in the orbit that crossed the equilibrium-point line. It calculates the intersection of the plane with the line between the two orbit points (on either sides of the plane).

6. Conclusion

We have shown the usefulness of DE, version rand/1/bin, to optimize the positive Lyapunov exponent of a multi-scroll chaotic oscillator based on saturated function series. It has been implemented using MATLAB. From the optimized results, computed from two to nine scrolls, we generated their respective bifurcation diagrams, where parameter c was chosen as the bifurcation parameter because it resulted to be the most sensitive coefficient.

The optimized Lyapunov exponents computed by DE and listed in Table 1, and their respective bifurcation diagrams help to provide the following insights: selecting small values for c, e.g. 0.1–0.3; and large for a, b, d_1, e.g. 0.7–0.9, allows guaranteeing higher positive Lyapunov exponent values.

Acknowledgement

This work has been partially supported by CONACyT/MEXICO under project 131839-Y.

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