Analog Realizations for Decomposition and Reconstruction of Real Time Signals using the Wavelet Transform

by

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Short Abstract

In this work, the analog implementations of the wavelet transform are explored. The decomposition of the input signal and the reconstruction of this signal from its wavelet representation are both considered. Attention is paid in the develop of a theoretical basis for the wavelet transform adequate for analog circuits implementations.

Between the most important results is shown that any bandpass rational filter develops the wavelet transform of the input signal, where the prototype wavelet is the inverse in time of the impulse response of the filter; an expression for the numerical estimation of the reconstruction error is shown; two methods to obtain decomposition-reconstruction wavelet systems with a reconstruction error as low as be required are explained.

The decomposition and reconstruction developed with the wavelet systems here described can be achieved with cero-delay. The classical wavelet transform systems do not posses this important property.

As an application example, the hard thresholding filtering technique has been adapted to be applied with the wavelet systems here developed.

In order to show the applicability of the obtained theoretical results, has been designed, manufactured, and measured a wavelet transform system in an analog integrated circuit with a CMOS technology of 0.5 μm. In this system has been also included the hard thresholding technique technique.
Breve Resumen

En este trabajo se exploran las implementaciones analógicas de la transformada wavelet. Se estudian tanto la descomposición de la señal entrante como la reconstrucción de ésta señal a partir de su representación en el dominio wavelet. Se hace énfasis en el desarrollo de una base teórica para la transformada wavelet que sea apropiada para su aplicación en circuitos analógicos.

Entre los resultados más importantes se demuestra que todo filtro racional pasabanda realiza la transformada wavelet de la señal entrante, donde la wavelet prototipo es el inverso en el tiempo de la respuesta en frecuencia del filtro; se muestra una fórmula para la estimación numérica del error de reconstrucción; se explican dos métodos para obtener sistemas de descomposición-reconstrucción basados en la transformada wavelet, con un error de reconstrucción tan bajo como sea requerido.

La descomposición y reconstrucción realizada con los sistemas wavelet aquí descritos se puede efectuar sin ningún retardo. Propiedad que no poseen los sistemas clásicos de transformada wavelet.

Como ejemplo de aplicación, se adaptó la técnica de filtrado hard thresholding para ser aplicada con los sistemas wavelet aquí desarrollados.

Como vehículo de prueba de los resultados teóricamente obtenidos, se diseñó, fabricó y midió un sistema de transformada wavelet en un circuito integrado analógico con tecnología CMOS de 0.5 µm. A este sistema también se le incluyó la función de filtrado.
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Dedication

To God and my precious family

Alaciel and Juan Andrés

abuelita María and Mamá Rorra

Dora, Nicole, César, Paola and Alejandra

Aurora and Rafael

Alicia and Graciela
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Preface

In this work, a new approach to implement the wavelet transform in analog circuits, using only a set of continuous filters working in parallel and a summing amplifier, is presented. With just these components the decomposition of a signal on time, its representation in the wavelet domain, and its reconstruction can be performed. The achieved reconstruction is not exact, but an error below any pre-specified bound can be obtained.

In this research, a new set of decomposition and reconstruction wavelets, implemented by continuous-time filters, has been identified. The implementation of the decomposition wavelets is inherently exact. The zero-delay processing capacity is a remarkable property reached with the wavelets here described. This property is not usually shared neither by the classical continuous nor by the discrete wavelets.

As an example of analog wavelet processing, a modification of the well-known thresholding denoising method is presented. To implement this denoising technique only some extra comparators and amplifiers are required.

By the application of the theoretical principles here developed, a decomposition-reconstruction wavelet system, with denoising-capability, has been designed and manufactured in an analog integrated circuit. The corresponding schematics, simulations, layouts, an measurements are explained in detail.

The eight chapters of this thesis deal with three different issues: theoretical principles, design of the integrated circuit, and chip measurements.

The theoretical bases of this work are presented in the first three chapters. In
chapter 1, some basic results of the wavelet theory oriented to the numerical implementation of the wavelet transform are presented. A review of some intents to adapt these principles to analog circuits is made, focussing on the drawbacks and the advantages of each different approach. Then, the main idea of this work is derived: looking for rational functions in the $s$-domain performing the wavelet transform as a convolution, where the impulse response of the filter is the prototype wavelet.

In chapter 2 is shown that any band-pass rational function in the $s$-domain performs the wavelet transform. Then, two different methods to find reconstruction filters are presented. To measure the reconstruction performance of the different wavelet systems, some metrics of the reconstruction error are proposed. The delay in the wavelet signal processing is here reviewed.

In chapter 3, a denoising method similar to the thresholding technique is proposed. The effect of the thresholding operation, and of the reconstruction filters, in the smoothness of the signal is analyzed. The threshold levels in each band are found for the case of the white noise. Finally, the denoising performance of the wavelet systems here proposed, is compared against the performance of other classical denoising techniques.

Chapters 4 to 7 cover the integrated circuit design of the implemented wavelet system. Chapter 4 covers the decomposition-reconstruction part of the design. The design procedure begins with the mathematical definition of the functions of the system. Next, the selection of the technology is made, and the typical and worst case models, the Monte Carlo parameters, and the simulations of the intrinsic noise, are considered. Then the details of the design are explained, including schematics, simulations and design criteria. In a similar way, in chapter 5 is presented the design of the denoising function. While in chapter 6 is covered the design of the layout.

Finally, in chapter 7 the measures of the chip are presented, and the conclusions of the work are presented in chapter 8.
Chapter 1

Introduction

The wavelet transform is specially helpful in the processing of noisy, intermittent, or transitory signals [1], because it allow a simultaneous time-frequency representation of the analyzed signal.

Several processing techniques have been developed to be applied in the wavelet domain. For instance: *wavelet pattern recognition*, applied in medical diagnosis, earth sciences, remote detection, machinery maintenance, etc. [1][2]; *wavelet denoising*, applied for noise reduction in one-dimensional or multi-dimensional signals [3][4]; *wavelet compression*, extensively used in digital images and video.

In the most publications about implementations and applications of the wavelet transform, the numeric calculation of the wavelet transform by software [5] or digital hardware [6] is supposed. There are lots of publications [7][8][9][10][11][12][13] where the theoretical bases oriented to the numerical or algorithmic implementation of the wavelet transform are established.

Despite the high development of its numerical implementations and due to technical or fundamental issues, the wavelet transform can not be applied in some situations of high-frequencies, low-consumption, or real-time processing. In these cases an analog implementation of the semidiscrete wavelet transform could be an good alternative.
But looking for analog implementations of the wavelet transform we could find only a few articles [14][15][16][17][18][19][20][21][22][23][24][25][26], and only one implementation intended for pattern recognition in electrocardiographic signals. Reviewing these works, in some of them we could find a lack of a strong theoretical basis oriented to implement the wavelet transform in analog systems [24][25][26].

Therefore, our first task in this chapter is a review of the wavelet transform theory. After that we make a review of the published works dealing with analog implementations of the wavelet transform, focusing on the advantages and disadvantages to implement the wavelet transform in analog circuits whit those approaches.

Now we present some of the notation used in this work:

- \( \mathbb{N} = \{0, 1, 2, \ldots\} \), \( \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\} \), and \( \mathbb{R} \) are the sets of the natural, integer and real numbers.

- The complex conjugated of \( f(x) \) is denoted by \( f^*(x) \).

- We say that \( f(x) \) is \( L^\infty \) on \( \mathbb{R} \), or is bounded, if \( |f(x)| \leq M \) for some \( M > 0 \).

- We say that \( f(x) \) is \( L^1 \) on \( \mathbb{R} \), or is integrable, if \( \int_{-\infty}^{\infty} |f(x)| \, dx < \infty \).

- We say that \( f(x) \) is \( L^2 \) on \( \mathbb{R} \), or is square-integrable, if \( \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \).

- The norm of \( f(x) \) is defined by \( \|f\| = \int_{-\infty}^{\infty} |f(x)|^2 \, dx \).

- The Fourier transform of \( f(t) \) is denoted with the respective uppercase letter or by the operator \( \mathcal{F}\{\cdot\} \). That is, \( \mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt \).

- An exception to the previous notation is \( H(s) = P(s)/Q(s) \), representing a rational function of the complex variable \( s \), where the impulse response of \( H(s) \) is denoted by \( h(t) \). Hence, the Fourier transform of \( h(t) \) is \( H(j\omega) \) instead of \( H(\omega) \).

- The convolution between the functions \( f(t) \) and \( g(t) \) is defined by \( f \ast g(t) = \int_{-\infty}^{\infty} f(x)g(t-x) \, dx \).
1.1 Principles of wavelet transform

The wavelet transform of a function $f(t)$ is the function $w(a,b)$, where the scale $a$ and the position or time translation $b$ can be continuous or discrete variables. In the continuous wavelet transform both variables are continuous. In the semidiscrete wavelet transform $a$ is continuous and $b$ is discrete; and both variables are discrete for the case of the discrete wavelet transform.

1.1.1 Continuous wavelet transform

The next definition of the continuous wavelet transform applies when we consider only positive values of the scale $a$. That is, when only positive frequencies are of interest.

From a function $\psi(t)$, selected as mother or prototype wavelet, we obtain the family of scaled and translated wavelets

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right), \quad (1.1)$$

for $a,b \in \mathbb{R}$ and $a > 0$.

The function $\psi(t)$ defined on $\mathbb{R}$ can be used as wavelet for the continuous wavelet transform only if it satisfies the so-called admissibility condition \[7\][8]

$$C_\psi = 2 \int_0^\infty \frac{|\Psi(\omega)|^2}{\omega} d\omega = 2 \int_0^\infty \frac{|\Psi(-\omega)|^2}{\omega} d\omega < \infty. \quad (1.2)$$

When $\psi(t)$ is $L^1$ on $\mathbb{R}$, then (1.2) implies that $\psi(t)$ must have zero mean \[7\].

The continuous wavelet transform of the signal $f(t)$ respect to the mother wavelet $\psi(t)$ is defined by

$$w(a,b) = \int_{-\infty}^{\infty} f(t) \psi^*_a,b(t) dt. \quad (1.3)$$

The value of the wavelet transform at the point $(a_o, b_o)$ give us a measure of the similitude between the wavelet $\psi_{a_o,b_o}(t)$ and the signal $f(t)$ at the proximities of $t = b_o$. 
The original signal $f(t)$ can be reconstructed from the values $w(a,b)$ applying the inverse continuous wavelet transform \[7\][8]

$$f(t) = \frac{2}{C_\psi} \int_0^\infty \int_{-\infty}^\infty w(a,b) \psi_{a,b}(t) \frac{db da}{a^2},$$  

which allows a perfect reconstruction.

### 1.1.2 Semidiscrete wavelet transform

In the semidiscrete wavelet transform \[8\][11] the position $b$ is a continuous variable, while the scale is a discrete variable restricted to the exponentially spaced values

$$a = r^m,$$  

(1.5)

for $m \in \mathbb{Z}$, where $r$ is given by

$$r = \sqrt[5]{2},$$  

(1.6)

for $D > 0$, and then $r > 1$.

We call the parameter $r$ the scale ratio because it give us the ratio between the size of wavelets in adjacent scales, and we call the parameter $D$ scale density since it give us the number of scales per octave of frequency.

From the selected mother wavelet $\psi(t)$, we obtain the next semidiscrete wavelet family

$$\psi_m(t) = \frac{1}{r^m} \psi \left( \frac{t}{r^m} \right).$$  

(1.7)

The semidiscrete wavelet transform of the function $f(t)$ respect to the mother wavelet $\psi(t)$ is defined by the next family of continuous functions

$$w_m(b) = w(r^m, b) = \int_{-\infty}^{\infty} f(t) \psi_m^*(t - b) \, dt,$$  

(1.8)

where $w(r^m, b)$, with $r^m = a$, has the same meaning as in the continuous case (1.3). We call $w_m(t)$ the $m$-th wavelet component.

The inverse semidiscrete wavelet transform is given by

$$f(t) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} w_m(b) \chi_m(t - b) \, db,$$  

(1.9)
where $\chi(t)$ is dual of the semidiscrete wavelet $\psi(t)$, as is explained below.

For a given $r > 1$, the function $\psi(t)$ can be used as wavelet for the semidiscrete wavelet transform if it satisfies condition (1.2), as well as the stability condition of the semidiscrete wavelet transform that requires that for $\omega > 0$

$$A \leq \sum_{m=-\infty}^{\infty} |\Psi(r^m \omega)|^2 \leq B,$$  \hspace{1cm} (1.10)

for some $A, B > 0$.

Additionally, if $\chi(t)$ satisfies

$$\sum_{m=-\infty}^{\infty} \Psi^*(r^m \omega) X(r^m \omega) = 1,$$  \hspace{1cm} (1.11)

we say that $\chi(t)$ is a dual wavelet of $\psi(t)$, and a perfect reconstruction can be achieved.

1.1.3 Discrete wavelet transform

Redundant discrete wavelet transform

In the discrete wavelet transform [7][8][9], scale and translation are restricted to the discrete values $a = r^m$ and $b = n b_0 r^m$, where $m, n \in \mathbb{Z}$, for some given parameters $b_0 > 0$ and $r > 1$. When these values are replaced in (1.1), the next wavelet family is obtained

$$\psi_{[m,n]}(t) = \psi_{r^m,n b_0 r^m}(t) = \frac{1}{\sqrt{r^m}} \psi \left( \frac{t - n b_0 r^m}{r^m} \right).$$  \hspace{1cm} (1.12)

We include squared brackets in $\psi_{[m,n]}$ to avoid confusion with the notation $\psi_{a,b}$ used in (1.1).

The discrete wavelet transform of a signal $f(t)$ respect to the wavelet $\psi(t)$ is defined by

$$w_{m,n} = w(r^m, n b_0 r^m) = \int_{-\infty}^{\infty} f(t) \psi^*_{[m,n]}(t) \, dt,$$  \hspace{1cm} (1.13)

where $w(r^m, n b_0 r^m)$, with $r^m = a$ and $n b_0 r^m = b$, has the same meaning as in the continuous case (1.3).
From the wavelet coefficients, \( w_{m,n} \), the original signal can be reconstructed applying the inverse discrete wavelet transform

\[
f(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} w_{m,n} \chi_{[m,n]}(t),
\]

where \( \chi(t) \) is a dual discrete wavelet of \( \psi(t) \).

For some given parameters \( b_0 > 0 \) and \( r > 1 \), a function \( \psi(t) \) can be used as prototype wavelet for the discrete wavelet transform if it satisfies condition (1.2), as well as the stability condition of the discrete wavelet transform that requires that for any function \( f(t) \), \( L^2 \) on \( \mathbb{R} \), is verified that

\[
A \|f\|^2 \leq \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |w_{m,n}|^2 \leq B \|f\|^2, \tag{1.15}
\]

for some \( A, B > 0 \).

**Non-redundant discrete wavelet transform**

When \( a = 2^m \) and \( b = n 2^m \) we say that the wavelet family \( \psi_{[m,n]} \) is arranged over a dyadic frame. For this special case, orthogonal and biorthogonal wavelet bases have been developed [7][8][9][10][1], implying that for any square-integrable function the coefficients \( w_{m,n} \), given by (1.13), do not contain redundant information. Usually, this wavelet transform is implemented with the next recursive algorithm known as the fast wavelet transform [1]

\[
S_{m+1,n} = \frac{1}{\sqrt{2}} \sum_k \alpha_{k-2n} S_{m,k},
\]

\[
w_{m+1,n} = \frac{1}{\sqrt{2}} \sum_k \beta_{k-2n} S_{m,k}, \tag{1.17}
\]

where \( \alpha \) and \( \beta \) are the coefficients of low-pass and band-pass FIR filters, which are related with the so-called scale function \( \phi(t) \) and the wavelet \( \psi(t) \).

For the application of this algorithm is required the sequence \( S_{m_0,n} \), called approximation coefficients of \( f(t) \) at the lower used scale \( m_0 \), given by

\[
S_{m_0,n} = \int_{-\infty}^{\infty} f(t) \phi_{m_0,n}(t) \, dt. \tag{1.18}
\]
The sequence $S_{m_0,n}$ can be recovered from the wavelet coefficients $w_{m,n}$ applying the inverse recursive algorithm

$$S_{m-1,n} = \frac{1}{\sqrt{2}} \sum_k \alpha_{n-2k} S_{m,k} + \frac{1}{\sqrt{2}} \sum_k \beta_{n-2k} w_{m,k},$$

(1.19)

where the coefficients $\alpha$ and $\beta$ are the same involved in the decomposition relations (1.16) if the wavelet basis is orthogonal, as the Daubechies wavelets, or are different if the basis is biorthogonal.

In addition to satisfy conditions (1.2) and (1.15), as any discrete wavelet, to be orthogonal $\psi(t)$ has to satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} \psi_{[j,k]}(t) \psi_{[l,m]}^*(t) dt = \begin{cases} 1 & \text{if } j = l \text{ and } k = m \\ 0 & \text{for other cases} \end{cases}.$$  

(1.20)

1.2 Analog implementations of the wavelet transform

In this section we make a review of the published works about analog implementations of the semidiscrete, discrete and fast wavelet transform, focusing on the advantages and disadvantages that the different approaches have for the implementation of the wavelet transform in analog systems.

We can tell in advance that for this finality, the semidiscrete wavelet transform has great advantages over the other types of wavelet transform, because in it the processing can be developed by continuous time signals over multiple frequency bands, which is the natural way of work of the analog circuits.

1.2.1 Analog implementation of the continuous wavelet transform

We have not yet found reported analog implementations of the continuous wavelet transform, but the numerical implementation of this transform is generally made
Figure 1.1: (a) Wavelet component $w_m(t)$ obtained by convolution. (b) Direct and inverse semidiscrete wavelet transform implemented with continuous filters.

approximating the continuous scales by means of very close discrete scales. This approach is similar to the implementation of the semidiscrete wavelet transform explained below, with the only difference being the way in which the prototype wavelet is scaled: linearly in the continuous case (1.1), and logarithmically in the semidiscrete case (1.7). Therefore some of the results obtained in this work for the semidiscrete wavelet transform are also applicable to the continuous case.

1.2.2 Analog implementation of the semidiscrete wavelet transform

From (1.8) we note that each wavelet component, $w_m(t)$, is the convolution between $f(t)$ and the function $\psi_m^*(-t)$. Then $w_m(t)$ can be obtained putting the signal $f(t)$ through a linear time-invariant system with impulse response $h(t) = \psi_m^*(t)$, as is shown in Figure 1.1(a).

This approach is employed in [20][21][22][23][24][25][26], but in all of them only
the decomposition is considered. However, as can be noted in (1.9), the convolution principle also can be used during the reconstruction, because at each scale the convolution between the wavelet component \( w_m(t) \) and the function \( \chi_m(t) \) must be performed. Therefore, the direct and the inverse semidiscrete wavelet transform can be implemented in a system like the one shown in Figure 1.1(b).

In some of the reviewed works have been developed methods to synthesize continuous filters whose impulse response is an approximation of some wavelet used for the continuous wavelet transform: Morlet [25], first derivative of Gaussian [26], first derivative of complex Gaussian [24]. In other work is made an approximation of a wavelet used for the discrete wavelet transform: Daubechies-5 [25]. One paper proposes an universal method to approximate wavelets using adaptive Laguerre filters [21]. Instead, others have created their own ideally exact wavelets, either by means of a linear summation of quadratic filters [20], or by the design of a new wavelet in the Fourier domain using Laguerre structures [23]. But, as previously has been commented, these works deal only with the wavelet decomposition of a signal.

It is convenient to emphasize that the recovery of the time signal from its wavelet representation requires to find and implement in analog circuits both, the decomposition \( \psi(t) \) and the reconstruction \( \chi(t) \) wavelets.

1.2.3 Analog implementation of the discrete wavelet transform

The main operations involved in the discrete wavelet transform (1.13) and its inverse (1.14) are wave generators, multipliers, and an integrator. In works [16][17][18] these operations are explicitly implemented in a block similar to the one shown in Figure 1.2(a). This basic block has a decomposition and a reconstruction part, and the control signals \( D1, D2, D3 \) and \( D4 \) are needed to synchronize the events.

In this approach, each wavelet coefficient is available at the output of the integrator at the end of the effective support of the wavelet. After that, a new cycle can begin.
Figure 1.2: Implementation of the Discrete wavelet transform. (a) Basic block. (b) The four basic Blocks needed at each scale when the wavelet has four coefficients in its effective support. Here \( f_{\text{out}}(t) = \sum_m f_m(t) \). \( D1, D2, D3 \) and \( D4 \) are control signals.

Hence, for each scale will be needed as many basic blocks as coefficients has the wavelet in its effective support. This fact is schematically illustrated in Figure 1.2(b), where are shown the four basic blocks needed in the scale \( m = 1 \) for a wavelet with four coefficients in its effective support.

In work [16] a similar approach is used, proposing a wavelet generated by modulating a complex sinusoid by means of another real sinusoid of lower frequency. But the reconstruction of the signal is not dealed with.

In works [17] and [18] the main idea is to produce a complex Morlet wavelet modulating a complex sinusoidal by a Gaussian wave. In work [17] only decomposition is implemented, while in [18] this complex wavelet is used for both decomposition and reconstruction, but this is a wrong approach because the complex Morlet wavelet is not a dual discrete wavelet of itself.

Comparing the amount of needed components, it must be clear that the semidiscrete implementation, Figure 1.1(b), is simpler than the discrete one, Figure 1.2(b),
because in that case just two continuous filters are needed at each scale.

1.2.4 Analog implementation of the fast wavelet transform

In Figure 1.3(a) is shown a filter bank accomplishing the fast wavelet transform algorithm and its inverse depicted in (1.16), (1.17) and (1.19). The involved operations are decimation (↓ 2), interpolation (↑ 2), addition (+), and the low-pass $A$ and high-pass $B$ FIR filters, composed with the coefficients $\alpha_k$ and $\beta_k$ of (1.16) and (1.17).

The internal structure of a FIR filter with four coefficients is illustrated in Figure 1.3(b). The needed operations are multiplications, additions and delays, which are implemented by analog circuits. The filter coefficients $k_i$, the values of the variables $S_{m,n}$, $w_{m,n}$, and the values through the steps of the FIR filters are represented by analog variables. The needed circuital blocks are adders, multipliers, delays and sample-hold structures.

This approach is used in work [19] for both decomposition and reconstruction, but without show a good quality reconstruction. The mayor advantage of this approach is that the theoretical base for the algorithmic fast wavelet transform is quite developed.
But this comfort can be seen as a disadvantage, because it prevents from seeking new wavelets and methodologies more adaptable to analog circuits.

Comparing the amount of needed components, can be seen that the implementation of the semidiscrete wavelet transform, Figure 1.1(b), is simpler than the implementation of the fast wavelet transform, Figure 1.3(b).

1.2.5 Remarks

We have reviewed three different analog implementations of the wavelet transform. The semidiscrete implementation requires less and simpler components, just continuous filters and an adder, while the discrete and fast implementations need adders, multipliers, integrators, sample-hold structures, and analog delays.

In some works the approach have been to approximate wavelets used in the continuous wavelet transform (Morlet, first derivative of Gaussian), or wavelets used in the fast wavelet transform (Daubechies-5). But a continuous wavelet generally can not be used in the other kinds of wavelet transform because more restrictive constraints must be fulfilled. While the orthogonal wavelets of the fast wavelet transform present advantages only in their numeric implementation, but their analog implementation is difficult.

Instead, in other works have been designed, with relative success, original wavelets whose implementations in analog circuits are ideally exact.

On the other hand, the analog implementation of the semidiscrete wavelet transform only have included the decomposition part. But the reconstruction implies to find and implement in analog circuits both, the decomposition $\psi(t)$ and the reconstruction $\chi(t)$ wavelets.

Two ideas from previous works have been taken in this research: 1) to define new wavelets with simple and ideally exact analog implementations, and 2) to implement the semidiscrete transformada wavelet by convolution in continuous time filters whose impulse response is the wavelet function.
1.3 Objectives of the thesis

The main goal in this work is to identify pairs of rational filters $H_\psi(s)$ and $H_\chi(s)$ performing the direct and the inverse semidiscrete wavelet transform, such that these filters could be easily and directly implemented in analog circuits following any of the known techniques to implement rational functions.

As a second goal, a semidiscrete wavelet transform system is implemented in an analog integrated circuit by means of the application of the reached theoretical results.

As an extra objective, we will present a denoising technique to be applied together with the analog implementations of the wavelet transform.
Chapter 2

Principles of Analog Wavelet Transform

As could be seen in the previous chapter, to implement a system to perform the direct and the inverse semidiscrete wavelet transform, the decomposition filter, the reconstruction filter and the scale density have to be specified.

In this chapter is proved that any band-pass rational function in the $s$-domain can be used as a decomposition filter for the semidiscrete wavelet transform, and two different methods to find reconstruction filters are proposed. An exact reconstruction can not be implemented with the obtained filters, but the error can be reduced below any pre-specified bound. This can be reached by increasing the scale density, first method, or increasing the complexity of the reconstruction filter, second method.

The general validity of these outcomes is proved for a wide set of rational functions using some basic results of the mathematical analysis, while their practical applicability is shown with several specific examples.

An undesirable band-pass behavior of the wavelet system, that appears when we take a finite amount of scales is analyzed. Instead, the non-delay processing is shown as a remarkable property that the systems here described have, which is not shared neither by the classical continuous nor by the discrete wavelet transform.
2.1 Wavelet decomposition

In this section we identify an extensive set of rational filters that perform the semidiscrete wavelet transform. The ability of the obtained wavelets to perform a sharp time-frequency analysis is reviewed. As an example of decomposition filters the set of the powers of the band-pass biquads is shown. At the final of this section, the semidiscrete wavelet transform is applied over some probe signals and is compared with continuous and discrete versions of the wavelet transform.

2.1.1 Decomposition filters in the Laplace domain

The main class of functions of interest in this work are now described as follows.

**Definition 2.1.** In this work a band-pass rational function is expressed as

\[ H(s) = \frac{p_{d-1}s^{d-1} + p_{d-2}s^{d-2} + \cdots + p_1 s}{s^d + q_{d-1}s^{d-1} + \cdots + q_1 s + q_0}, \]  

(2.1)

which is the quotient of two polynomials of the complex variable \( s \), with real coefficients \( p_1, \ldots, p_{d-1}, q_0, \ldots, q_{d-1} \), where at least one \( p \) is different to zero, and the denominator is a strictly Hurwitz polynomial of degree \( d \geq 2 \).

A polynomial \( Q(s) \) is strictly Hurwitz if all its zeros are at the left side of the imaginary axis, implying that \( Q(j\omega) \neq 0, \omega \in \mathbb{R} \).

The next lemma remembers us that the Bode plot of any band-pass filter has asymptotes with slopes of at least \( 20 \frac{\text{dB}}{\text{dec}} \) and \( -20 \frac{\text{dB}}{\text{dec}} \). This result will be used to prove the convergence of infinite summations.

**Lemma 2.2.** Any band-pass rational function \( H(s) \) satisfies

\[ |H(j\omega)| \leq \begin{cases} 
L\omega, & \omega \in (0, 1] \\
L/\omega, & \omega \in [1, \infty) 
\end{cases} \]  

(2.2)

for some \( L > 0 \).
For any scale ratio $r > 1$, any band-pass rational function $H(s)$ has an impulse response $h(t)$, such that $\psi(t) = h(-t)$ satisfies the admissibility condition (1.2), and satisfies the stability condition of the semidiscrete wavelet transform (1.10).
Proof: Admissibility condition: \( \psi(t) = h(-t) \) is a real valued function, then \( |\Psi(\omega)| = |H(j\omega)| = |H(-j\omega)| \). From Lemma (2.2) we obtain (1.2) as follows

\[
C_\psi = 2 \int_{0}^{\infty} \frac{|H(j\omega)|^2}{\omega} d\omega \leq 2 \int_{0}^{1} \frac{|L\omega|^2}{\omega} d\omega + 2 \int_{1}^{\infty} \frac{|L/\omega|^2}{\omega} d\omega = 2L^2,
\]

for some \( L > 0 \).

Stability condition: From Lemma 2.3, \( \sum_{m=-\infty}^{\infty} |\Psi(r^m\omega)|^2 = \sum_{m=-\infty}^{\infty} |H(r^m j\omega)|^2 < B \), for some \( B > 0 \). On the other hand, from Definition 2.1, \( H(s) \) can have a maximum of \( d - 1 \) zeros, and it has at least one zero at \( \omega = 0 \). If \( \omega_{\text{max}} \) is the position of the upper zero over the imaginary axis, then \( H(j\omega) \neq 0 \) in the interval \( \omega \in [\omega_c, r\omega_c] \) for some \( \omega_c > \omega_{\text{max}} \). But, since \( \sum_{m=-\infty}^{\infty} |\Psi(r^m\omega)|^2 = \sum_{m=-\infty}^{\infty} |H(r^m j\omega)|^2 \) we can take

\[
A = \inf \left\{ |H(j\omega)|^2 : \omega \in [\omega_c, r\omega_c] \right\}.
\]

These \( A \) and \( B \) values satisfy the inequalities of condition (1.10). \( \square \)

2.1.2 Sharpness of the time-frequency analysis

A signal can be analyzed with total precision in the time-domain, or in the frequency-domain after the Fourier transform has been applied. But, when the signal is simultaneously analyzed in time and frequency there is a fundamental limit on the precision that can be achieved. In the case of the wavelet transform, the reached sharpness or precision is a property of the selected prototype-wavelet.

To compare the analysis sharpness reached by the different prototype wavelets we use the next definitions.

Definition 2.5. The center \( \bar{t} \) and the radius \( \Delta_t \) of the wavelet \( \psi(t) \), such that \( \psi(t) \) and \( t\psi(t) \) are \( L^2 \) on \( \mathbb{R} \), are defined to be [8]

\[
\bar{t} = \frac{1}{\|\psi\|^2} \int_{-\infty}^{\infty} t |\psi(t)|^2 dt, \quad (2.3)
\]

\[
\Delta_t = \frac{1}{\|\psi\|} \sqrt{\int_{-\infty}^{\infty} (t - \bar{t})^2 |\psi(t)|^2 dt}. \quad (2.4)
\]

In a similar way, if \( \Psi(\omega) \) is the Fourier transform of \( \psi(t) \), and \( \Psi(\omega) \) and \( \omega \Psi(\omega) \) are \( L^2 \) on \( \mathbb{R} \), we define the center \( \bar{\omega} \) and the radius \( \Delta_\omega \) of \( \Psi(\omega) \). But in the case of real
valued wavelets is verified that $|\Psi(\omega)| = |\Psi(-\omega)|$, then to obtain $\bar{\omega}$ and $\Delta_\omega$ values with a useful meaning we must consider only positive frequencies as follows

$$\bar{\omega} = \frac{\sqrt{2}}{\|\Psi\|^2} \int_0^\infty \omega |\Psi(\omega)|^2 d\omega, \quad (2.5)$$

$$\Delta_\omega = \frac{\sqrt{2}}{\|\Psi\|} \sqrt{\int_0^\infty (\omega - \bar{\omega})^2 |\Psi(\omega)|^2 d\omega}. \quad (2.6)$$

The product $\Delta_t \Delta_\omega$ can be considered as a measure of the sharpness of the time-frequency analysis that can be made with some specific wavelet. In this case, the sharpness is higher as the product is lower. A limit in this sharpness capacity is given by the next result.

**Theorem 2.6** (Uncertainty principle [8, Theorem 3.5]). Let $\psi(t)$, $L^2$ on $\mathbb{R}$, be chosen such that both $t \psi(t)$ and $\omega \Psi(\omega)$ are $L^2$ on $\mathbb{R}$, then

$$\Delta_t \Delta_\omega \geq \frac{1}{2}. \quad (2.7)$$

The next definition of selectivity in frequency can be helpful to compare the performance of the different prototype wavelets.

**Definition 2.7.** The frequency selectivity, or just selectivity, of a filter $\Psi(\omega)$, $L^2$ on $\mathbb{R}$, with $\omega \Psi(\omega)$ $L^2$ on $\mathbb{R}$, is defined by

$$Q = \frac{\bar{\omega}}{\Delta_\omega}. \quad (2.8)$$

### 2.1.3 Examples of decomposition filters

As example of decomposition filters, we consider the powers of the band-pass biquadratic filter using the next notation.

**Definition 2.8.** We specify the power of a particular biquadratic filter by

$$\text{biqu}(q, n) = \left(\frac{s/q}{s^2 + s/q + 1}\right)^n, \quad (2.9)$$

while the expression wavelet biqu$(q, n)$ refers to the corresponding wavelet, i.e., $\psi(t) = h_{\psi}(-t)$, where $h_{\psi}(t)$ is the impulse response of the filter.
Example 2.9. In Figure 2.1 are shown the plots of the analysis sharpness, $\Delta_t \Delta_\omega$, as a function of the $q$-parameter for the cases $n = 2, 3, \ldots, 8$. In these plots, the behavior of the frequency selectivity $Q$ of the family of filters biqu($q, n$) is also shown. Note that $Q$ is dependent of the $q$-values.

Example 2.10. In Figure 2.2(a) is shown the frequency spectrum of the filters biqu($q, n$) for $n = 2, 3, \ldots, 8$ and $Q = 2.5$. The corresponding wavelets are shown in Figure 2.2(b). Spectrums and wavelets have been normalized to have the same norm. Besides the most different case $n = 2$, these wavelets are very similar, but differ in sharpness capacity and in the number of continuous derivatives that they have.

One of these wavelets, biqu(0.698, 3), was plotted again in Figure 2.2(c) to be compared with the Daubechies wavelet db4, and the Morlet wavelet, $\psi(t) = e^{-t^2/2} \cos 5t$, shown in Figure 2.2(d) and 2.2(e).

2.1.4 Decomposition examples

Example 2.11. In this first example we use the probe signal tone2 shown in Figure 2.3(a), whose frequency spectrum is shown in Figure 2.3(b). This probe signal
Figure 2.2: (a) Frequency spectrums and (b) wavelets of the filters biqu(\(q, n\)) for \(n = 2, 3, \ldots, 8\) and selectivity \(Q = 2.5\). (c) Wavelet biqu(0.698, 3). (d) Daubechies wavelet, db4. (e) Morlet wavelet, \(\psi(t) = e^{-t^2/2} \cos 5t\).

comprises two modulated sinusoidal pulses of 0.5Hz and 2Hz.

For comparative purposes, four different wavelet transforms were obtained. In all the cases, the wavelet transform was computed over 7 octaves, covering a similar frequency range, from approximately 88mHz to 11Hz.

In Figure 2.3(c) is shown a semidiscrete wavelet transform, obtained with the wavelet biqu(0.689, 3) and a scale density of 1 scale per octave, \(D = 1\). Even though this plot does not include amplitude units, all the shown components maintain the correct proportions between them.

In Figure 2.3(d) is shown other semidiscrete wavelet transform, obtained with the wavelet biqu(0.689, 3) and a scale density of \(D = 3\). In this case, the absolute value of the 22 wavelet components were plotted together as a surface image, in this way is possible a visual identification of the time-frequency signal characteristics.

In Figure 2.3(e) is shown a continuous wavelet transform, obtained with the Morlet wavelet, \(\psi(t) = e^{-t^2/2} \cos 5t\). The absolute value of the computed 71 scales have been plotted as an image.
Figure 2.3: Four wavelet transform of the probe signal \textit{tone2} (a) with frequency spectrum (b). (c) Semidiscrete wavelet transform with biqu(0.689, 3) and $D = 1$. (d) Semidiscrete wavelet transform with biqu(0.689, 3) and $D = 3$. (e) Continuous wavelet transforms with the Morlet wavelet. (f) Discrete wavelet transforms with the Daubechies wavelet, db4. (Matlab simulations.)

In Figure 2.3(f) is shown a discrete wavelet transform, obtained with the Daubechies wavelet db4, along 8 dyadic scales. To compute the wavelet coefficients, the fast-wavelet transform algorithm has been applied. The absolute value of the coefficients have been plotted as an image for comparison purposes.

In all of these image plots the lower used scale is considered as having a scale value of $a = 1$.

\textbf{Example 2.12.} For this example we use the probe signal \textit{sin2} shown in Figure 2.4(a). This signal is comprised by three joined sinusoidal segments with equal amplitudes and frequencies of 0.5Hz and 2.0Hz. The frequency spectrum of this signal is shown in Figure 2.4(b). Two wavelet transform were calculated in this case.

In Figure 2.4(c) is shown the absolute value of the continuous wavelet transform
of \( \sin^2 \) plotted as an image, using the Morlet wavelet \( \psi(t) = e^{-t^2/2} \cos 5t \).

In Figure 2.4(d) is shown the absolute value of the semidiscrete wavelet transform plotted as an image, using the wavelet biqu(2, 2) and a scale density of \( D = 3 \). (Matlab simulations.)

In this example we can note an important difference between the Morlet and the biquadratic wavelets. In the first case, a symmetric wavelet transform is obtained, due to the symmetry of the Morlet wavelet. While, in the case of the biquadratic wavelet, the wavelet transform is asymmetric, stretched to the right.

**Example 2.13.** For this example we use the probe signal \( \text{chirp}^2 \) shown in Figure 2.5(a), which is a double chirp signal going from high to low frequencies and returning to high frequencies. This signal allows us to probe the system along all its frequency range. The frequency spectrum of the signal is shown in Figure 2.5(b). Two wavelet transforms were calculated in this case. Again, the asymmetric behavior of the semidiscrete wavelet transform is evident.

In Figure 2.5(c) is shown the absolute value of the continuous wavelet transform of \( \text{chirp}^2 \) plotted as an image, using the Morlet wavelet \( \psi(t) = e^{-t^2/2} \cos 5t \).

In Figure 2.5(d) is plotted the the absolute value of the semidiscrete wavelet
Figure 2.5: Two wavelet transforms of the probe signal \textit{chirp2} (a) with frequency spectrum (b). (c) Continuous wavelet transform with the Morlet wavelet. (e) Semidiscrete wavelet transform with biqu(1.414, 2) and $D = 3$. (Matlab simulations.)

transform of \textit{chirp2} plotted as an image, using the wavelet biqu(1.414, 2) and $D = 3$.

2.2 Wavelet reconstruction

We have yet solved the problem of wavelet decomposition with rational filters. Now, to perform reconstruction, in this section we provide two different approaches to identify appropriate dual filters. An exact reconstruction can not be implemented with the obtained filters, but the error can be reduced below any pre-specified level. This can be reached by increasing the number of scales per octave, first method, or increasing the complexity of the reconstruction filter, second method.

Our first task is to provide a more manageable condition equivalent to (1.11). To achieve this, we work in the logarithmic-frequency domain instead of the normal frequency domain. This allows that the infinite summation $\sum_m F(r^m \omega)$, in (1.11), can become a periodic function, and then, we can apply the theory of Fourier series to it.

Another important task in this section is to define a measure of the reconstruction
error. We prefer to work with the maximum bound of the error instead of the mean quadratic error, because the uniform norm, \( \| f \|_\infty = \operatorname{sup} \{ f(x) : x \in \mathbb{R} \} \), maintain the same meaning when we pass from the logarithmic frequency domain to the normal frequency domain. In contrast with the mean square norm that is affected for the domain change.

### 2.2.1 Condition for perfect reconstruction

For the application of the inverse wavelet transform is needed a reconstruction filter \( H_\chi(s) \), whose impulse response, \( \chi(t) \), is an adequate dual function of the wavelet \( \psi(t) \), used in the decomposition. To be dual, the pair of wavelets have to satisfy the condition (1.11). However, this condition is hard to handle directly, since it has an infinite summation of scaled filters, i.e., filters exponentially shifted in frequency. In this subsection a more manageable equivalent condition is provided. But, first some preliminary results have to be proved.

With the next result, we ensure the uniform convergence of the infinite summation of condition (1.11).

**Lemma 2.14.** For any \( r > 1 \), if \( H(s) \) is a band-pass rational function then the series \( \sum_{m=-\infty}^{\infty} H(r^m j \omega) \) has uniform convergence and is continuous for \( \omega \in (0, \infty) \).

**Proof:** From Lemma 2.3, \( \sum_{m=-\infty}^{\infty} H(r^m j \omega) \leq \sum_{m=-\infty}^{\infty} |H(r^m j \omega)| \) converges.

For Lemma 2.2 we have that \( |H(j^x)| \leq L r^x \) for \( x \in (0, \infty) \), \( |H(j^x)| \leq L r^{-x} \) for \( x \in (0, \infty) \), for some \( L > 0 \). Then, for any interval \( x \in [m, m+1], m \in \mathbb{Z} \), we have that \( |H(j^x)| \leq L r^{m+1} \) for \( m < 0 \), and \( |H(j^x)| \leq L r^{-m} \) for \( m \geq 0 \). Then in the interval \( x \in [0, 1] \) is verified that

\[
\sum_{m=-n}^{n} |H(j^{r^{m+x}})| \leq \sum_{m=-n}^{-1} L r^{m+1} + \sum_{m=0}^{n} L r^{-m} < 2L \sum_{m=0}^{n} r^{-m} = 2L \frac{1 - 1/r^n}{1 - 1/r}.
\]

For \( N > n \), is verified that

\[
\sum_{|m|=n+1}^{N} |H(j^{r^{m+x}})| \leq \sum_{m=-N}^{N} |H(j^{r^{m+x}})| - \sum_{m=-n}^{n} |H(j^{r^{m+x}})| \leq 2L \frac{1/r^n - 1/r^N}{1 - 1/r},
\]
and making $N \to \infty$ we prove that

$$\left| \sum_{m=-\infty}^{\infty} H(jr^{m+x}) - \sum_{m=-n}^{n} H(jr^{m+x}) \right| = \sum_{|m|=n+1}^{\infty} H(jr^{m+x}) \leq 2L \frac{1/r^n}{1-1/r}.$$ 

And then, $\sum_{m=-\infty}^{\infty} H(jr^{m+x})$ converges uniformly on $x \in [0, 1]$.

From Definition 2.1, note that $H(j\omega)$ is continuous in $\omega \in (0, \infty)$, because numerator and denominator are continuous and the denominator is never equal to zero. Then, using $\omega = r^x$, we have that $H(jr^x)$ is continuous in $x \in (-\infty, \infty)$, and then, $H(jr^{m+x})$ is continuous in $x \in [0, 1]$ for any $m \in \mathbb{Z}$.

We have that: If $f_n(x) \to f(x)$ uniformly on the interval $I$, and if each $f_n(x)$ is continuous on $I$, then $f(x)$ is continuous on $I$. [10, Theorem 1.29] With this result we prove the continuity of $\sum_{m=-\infty}^{\infty} H(jr^{m+x})$ in $x \in [0, 1]$.

Finally, we have that $\sum_{m=-\infty}^{\infty} H(r^m j\omega) = \sum_{m=-\infty}^{\infty} H(jr^{m+x})$ converges uniformly and is continuous in $x \in (-\infty, \infty)$, that is, in $\omega \in (0, \infty)$, since $\sum_{m=-\infty}^{\infty} H(jr^{m+x})$ is periodic in $x$ with period 1.

The next result says that when we work with condition (1.11) in the logarithmic frequency domain, the infinite summation $\sum_{m} H(r^m j\omega) = \sum_{m} \Psi^*(r^m \omega)X(r^m \omega)$ becomes periodic and, very important, its Fourier series converges uniformly. The periodicity of $\sum_{m} H(r^m j\omega)$, with $\omega = r^{x/2\pi}$, is illustrated in Figure 2.6.

**Lemma 2.15.** For some $r > 1$, if $H(s)$ is a band-pass rational function then the function $\sum_{m=-\infty}^{\infty} H(jr^{m+x}/2\pi)$, which is periodic in $x$ with period $2\pi$, has a Fourier
series $\sum_{n=-\infty}^{\infty} c_n e^{jnx}$ of uniform convergence, where
\[
c_n = \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{m=-\infty}^{\infty} H(jr^{m+x/2\pi}) e^{-jnx} \, dx.
\] (2.10)

**Proof:** To prove the uniform convergence we use the result: *Suppose that $f(x)$ has period $2\pi$ and is continuous and piecewise differentiable on $\mathbb{R}$. Then the sequence of partial summations $S_N(x) = \sum_{n=-N}^{N} c_n e^{jnx}$, with $c_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) e^{-jnx} \, dx$, converges uniformly to $f(x)$. [10, Theorem 2.17]*

From Lemma 2.14 we have that $\sum_{m=-\infty}^{\infty} H(jr^{m+x/2\pi})$, with $r^{x/2\pi} = \omega$, is continuous, then we just have to prove that $H(jr^{m+x/2\pi})$ is piecewise differentiable.

With $H(s) = \frac{P(s)}{Q(s)}$ given by (2.1), and making $s = jr^{x/2\pi}$, we have that $\frac{ds}{dx} = ks$, with $k = \frac{lnr}{2\pi}$, and then we obtain
\[
\frac{dH(s)}{dx} = \frac{dH(s)}{ds} \frac{ds}{dx} = \frac{Q(s) \frac{dP(s)}{ds} - P(s) \frac{dQ(s)}{ds}}{Q^2(s)} \frac{ds}{dx} = \frac{\bar{p}_2d-1}s^{2d-1} + \bar{p}_{2d-2}s^{2d-2} + \cdots + \bar{p}_2s^2 + \bar{p}_1s}{s^{2d} + q_{2d-1}s^{2d-1} + \cdots + q_1s + q_0},
\]
where $\bar{p}_1 = kp_1q_0$, $\bar{p}_2 = 2kp_2q_0$, $\ldots$, $\bar{p}_{2d-2} = -2kp_{d-2}$, $\bar{p}_{2d-1} = -kp_{d-1}$, $\bar{q}_0 = q_0^2$, $\bar{q}_1 = 2q_0q_1$, $\ldots$, $\bar{q}_{2d-2} = 2q_{d-2} + q_{d-2}^2$, $\bar{q}_{2d-1} = 2q_{d-1}$.

As can be seen, $\frac{dH(s)}{dx}$, expressed as a function of $s$, is also a band-pass rational function. Then, by Lemma 2.14, $\sum_{m=-\infty}^{\infty} \frac{dH(jr^{m+x/2\pi})}{dx} = \sum_{m=-\infty}^{\infty} \frac{dH(jr^{m+x/2\pi})}{dx}$ converges uniformly and can be integrated term by term over any finite range, and particularly over the period $x \in [0, 2\pi]$. Therefore, $\sum_{m=-\infty}^{\infty} H(jr^{m+x/2\pi})$ is differentiable for $x \in [0, 2\pi]$, since it is differentiable term by term.

The next results give us an optional way to compute the Fourier coefficients of the infinite summation $\sum_{m} H(jr^{m+x/2\pi})$, defining the Fourier coefficients as samples of a Fourier transform defined in the logarithmic frequency domain.

**Corollary 2.16 (Poisson summation formula [27, (5.75)]).** The Fourier coefficients
\[
c_n = \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{m=-\infty}^{\infty} H(jr^{m+x/2\pi}) e^{-jnx} \, dx,
\]
where $H(s)$ is a band-pass rational function, also can be expressed as
\[
c_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(jr^{x/2\pi}) e^{-jnx} \, dx.
\] (2.11)
The next result establishes the easier manageable condition, equivalent to (1.11), which we have been looking for.

**Theorem 2.17.** For any $r > 1$, if $H(s)$ is a band-pass rational function such that $H(j\omega) = \Psi^* (\omega)X(\omega)$, then the condition for perfect reconstruction (1.11) will be fulfilled if and only if the Fourier coefficients $c_n = \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=-\infty}^{\infty} H(jr^{m+x/2\pi}) e^{-jnx} dx$ are $c_n = 0$ for all $n \in \mathbb{Z}$, except $c_0 = 1$.

**Proof:** If condition (1.11) is satisfied then we have

\[
1 = \sum_{m=-\infty}^{\infty} \Psi^*(r^m \omega) X(r^m \omega) = \sum_{m=-\infty}^{\infty} H(r^m j\omega) = \sum_{m=-\infty}^{\infty} H(jr^{m+x/2\pi}),
\]

where $\omega = r^{x/2\pi}$. Now, applying $c_n = \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=-\infty}^{\infty} H(jr^{m+x/2\pi}) e^{-jnx} dx$, we obtain $c_n = 0$ for all $n$ except $c_0 = 1$.

A set of coefficients of a Fourier series of a periodic function has the uniqueness property, then, only when $c_0 = 1$ and all the other coefficients are zero the periodic function can be equal to 1. 

The next result establishes the same previous condition, equivalent to (1.11), but using real coefficients.
**Corollary 2.18.** For any \( r > 1 \), if \( H(s) \) is a band-pass rational function such that \( H(j\omega) = \Psi^*(\omega)X(\omega) \), then the condition for perfect reconstruction (1.11) will be fulfilled if and only if the real Fourier coefficients

\[
\alpha_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re} H(jr^{x/2\pi}) \cos nx \, dx ,
\]
\[
\beta_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Im} H(jr^{x/2\pi}) \sin nx \, dx ,
\]
\[
A_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Im} H(jr^{x/2\pi}) \cos nx \, dx ,
\]
\[
B_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re} H(jr^{x/2\pi}) \sin nx \, dx ,
\]

(2.12)

satisfy \( \alpha_n = \beta_n = A_n = B_n = 0 \) for all \( n \in \mathbb{N} \) except \( \alpha_0 = 1 \).

**Proof:** Replacing in (2.11) the identities \( H(jr^{x/2\pi}) = \text{Re} H(jr^{x/2\pi}) + j \text{Im} H(jr^{x/2\pi}) \) and \( e^{-jn\omega} = \cos nx - j \sin nx \) we obtain \( c_n = (\alpha_n + \beta_n) + j(A_n - B_n) \), for \( n \in \mathbb{Z} \), where \( \alpha_n, \beta_n, A_n \) and \( B_n \) are given by (2.12). But, from (2.12) we have that \( \alpha_{-n} = \alpha_n, \beta_{-n} = -\beta_n, A_{-n} = A_n, \) and \( B_{-n} = -B_n \). Then

\[
c_0 = \alpha_0 + jA_0 ,
\]
\[
c_n = (\alpha_n + \beta_n) + j(A_n - B_n) ,
\]
\[
c_{-n} = (\alpha_n - \beta_n) + j(A_n + B_n) ,
\]

(2.13)

for \( n \geq 1 \), and also

\[
\alpha_0 = \text{Re} c_0 ,
\]
\[
A_0 = \text{Im} c_0 ,
\]
\[
\alpha_n = \text{Re}(c_n + c_{-n})/2 ,
\]
\[
\beta_n = \text{Re}(c_n - c_{-n})/2 ,
\]
\[
A_n = \text{Im}(c_n + c_{-n})/2 ,
\]
\[
\beta_n = \text{Im} - (c_n - c_{-n})/2 ,
\]

(2.14)

for \( n \geq 1 \). And this proof is finished by applying Theorem 2.17. \( \square \)

Theorem 2.17 says that a band-pass rational function \( H(j\omega) = \Psi^*(\omega)X(\omega) \) satisfies the condition for perfect reconstruction (1.11) when its first Fourier coefficient \( c_0 \) is equal to 1 and all the others are zero. But in general, the coefficients of a band-pass rational function will not be equal to zero. As a consequence, condition (1.11) will not be completely satisfied, and will not be possible a perfect reconstruction of the
original signal. But, using any of the two methods below explained, can be reached a reconstruction error as low as desired.

### 2.2.2 Reconstruction relative error

In this subsection is established a method to quantify the maximum possible reconstruction error as a function of the ripple present in the infinite summation of condition (1.11). But first, we have to specify a measure for this type of error.

**Definition 2.19.** The **reconstruction relative error** of the reconstructed signal \( f_{\text{out}}(t) \) respect to the original signal \( f_{\text{in}}(t) \), where \( f_{\text{in}}(t) \) and \( f_{\text{out}}(t) \) are \( L^2 \) on \( \mathbb{R} \), is given by

\[
\varepsilon = \frac{\| f_{\text{out}} - f_{\text{in}} \|}{\| f_{\text{in}} \|}.
\]

**Theorem 2.20.** The reconstruction relative error satisfies the relation

\[
\varepsilon \leq \varepsilon_{\text{max}} = \sup \left\{ \left| \sum_{m=-\infty}^{\infty} H(jr^m \omega) - 1 \right| : \omega \in [\omega_c, r\omega_c] \right\},
\]

for any \( \omega_c > 0 \), where \( H(j\omega) = \Psi^*(\omega)X(\omega) \).

**Proof:** From (1.8) and (1.9) we have that

\[
W_m(\omega) = F_{\text{in}}(\omega)\Psi_m^*(\omega) \quad \text{and} \quad F_{\text{out}}(\omega) = \sum_{m=-\infty}^{\infty} W_m(\omega)X_m(\omega),
\]

where \( \Psi_m(\omega) = \Psi(r^m \omega) \) and \( X_m(\omega) = X(r^m \omega) \). Combining these results we obtain

\[
F_{\text{out}}(\omega) = F_{\text{in}} \sum_{m=-\infty}^{\infty} \Psi^*(r^m \omega)X(r^m \omega) = F_{\text{in}}(\omega) \sum_{m=-\infty}^{\infty} H(jr^m \omega).
\]

From Definition 2.19 we have that

\[
\varepsilon = \frac{\| f_{\text{out}} - f_{\text{in}} \|}{\| f_{\text{in}} \|} = \frac{\| F_{\text{out}} - F_{\text{in}} \|}{\| F_{\text{in}} \|} = \sqrt{\int_{-\infty}^{\infty} \left| F_{\text{in}}(\omega) \sum_{m=-\infty}^{\infty} H(jr^m \omega) - F_{\text{in}}(\omega) \right|^2 d\omega} \| F_{\text{in}} \| = \sqrt{\int_{-\infty}^{\infty} |F_{\text{in}}(\omega)|^2 \left| \sum_{m=-\infty}^{\infty} H(jr^m \omega) - 1 \right|^2 d\omega} \| F_{\text{in}} \| \leq \sqrt{\int_{-\infty}^{\infty} |F_{\text{in}}(\omega)|^2 \sup \left\{ \sum_{m=-\infty}^{\infty} H(jr^m \omega) - 1 \left|^{2} : \omega \in (0, \infty) \right. \right\} d\omega} \| F_{\text{in}} \| = \sup \left\{ \sum_{m=-\infty}^{\infty} H(jr^m \omega) - 1 \left|^{2} : \omega \in (0, \infty) \right. \right\} = \varepsilon_{\text{max}}.
\]
But making \( w = r^x \), we have that \( \sum_{m=-\infty}^{\infty} H(jr^m \omega) = \sum_{m=-\infty}^{\infty} H(jr^{m+x}) \) is periodic in \( x \) with period 1, then
\[
\varepsilon_{max} = \sup \left\{ \sum_{m=-\infty}^{\infty} H(jr^{m+x}) - 1 : x \in [\log_r \omega_c, \log_r \omega_c + 1] \right\}
\]
\[
= \sup \left\{ \sum_{m=-\infty}^{\infty} H(jr^m \omega) - 1 : \omega \in [\omega_c, r\omega_c] \right\},
\]
for any \( \omega_c > 0 \).

To find \( \sup \left\{ \sum_{m=-\infty}^{\infty} H(jr^m \omega) - 1 \right\} \) we just have to examine in some range \( \omega \in [\omega_c, r\omega_c] \), for example in \( \omega \in [1, r] \). The importance of this theorem is in allow the numeric estimation of the maximum error bound, \( \varepsilon_{max} \), for any function \( f_{in}(t) \).

In the proof of the theorem, we could see that \( F_{out}(\omega) = F_{in}(\omega) \sum_{m=-\infty}^{\infty} H(jr^m \omega) \).
Then, only when \( \sum_{m=-\infty}^{\infty} H(r^m j \omega) = 1 \), the input and the output signals can be equal. This perfect reconstruction is illustrated in Figure 2.7(a), while in Figure 2.7(b) the reconstruction can not be perfect because the sum is not equal to 1. As can be seen, the error is due to the ripple contained in the infinite summation.

Another way to measure the reconstruction error is with the next definition.

\textbf{Definition 2.21.} The \textit{signal to error ratio} is given by
\[
\text{SER} = 20 \log \frac{\|f_{in}\|}{\|f_{out} - f_{in}\|} \text{ (dB)}.
\]

(2.17)

From (2.17) and (2.16) we can derive the minimum signal to error ratio
\[
\text{SER}_{\text{min}} = 20 \log \frac{1}{\varepsilon_{max}} \text{ (dB)}.
\]

(2.18)

\textbf{2.2.3 Rate of convergence of the Fourier coefficients}

The efficacy of the methods to find reconstruction wavelets, which are described in subsequent sections, is supported by the fast rate of convergence of the Fourier coefficients of the infinite summation \( \sum_m H(jr^{m+x}/2\pi) \).
In this subsection we show that these coefficients have a geometric rate of convergence. To prove this, we make use of the concept of analyticity of a function in a complex domain, when this domain is an extension of the logarithmic frequency. For this, we follow a similar argument to the one given for the strip of convergence for Fourier series [28, Theorem 5].

In the next result we find the domain of analyticity of the function $H(jr^z/2\pi)$.

**Lemma 2.22.** For a given $r > 1$, and if $H(s)$ is a band-pass rational function with poles at $\{r e^{j\theta_k}\}_{1 \leq k \leq d}$, then $H(jr^z/2\pi)$, where $z = x + jy$, is analytic in the domain $x \in (-\infty, \infty)$, $y \in (y_{\text{min}}, y_{\text{max}})$, where

$$
\begin{align*}
y_{\text{max}} &= \frac{\pi^2}{\ln r} \left( \frac{\theta_{\text{up}}}{\pi/2} - 1 \right) \quad \text{where} \quad \theta_{\text{up}} = \min \{\theta_k \in (\pi/2, \pi]\} \\
y_{\text{min}} &= \frac{\pi^2}{\ln r} \left( \frac{\theta_{\text{low}}}{\pi/2} - 1 \right) \quad \text{where} \quad \theta_{\text{low}} = \max \{\theta_k \in [-\pi, -\pi/2)\}
\end{align*}
$$

(2.19)

**Proof:** The function $H(jr^z/2\pi)$ is analytic in the real axis $z = x$ because it is compound...
of exponentials, polynomials and quotient functions of the analytic variable \( z \), and the quotient is different to zero in all the real domain.

The unique singularities of \( H(jr^{x/2\pi}) \) are the poles of \( H(s) \) mapped to the \( z \)-plane. To find them, we use \( j = e^{j\pi/2} \) to obtain \( H(jr^{x/2\pi}) = H(r^{x/2\pi}e^{j(\ln r/2\pi)\gamma + \pi/2}) \). If we compare this expression with the evaluation of \( H(s) \) in \( s = \rho e^{j\theta} \), then we obtain

\[
\theta = \frac{\ln r}{2\pi} y + \frac{\pi}{2} + 2\pi l,
\]

where the term \( 2\pi l \), for \( l \in \mathbb{Z} \), has been included to consider the periodicity of \( \theta \). Then

\[
y = \frac{\pi^2}{\ln r} \left( \frac{\theta}{\pi/2} - 1 + 4l \right).
\]

Now, we are interested in the \( y \)-values of the nearest singularities above and below the real axis.

From Definition 2.1, we know that the argument of the poles of the function \( H(s) \) lay in the ranges \( \theta \in (\pi/2, \pi] \) and \( \theta \in (-\pi, -\pi/2] \). Then, by direct substitution, is verified that the ordinate, \( y_{\text{max}} \), of the nearest singularity above the real axis is obtained applying the last equation to the values \( \min\{\theta_k \in (\pi/2, \pi]\} \) and \( l = 0 \), and the ordinate, \( y_{\text{min}} \), of the nearest singularity below the real axis is obtained applying the last equation to the values \( \max\{\theta_k \in (-\pi, -\pi/2]\} \) and \( l = 0 \).

In the next theorem we obtain the exact rate of convergence of the Fourier transform of the function \( H(jr^{m+x/2\pi}) \).

**Theorem 2.23.** For a given \( r > 1 \), and if \( H(s) \) is a band-pass rational function, then the function \( \hat{H}(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(jr^{x/2\pi})e^{-j\gamma x} \, dx \) satisfies the asymptotic rate of geometric convergence given by

\[
\lim_{\gamma \to \infty} \frac{|\hat{H}(\gamma)|}{C_1 e^{\gamma y_{\text{min}}}} = 1
\]

(2.20)

\[
\lim_{\gamma \to -\infty} \frac{|\hat{H}(\gamma)|}{C_2 e^{\gamma y_{\text{max}}}} = 1
\]

for some constants \( C_1, C_2 > 0 \), where \( y_{\text{min}} \) and \( y_{\text{max}} \) are given by (2.19).

**Proof:** From Lemma 2.2, we have that

\[
\int_{-\infty}^{\infty} \left| H(jr^{x/2\pi}) \right| \leq \int_{-\infty}^{0} Lr^{x/2\pi} \, dx + \int_{0}^{\infty} Lr^{-x/2\pi} \, dx = \frac{2L \ln r}{2\pi},
\]

and...
meaning that \( H(jr^{\pi/2\pi}) \) is \( L^1 \) on \( \mathbb{R} \).

We define
\[
\hat{H}(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(jr^{\pi/2\pi})e^{-j\gamma x} \, dx,
\]
but we have that if \( f(x) \) is continuous and \( L^1 \) on \( \mathbb{R} \), and \( F(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-j\gamma x} \, dx \), then for each \( x \in \mathbb{R} \), \( \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} \hat{H}(\gamma) e^{-\pi\tau\gamma^2} e^{j\gamma x} d\gamma = f(x) \). [10, Theorem 3.11]

Then
\[
H(jr^{\pi/2\pi}) = \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} \hat{H}(\gamma) e^{-\pi\tau\gamma^2} e^{j\gamma z} d\gamma,
\]
where we know that \( \hat{H}(\gamma) \) is well defined in \( \gamma \in \mathbb{R} \) because if \( f(x) \) is \( L^1 \) on \( \mathbb{R} \), then its Fourier transform is uniformly continuous on \( \mathbb{R} \). [10, Theorem 3.8]

On the other hand, a function analytic in a domain \( S \) is uniquely determined by its values in any domain or arc inside \( S \). [29, Theorem 1. Sec 107.] Then, replacing \( x \) by \( z = x + jy \) we obtain
\[
H(jr^{\pi/2\pi}) = \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} \hat{H}(\gamma) e^{-\pi\tau\gamma^2} e^{j\gamma z} d\gamma = \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} \hat{H}(\gamma) e^{-\pi\tau\gamma^2} e^{-\gamma y} e^{j\gamma x} d\gamma,
\]
that converges for \( y \in (y_{\min}, y_{\max}) \), where \( y_{\min} < 0 \) and \( y_{\max} > 0 \) are given by (2.19).

(From here we follow the proof given in [28, Theorem 5].) Suppose that \( |\hat{H}(\gamma)| = C_1 e^{\gamma(y_{\min} + \epsilon_1)} \) when \( \gamma \to \infty \), for some \( C_1 \). If \( \epsilon_1 > 0 \) then the previous inverse Fourier transform would not converge for \( y = y_{\min} + \epsilon_1 \), and if \( \epsilon_1 < 0 \) then the inverse transform would converge for \( y = y_{\min} \). Then, for sufficiently large \( \gamma \), the behavior of \( |\hat{H}(\gamma)| \) must be like \( C_1 e^{\gamma y_{\min}} \) in order to satisfy the convergence of the inverse Fourier transform in the analyticity domain of \( H(jr^{\pi/2\pi}) \).

Suppose that \( |\hat{H}(\gamma)| = C_2 e^{\gamma(y_{\max} + \epsilon_2)} \) when \( \gamma \to -\infty \), for some \( C_2 \). If \( \epsilon_2 < 0 \) then the previous inverse Fourier transform would not converge for \( y = y_{\max} - \epsilon \), and if \( \epsilon_2 > 0 \) then the inverse transform would converge for \( y = y_{\max} \). Then, for negative values of \( \gamma \) and sufficiently large \(|\gamma|\), the behavior of \( |\hat{H}(\gamma)| \) must be like \( C_2 e^{\gamma y_{\max}} \) in order to satisfy the convergence of the inverse Fourier transform in the analyticity domain of \( H(jr^{\pi/2\pi}) \).

With this last theorem, the rate of convergence of the coefficients of \( \sum_m H(jr^{m+x/2\pi}) \) are also specified, since, by Corollary 2.16, they are just samples of the Fourier transform given in the last theorem.
2.2.4 Method 1: Increasing the scale density

In this section and the next one are explained two original methods to design decomposition-reconstruction analog wavelet systems. With these methods we can select a decomposition filter, a reconstruction filter and a scale density producing a reconstruction error below any pre-specified bound.

Suppose that we have already selected a band pass function and a stable function as decomposition and reconstruction filters of a wavelet system with a scale density $D > 0$, then, we obtain the error bound applying (2.20). Now, if we increase the scale density, holding the same decomposition and reconstruction filters, the reconstruction error bound, together with the ripple of the summation, goes down as the spectrums of the filters are closer and closer. This is illustrated in Figure 2.8 where the scale density goes from 1 to 2 scales per octave. In this section we prove the generality of this principle.

To compare the performance of systems with identical prototype filters but different scale density, we use the next definition.

**Definition 2.24.** The product between the decomposition and the reconstruction...
filters $k H_\psi(s) H_\chi(s)$ is normalized, for a given scale ratio $r > 0$, selecting the real constant $k$ that satisfies

$$\text{Re} c_0 = \text{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} k H_\psi \left(j r^{x/2\pi}\right) H_\chi \left(j r^{x/2\pi}\right) dx = 1. \tag{2.21}$$

With the next theorem we prove that, as the scale density increases, the reconstruction error tends to zero.

**Theorem 2.25.** For a pair of given decomposition and reconstruction filters $H_\psi(s)$ and $H_\chi(s)$, such that $\text{Im} \int_{-\infty}^{\infty} H_\psi \left(j r^{x/2\pi}\right) H_\chi \left(j r^{x/2\pi}\right) dx = 0$, the reconstruction relative error $\varepsilon_{\text{max}} \to 0$ as the scale density $D \to \infty$.

**Proof:** For an initially given $D_o > 0$, suppose that $H_o(s) = k H_\psi(s) H_\chi(s)$ is already normalized for a scale ratio $r_o = 2^{1/D_o}$.

We now define $\hat{H}(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_o \left(j r_o^{x/2\pi}\right) e^{-j\gamma x} dx$. And, by Corollary 2.16, the Fourier coefficients of the wavelet system $\sum_{m=-\infty}^{\infty} H_o \left(j r_o^{m+x/2\pi}\right)$ are given by $c_n = \hat{H}(n)$.

Now, we select a new scale density $D > D_o$, producing a scale ratio of $r = 2^{1/D} = r_o^{D/D_o}$. By Corollary 2.16, the Fourier coefficients of the new and normalized wavelet system $\sum_{m=-\infty}^{\infty} k H_o \left(j r^{m+x/2\pi}\right)$ are given by

$$d_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} k H_o \left(j r^{(D/D_o)(x/2\pi)}\right) e^{-jnx} dx = \frac{k}{2\pi} \frac{D}{D_o} \int_{-\infty}^{\infty} H_o \left(j r^{\zeta/2\pi}\right) e^{-j(D/D_o)n\zeta} d\zeta$$

$$= k \frac{D}{D_o} \hat{H} \left( \frac{D}{D_o} n \right),$$

where we have used $\zeta = x \frac{D}{D_o}$, and the normalization factor must be $k = \frac{D_o}{D}$ to make $\text{Re} d_0 = \text{Re} c_0 = 1$. Then

$$d_n = \hat{H} \left( \frac{D}{D_o} n \right).$$

The hypothesis $\text{Im} \int_{-\infty}^{\infty} H_\psi \left(j r^{x/2\pi}\right) H_\chi \left(j r^{x/2\pi}\right) dx = 0$ implies that $d_0 = \text{Re} d_0 = 1$. 
Now, for Theorem 2.20, applying Theorem 2.23, and replacing \( \omega = r^{x/2}\pi \), we have that

\[ \varepsilon_{\text{max}} = \sup \left\{ \left| \sum_{m=-\infty}^{\infty} k H_o \left( j r^{m+x/2}\pi \right) - 1 \right| \right\} = \sup \left\{ \left| \sum_{n=-\infty}^{\infty} d_n e^{jnx} - 1 \right| \right\} \]

\[ \leq \sum_{n=1}^{\infty} (|d_{-n}| + |d_n|) = \sum_{n=1}^{\infty} \left( |\hat{H} \left( -\frac{D}{D_o} n \right)| + |\hat{H} \left( \frac{D}{D_o} n \right)| \right) \]

\[ \leq \sum_{n=1}^{\infty} \left( k_1 e^{y_{\text{min}}nD/D_o} + k_2 e^{-y_{\text{max}}nD/D_o} \right) = C_1 \frac{e^{y_{\text{min}}D/D_o}}{1 + e^{y_{\text{min}}D/D_o}} + C_2 \frac{e^{-y_{\text{max}}D/D_o}}{1 + e^{-y_{\text{max}}D/D_o}}, \]

for some \( C_1, C_2 > 0, \ y_{\text{min}} < 1 \) and \( y_{\text{max}} > 1 \). As can be seen \( \varepsilon_{\text{max}} \to 0 \) as \( D \to \infty \). \( \square \)

In conclusion, for a given pair of filters \( H_\psi(s) \) and \( H_\chi(s) \), as the scale density increases, the ripple, together with the maximum reconstruction error \( \varepsilon_{\text{max}} \), tends to decrease with the condition that \( \text{Im} \int_{-\infty}^{\infty} H_\psi \left( j r^{x/2}\pi \right) H_\chi \left( j r^{x/2}\pi \right) dx = 0 \). This condition has always been corroborated for any band-pass rational filter that we have used in the numerical simulations. Nevertheless, we have not already proved the generality of this fact, except for the next subset of rational filters.

**Theorem 2.26.** For a given \( q > 0 \) and an integer \( N \geq 1 \), the function \( H_N(s) = \left( \frac{s/q}{s^2 + s/q+1} \right)^N \) satisfies \( \text{Im} \int_{-\infty}^{\infty} H_N(j r^{x/2}\pi) dx = 0 \) for any \( r > 1 \).

**Proof:** Making \( s = j r^{x} \) we obtain

\[ H_1(j r^{x}) = \frac{j r^{x}/q}{-r^{2x} + j r^{x}/q + 1} = \frac{-j r^{3x}/q + r^{2x}/q^2 + j r^{x}/q}{r^{4x} + (1/q^2 - 2) r^{2x} + 1}. \]

On the other hand,

\[ H_1(j r^{-x}) = \frac{-j r^{-3x}/q + r^{-2x}/q^2 + j r^{-x}/q}{r^{-4x} + (1/q^2 - 2) r^{-2x} + 1} = \frac{j r^{3x}/q + r^{2x}/q^2 - j r^{x}/q}{r^{4x} + (1/q^2 - 2) r^{2x} + 1}, \]

and then \( \text{Re} H_1(j r^{x}) \) is even and \( \text{Im} H_1(j r^{x}) \) is odd.

We have that \( H_{n+1}(j r^{x}) = H_n(j r^{x}) H_1(j r^{x}) \) and supposing that \( \text{Re} H_n(j r^{x}) \) is even and \( \text{Im} H_n(j r^{x}) \) is odd, we have that

\[ \text{Re} H_{n+1}(j r^{x}) = \text{Re} H_n(j r^{x}) \text{Re} H_1(j r^{x}) - \text{Im} H_n(j r^{x}) \text{Im} H_1(j r^{x}) \]
Figure 2.9: Reconstruction error, $\varepsilon_{\text{max}}$, as a function of the scale density, $D$, of a wavelet system, where the product $H(s) = H_\psi(s) H_\chi(s)$ is a filter biqu($q, n$), for several $q$-values and $n = 1, 2, 3, 4$.

is an even function, and

$$\text{Im} \ H_{n+1}(jr^x) = \text{Re} \ H_n(jr^x) \text{Im} \ H_1(jr^x) + \text{Im} \ H_n(jr^x) \text{Re} \ H_1(jr^x)$$

is an odd function, and by mathematical induction $H_N(jr^x)$ is odd for all integer $N \geq 1$.

Then

$$\text{Im} \int_{-\infty}^{\infty} H_N(jr^x) dx = \text{Im} \frac{1}{2\pi} \int_{-\infty}^{\infty} H_N(jr^{x/2\pi}) dx = 0$$

for all integer $N \geq 1$.

**Example 2.27.** In Figure 2.9 is shown the reconstruction error $\varepsilon_{\text{max}}$ of a wavelet transform system, where the filter $H(s) = H_\psi(s) H_\chi(s)$ is a filter biqu($q, n$), for different scale densities $D$. Note the fast convergence as $D$ is increased.

**Example 2.28.** A wavelet system with $D = 2$, that is, $r = 2^{1/D} = 1.414$, and $H(s) = H_\psi(s) H_\chi(s) = \text{biqu}(\sqrt{2}, 3) = \left(\frac{s/\sqrt{2}}{s^2 + s/\sqrt{2} + 1}\right)^3$, has a bound reconstruction error $\varepsilon_{\text{max}} = 20.3\%$. If now we make $D = 4$ we obtain $\varepsilon_{\text{max}} = 1.38\%$, and making $D = 5$ we have that $\varepsilon_{\text{max}} = 0.082\%$. 
2.2.5 Method 2: Building custom filters

The method of this section applies when the decomposition filter and the scale density are already given, and a reconstruction filter maintaining the same scale density is required. In this case, making use of the previously explained Fourier coefficients, we obtain a reconstruction filter with a linear combination of rational functions. The initial real Fourier coefficients of the resulting system, which are the coefficients of most influence in reconstruction error, have a value of zero except $\alpha_0 = 1$ as is required by the equivalent condition of Corollary 2.18.

Method: Given a value $r > 1$ and a band-pass rational function, $H_\psi(s)$, used as decomposition filter for the semidiscrete wavelet transform, follow the next steps in order to find a realizable rational function, $H_\chi(s)$, acting as reconstruction filter, such that $\Psi^*(\omega)X(\omega) = H_\psi(j\omega)H_\chi(j\omega)$ satisfies the condition (1.11) with certain degree of approximation:

1st Take a set of realizable and stable rational functions $\{H_M(s)\}_{1 \leq M \leq 4N+2}$, with $N \in \mathbb{N}$, here called component functions, which could be or not band-pass.

2nd Making $\omega = r^{x/2\pi}$, for $0 \leq n \leq N$ calculate numerically the coefficients

\begin{align}
\alpha_{M,n} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re} \, H_\psi(jr^{x/2\pi}) H_M(jr^{x/2\pi}) \cos nx \, dx , \\
\beta_{M,n} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Im} \, H_\psi(jr^{x/2\pi}) H_M(jr^{x/2\pi}) \sin nx \, dx , \\
A_{M,n} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Im} \, H_\psi(jr^{x/2\pi}) H_M(jr^{x/2\pi}) \cos nx \, dx , \\
B_{M,n} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re} \, H_\psi(jr^{x/2\pi}) H_M(jr^{x/2\pi}) \sin nx \, dx .
\end{align}

(2.22)

3rd Verify that the next linear equation system is linearly independent. If it is not,
go back to $1^{st}$ step and select a different set of component functions.

\[
\begin{pmatrix}
\alpha_{1,0} & \alpha_{2,0} & \cdots & \alpha_{4N+2,0} \\
A_{1,0} & A_{2,0} & \cdots & A_{4N+2,0} \\
\alpha_{1,n} & \alpha_{2,n} & \cdots & \alpha_{4N+2,n} \\
\beta_{1,n} & \beta_{2,n} & \cdots & \beta_{4N+2,n} \\
A_{1,n} & A_{2,n} & \cdots & A_{4N+2,n} \\
B_{1,n} & B_{2,n} & \cdots & B_{4N+2,n}
\end{pmatrix}
\begin{pmatrix}
L_1 \\
L_2 \\
\vdots \\
L_{4N+2}
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad (2.23)
\]

4\textsuperscript{th} Solve the above linear system to find the values \(\{L_M\}_{1 \leq M \leq 4N+2}\) and compute the function \(H_\chi(s) = \sum_{M=1}^{4N+2} L_M H_M(s)\), which has an impulse response \(\chi(t)\) that is an approximated dual wavelet of \(\psi(t)\).

5\textsuperscript{th} Making \(H(j\omega) = H_\psi(j\omega)H_\chi(j\omega)\) apply (2.16) to find the maximum reconstruction relative error, \(\varepsilon_{\text{max}}\). If less error is required go back to $1^{st}$ step, select another set of component functions with a greater $N$ value and repeat the procedure.

\textit{Justification of the method:} If \(H_\psi(s)\) is a band-pass rational function and \(H_\chi(j\omega)\) is realizable and stable, then the product \(H(j\omega) = H_\psi(j\omega)H_\chi(j\omega)\) also will be a band-pass rational function.

In agree with Corollary 2.18, all the coefficients \(\alpha_n, \beta_n, A_n,\) and \(B_n\) have to be equal to zero except \(\alpha_0 = 1\). Since \(\chi(t)\) is not an exact dual wavelet of \(\psi(t)\), in general all these coefficients will be different to zero. But the geometric rate of convergence proved in Theorem 2.23 justifies why the attention must be paid in make zero the coefficients at the beginning of the Fourier series, because they cause the grater effect in the ripple of the infinite summation (1.11), which is the source of the reconstruction error.

Now, when the system (2.23) is solved, it is found a linear combination

\[
H(j\omega) = \sum_{M=1}^{4N+2} L_M H_\psi(\omega) H_M(j\omega) = H_\psi(j\omega) \sum_{M=1}^{4N+2} L_M H_M(j\omega) = H_\chi(j\omega) H_\psi(j\omega)
\]
Table 2.1: Coefficients (2.22) for $r = 1.414$, $H_\psi(s) = \text{biqu}(\sqrt{2}, 3) = \left(\frac{s/\sqrt{2}}{s^2 + s\sqrt{2} + 1}\right)^3$, and component functions $H_1(s) = \frac{s}{(s+1)(s+0.5)}$, $H_2(s) = \frac{s}{(s+1)^2}$, and $H_3(s) = \frac{s}{(s+1)(s+2)}$.

<table>
<thead>
<tr>
<th>$\alpha_{1,0}$</th>
<th>0.027637206405</th>
<th>$\alpha_{2,0}$</th>
<th>0.020514987261</th>
<th>$\alpha_{3,0}$</th>
<th>0.013818603203</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{1,0}$</td>
<td>$-0.39931 \times 10^{-15}$</td>
<td>$A_{2,0}$</td>
<td>$-0.04764 \times 10^{-15}$</td>
<td>$A_{3,0}$</td>
<td>$0.03585 \times 10^{-15}$</td>
</tr>
<tr>
<td>$\alpha_{1,1}$</td>
<td>0.080317652400</td>
<td>$\alpha_{2,1}$</td>
<td>0.070923030365</td>
<td>$\alpha_{3,1}$</td>
<td>0.040158826200</td>
</tr>
<tr>
<td>$\beta_{1,1}$</td>
<td>$-0.080317652400$</td>
<td>$\beta_{2,1}$</td>
<td>$-0.070923030365$</td>
<td>$\beta_{3,1}$</td>
<td>$-0.040158826200$</td>
</tr>
<tr>
<td>$A_{1,1}$</td>
<td>$-0.033976982619$</td>
<td>$A_{2,1}$</td>
<td>$0.000000000000$</td>
<td>$A_{3,1}$</td>
<td>$0.016988491309$</td>
</tr>
<tr>
<td>$B_{1,1}$</td>
<td>$-0.033976982619$</td>
<td>$B_{2,1}$</td>
<td>$0.000000000000$</td>
<td>$B_{3,1}$</td>
<td>$0.016988491309$</td>
</tr>
</tbody>
</table>

whose first coefficients have the correct values. Then $H(j\omega) = \Psi^*X(\omega)$ satisfies the condition (1.11) with certain degree of approximation, where the maximum reconstruction relative error, $\varepsilon_{\text{max}}$, is given by (2.16).

**Remark 2.29.** Since the poles of a band-pass rational function are real or are given in complex conjugate pairs, from Lemma 2.22 we have that $|y_{\text{min}}| = |y_{\text{max}}| + \frac{2\pi^2}{\ln r}$. Then by Corollary 2.16 and Theorem 2.23 we obtain $\lim_{n \to \infty} \frac{|c_n|}{|c_{-n}|} = \lim_{n \to \infty} \frac{C_1}{C_2} e^{-2\pi^2 n / \ln r} = 0$. Then by (2.14), we have that $\beta_{M,n} \approx -\alpha_{M,n}$, and $B_{M,n} \approx A_{M,n}$ for sufficiently large $n$. When this is the case, the rows containing $\beta_{M,n}$ and $B_{M,n}$ can be removed of the system (2.23).

And, in spite of the fact that we could not prove that $\text{Im} c_0 = A_0 = 0$ for any band-pass rational function, we have always corroborated this in the numerical computations. When this is the case the row containing $A_{M,0}$ can also can be removed of the equation system (2.23).

**Example 2.30.** In Table 2.1 is shown the calculation of coefficients (2.22) for $r = 1.414$, $H_\psi(s) = \text{biqu}(\sqrt{2}, 3) = \left(\frac{s/\sqrt{2}}{s^2 + s\sqrt{2} + 1}\right)^3$, and component functions $H_1(s) = \frac{s}{(s+1)(s+0.5)}$, $H_2(s) = \frac{s}{(s+1)^2}$, and $H_3(s) = \frac{s}{(s+1)(s+2)}$, for $n = 1, 2$. As can be seen, these coefficients satisfy the conditions of Remark 2.29.

Taking only the three rows corresponding to $\alpha_{M,0}, \alpha_{M,1}$ and $A_{M,1}$ to solve the system (2.23), we obtain the coefficients $L_1 = 113.514$, $L_2 = -257.100$, and $L_3 =$
and the built filter $H_x(s) = \frac{83.44s^3 + 38.33s^2 + 83.44s}{s^4 + 4.5s^3 + 7.0s^2 + 4.5s + 1.0}$, In this case $\varepsilon_{\text{max}} = 0.58\%$.

<table>
<thead>
<tr>
<th>$\alpha_{M,0}$</th>
<th>1.000000000002</th>
<th>$\alpha_{M,1}$</th>
<th>$-5.07371 \times 10^{-12}$</th>
<th>$\alpha_{M,2}$</th>
<th>$-0.002676347466$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{M,1}$</td>
<td>5.31991 $\times 10^{-12}$</td>
<td>$\beta_{M,2}$</td>
<td>0.002676347465</td>
<td>$A_{M,1}$</td>
<td>$-9.56567 \times 10^{-13}$</td>
</tr>
<tr>
<td>$A_{M,2}$</td>
<td>$-5.82030 \times 10^{-12}$</td>
<td>$B_{M,1}$</td>
<td>1.02643 $\times 10^{-12}$</td>
<td>$B_{M,2}$</td>
<td>$-9.61818 \times 10^{-14}$</td>
</tr>
</tbody>
</table>

Table 2.2: Coefficients (2.12) for $r = 1.414$, $H_\psi(s) = \text{biqu}(\sqrt{2}, 3) = \left(\frac{s/\sqrt{2}}{s^2 + s/\sqrt{2} + 1}\right)^3$,

whose resulting real Fourier coefficients (2.12) are shown in Table 2.2. We can see how the first six coefficients are equal to zero, within the achievable numerical accuracy, except $\alpha_0 = 1$. The resulting system has a bound in the reconstruction error of $\varepsilon_{\text{max}} = 0.58\%$.

### 2.3 Delay in wavelet signal processing

When the wavelet transform is calculated in digital systems, some delay in the process can be tolerated because the signal can be temporally stored at memory. But in analog systems, it is better to make the signal processing in real-time, that is, with zero-delay. In this way, the signal does not have to be stored.

In this section are explained some constraints that must be satisfied by the decomposition and reconstruction wavelets in order to make possible a real-time processing. These constraints are satisfied by the wavelets presented in this work, but not by other classical wavelets. For example, when the Morlet wavelet is used, a delay proportional to the largest implemented scale is inevitable. Another presented result is that the fast algorithm of the discrete wavelet transform implies a delay, also proportional to the largest implemented scale.
2.3.1 Decomposition delay

From the definitions of the continuous (1.3), semidiscrete (1.8), and discrete (1.13) wavelet transforms, can be made a unified definition

\[ w(a, b) = k_a \int_{-\infty}^{\infty} f(t) \psi^* \left( \frac{t-b}{a} \right) dt , \]

(2.24)

where: \( k_a = \frac{1}{\sqrt{a}} \) for the continuous case; \( a = r^m \) and \( k_a = \frac{1}{r^m} \) for the semidiscrete case; and \( a = r^m, b = nb_0 r^m \) and \( k_a = \frac{1}{\sqrt{a}} \) for the discrete case.

Considering that the origin can be placed anywhere, with some cautions in the discrete case due to the time discretization, we only examine the value of \( w(a, b) \) at time \( b = 0 \). Now, suppose that the wavelet \( \psi(t) \) has an effective support in the range \( t \in (-c, d) \), with \( c, d > 0 \). Then we have that

\[ w(a, 0) = k_a \int_{-ac}^{0} f(t) \psi^* \left( \frac{t}{a} \right) dt + k_a \int_{0}^{ad} f(t) \psi^* \left( \frac{t}{a} \right) dt . \]

(2.25)

Considering that at the time \( b = 0 \) we have only the values of \( f(t) \) corresponding to \( t \leq 0 \) we could already calculate the left integral, but we had to wait a time \( ad \) to be able to compute the right integral. It implies that to make possible a real-time decomposition the wavelet must satisfy

\[ \psi(t) = 0 \quad \text{for} \quad t > 0 . \]

(2.26)

The wavelet corresponding to any rational filter, which is the inverse in time of the impulse response, satisfies this condition.

The Morlet wavelet does not satisfy this condition because it is symmetrical respect to the origin. Then we need to wait a time \( D = ad \) to have all the needed information to compute the right-hand integral. This delay is scale dependent and the worst case corresponds to the higher scale.

2.3.2 Reconstruction delay

The integrals \( \int_{-\infty}^{\infty} w(a, b) \psi_{a,b}(t) db \) and \( \int_{-\infty}^{\infty} w_m(b) \chi_m(t-b) db \) that appear in the inverse continuous (1.4) and in the inverse semidiscrete (1.9) wavelet transforms can
be established together as
\[ f_a(t) = k_a \int_{-\infty}^{\infty} w(a, b) \phi \left( \frac{t-b}{a} \right) \, db, \]  
(2.27)

where: \( \phi(t) = \psi(t) \) and \( k_a = \frac{1}{\sqrt{a}} \) for the continuous case; and \( \phi(t) = \chi(t) \), \( a = r^m \) and \( k_a = \frac{1}{r^m} \) for the semidiscrete case.

Considering that the origin can be placed anywhere, we only examine the value of \( f_a(t) \) at time \( t = 0 \). Now, suppose that the wavelet \( \phi(t) \) has an effective support in the range \( t \in (-c, d) \), with \( c, d > 0 \), we have that
\[ f_a(0) = k_a \int_{-ad}^{0} \phi \left( \frac{-b}{a} \right) \, db + k_a \int_{0}^{ac} \phi \left( \frac{-b}{a} \right) \, db. \]  
(2.28)

Considering that at the time \( t = 0 \) we have only the values of \( w(a, b) \) corresponding to \( b \leq 0 \) we could already calculate the left integral, but we had to wait a time \( D = ac \) to be able to compute the right integral. It implies that to make possible a real-time decomposition the wavelet must satisfy
\[ \phi(t) = 0 \quad \text{for} \quad t < 0. \]  
(2.29)

The wavelet corresponding to any rational filters, which is its impulse response, satisfies this condition.

The Morlet wavelet does not satisfy this condition because it is symmetrical respect to the origin. We have to wait a time \( D = ac \) to have the needed information to compute the right-hand integral. This delay is scale dependent and the worst case corresponds to the higher scale.

With a slightly more elaborated argument, due to the time discretization, can be shown the same condition for the case of the inverse discrete wavelet transform.

We can see that a wavelet can not satisfy (2.26) and (2.29) simultaneously. Then, to achieve zero-delay decomposition-reconstruction we have to use a wavelet for decomposition and a different wavelet for reconstruction.
2.3.3 Fast algorithm delay

Suppose that we already have the pair of FIR filters to implement the fast algorithm of the discrete wavelet transform. To begin the decomposition we need the sequence $S_{m_0,n}$ that is computed with (1.18). Using this equation and applying the above argument, we could show that the scale function must satisfy $\phi(t) = 0$ for $t > 0$ as a condition to achieve a zero-delay decomposition. However, the common practice is not to obtain the sequence by (1.18), but it is just taken as the mean value for each sample period. Then, in this section is studied the delay effect due to the decomposition and reconstruction algorithms by themselves, without considering how the sequence $S_{m_0,n}$ is obtained.

In Figure 2.10 is schematically shown the application of the decomposition algorithm, (1.16) and (1.17), for filters with 2 coefficients like in the Haar wavelet case, Figure 2.10(a), and for filters with 4 coefficients like in the Daubechies db2 case, Figure 2.10(b). The arrows indicate the scale coefficients that are needed to compute the scale and wavelet coefficients of the next level. The emphasis is in the exact time at which the coefficients are available. In both cases, the decomposition is done along 3 levels, from the sequence $S_{0,n}$ until the sequence $S_{3,n}$. The sample period of the sequence at the lowest scale is $b_0$.

Note that the decomposition is made in blocks of coefficients. These blocks are overlapped in the case (b). In these schemes the worst case delay corresponds to coefficients at the beginning of each block, like $S_{0,0}, S_{0,8}, S_{0,16}, \ldots$ when we consider three levels of decomposition, or like $S_{0,0}, S_{0,4}, S_{0,8}, \ldots$ when we consider two levels.

For example, for three decomposition levels, the worst case delay is the time from $S_{0,0}$ to $S_{3,0}$, that is $7b_0$ for the case of Figure 2.10(a), and $21b_0$ for the case of Figure 2.10(b).

When one coefficient of the higher level is obtained via decomposition, we already have all the needed coefficients to reconstruct a group of scale coefficients of the lowest scale, included the coefficient with the worst delay in the decomposition. Then no
Figure 2.10: Schematic application of the fast wavelet transform algorithm using FIR filters with 2 coefficients (a), and using FIR filters with 4 coefficients (b).

extra delay is needed to perform the reconstruction algorithm.

Let $D_L$ be the worst case delay at the decomposition level $L$, then directly from Figure 2.10 we have that

$$D_1 = b_0(N - 1)$$

$$D_2 = b_0\left[(N - 1) + (N - 1)2\right]$$

$$D_3 = b_0\left[(N - 1) + (N - 1)2 + (N - 1)2^2\right]$$

$$\vdots$$

where $N$ is the number of coefficients of the FIR filter. Then in general, for $L \geq 1$, we have that

$$D_L = b_0 \sum_{M=1}^{L} (N - 1)2^{M-1} = b_0(N - 1)(2^L - 1).$$

In conclusion, the decomposition-reconstruction delay in the fast algorithm of the
discrete wavelet transform depends on the levels of decomposition and on the number of coefficients of the FIR filters.

2.4 Finite number of scales

In previous sections were explained two methods to design systems performing the semidiscrete wavelet transform. With these methods can be found decomposition and reconstruction filters to achieve reconstruction errors as low as be required, but with the implicit condition of implement an infinite number of scales.

Practical systems can have only a finite number of scales over a limited frequency band. Hence the condition (1.11) is not fulfilled any more, and instead of an all-pass flat response, the finite summation has a band-pass behavior. In this section we are interested in the behavior of the reconstruction error over the work frequency band. To study this behavior we make use of some specific examples. We give a simple compensation to this problem and some directions for future research.

A way to quantify the reconstruction error, due exclusively to the band-pass behavior, is explained next.

2.4.1 Reconstruction error in steady-state

When the sinusoidal signal \( f_{in}(t) = \sin\left(2\pi \frac{t}{T}\right) \) is applied to a linear system in steady state, the output signal \( f_{out}(t) = A \sin\left(2\pi \frac{t}{T} + \phi\right) \) is obtained, where \( T \) is the period of both signals, and \( A \) and \( \phi \) are the amplitude and phase of the output signal. In this case, the reconstruction relative error (2.15) is given by

\[
\varepsilon(A, \phi) = \frac{\|f_{out} - f_{in}\|}{\|f_{in}\|} = \frac{\sqrt{\frac{1}{T} \int_0^T |A \sin\left(2\pi \frac{t}{T} + \phi\right) - \sin\left(2\pi \frac{t}{T}\right)|^2 dt}}{\sqrt{\frac{1}{T} \int_0^T |\sin\left(2\pi \frac{t}{T}\right)|^2 dt}}
\]

\[
= \sqrt{A^2 - 2A \cos \phi + 1},
\]
where the defined integrals have been solved applying the fact that \(\frac{B}{\sqrt{2}}\) is the rms value of a sinusoidal signal of amplitude \(B\), and the identity \(k \sin(ft + \theta) = k_1 \sin(ft) + k_2 \cos(ft)\), where \(\cos \theta = \frac{k_2}{k}\) and \(\sin \theta = \frac{k_1}{k}\).

Using the previous result the reconstruction error can be expressed as a function of the frequency

\[
\varepsilon(\omega) = \varepsilon(A(\omega), \phi(\omega)).
\]  

(2.31)

### 2.4.2 Band-pass behavior

**Example 2.31.** The system of this example has 16 scales, and has a scale density of \(D = 2\), implying a scale ratio of \(r = \sqrt{2}\). The decomposition filter is biqu(1.414, 1) and the reconstruction filter is only a normalization constant, resulting in the normalized product \(H_m(s) = H_{\psi_m}(s)H_{\chi_m}(s) = \frac{0.2683 \omega_m s}{s^2 + 0.7071 \omega_m s + \omega_m^2}\), where \(\omega_m = \frac{(2\pi)(19.027 \text{ Hz})}{r^m}\), for \(m = 1, 2, 3, \ldots, 16\).

The frequency response of each filter \(H_m(j\omega)\), and of the total summation \(\sum_{m=1}^{16} H_m(j\omega)\), are shown in Figure 2.11(a) for the real parts, and in Figure 2.11(b) for the imaginary parts.

In Figure 2.11(d) is shown the reconstruction error \(\varepsilon(\omega)\) of the system in a steady state with a sinusoidal input. It was obtained applying (2.31) to the frequency response of the system shown in Figure 2.11(c).

**Example 2.32.** The system of this example has 41 scales, and a scale density of \(D = 5\), i.e., a scale ratio of \(r = \sqrt[4]{2}\). The normalized product between the decomposition and the reconstruction filters is given by \(H_m(s) = H_{\psi_m}(s)H_{\chi_m}(s) = \frac{4.323 (\omega_m s)^3}{(s^2 + 0.7071 \omega_m s + \omega_m^2)^3}\), where \(\omega_m = \frac{(2\pi)(18.379 \text{ Hz})}{r^m}\), for \(m = 1, 2, 3, \ldots, 41\).

The frequency response of each filter \(H_m(j\omega)\), and of the total summation \(\sum_{m=1}^{41} H_m(j\omega)\), are shown in Figure 2.12(a) for the real parts, and in Figure 2.12(b) for the imaginary parts.
Figure 2.11: Frequency response of the system described in Example 2.31. Real (a) and imaginary (b) parts of the individual filters and their summation. (c) Magnitude and phase of the whole system. (d) Reconstruction error of a sinusoidal input in steady state. (Matlab simulations.)

In Figure 2.12(d) is shown the reconstruction error $\varepsilon(\omega)$ of the system in a steady state with a sinusoidal input. It was obtained applying (2.31) to the frequency response of the system shown in Figure 2.12(c).

2.4.3 Simple compensation

Now we present a simple way to compensate the band-pass behavior of a semidiscrete wavelet system with a finite number of scales.

Suppose that $H(s) = k H_\psi(s) H_\chi(s)$ is the normalized product between the decomposition and reconstruction filters. The frequency response of the wavelet system with a finite number of scales is given by

$$H_f(j\omega) = \sum_{m=m_a}^{m_b} H(r^m j\omega),$$

where $m_a < m_b$ are the indexes of the first and the last scales considered in the finite summation. A simple way to compensate the bandpass behavior of this finite
Figure 2.12: Frequency response of the system described in Example 2.32. Real (a) and imaginary (b) parts of the individual filters and their summation. For clarity, instead of 5, only 1 scale per octave has been plotted. (c) Magnitude and phase of the whole system. (d) Reconstruction error for a sinusoidal input in steady state. (Matlab simulations.)

The meaning of (2.33) and (2.34) is that the filters $H_a(j\omega)$ and $H_b(j\omega)$ approximate the corresponding lateral partial summations only inside the work frequency band.

We illustrate the method by an example.

**Example 2.33.** For the same wavelet system described in Example 2.31 we use the lateral filters

$$H_a(s) = \frac{0.780s^2 + 0.2923s}{s^2 + 0.3748s + 0.0984},$$

$$H_b(s) = \frac{117.2s + 12340}{s^2 + 150.3s + 15830}.$$

$$(2.36)$$
Figure 2.13: Fitted lateral filters $H_b(j\omega)$ and $H_a(j\omega)$.

whose frequency response is shown in Figure 2.13.

These filters were numerically fitted to the partial summations $\sum_{m=-\infty}^{0} H(r^m j\omega)$ and $\sum_{m=17}^{\infty} H(r^m j\omega)$ in the interval (74.33mHz, 13.45Hz).

In Figure 2.14 can be appreciated how the left and right lateral filters compensate the band-pass behavior of the partial summation.

The low-pass and high-pass lateral filters can be very simple, as in this case, because we are only looking for approximations in the frequency interval of interest.

2.4.4 Further research

The simple way to compensate the above explained band-pass behavior could work in some applications, but it is not the best approach.

As will be seen in the next chapter, the utility of the wavelet transform consists in the possibility of performing non-linear processing when the signal is in the wavelet domain, i.e., between the decomposition and the reconstruction filters where some wavelet components can be selectively processed.

In the case of the denoising task, the components to be processed are selected depending on their absolute value. As a result, only some components pass through the reconstruction filters until the sum-point. Then, the lower $m_a$ and the higher
Figure 2.14: (a) Frequency response of the wavelet system of Examples 2.34 with uncompensated and compensated band-pass behavior. (b) Relative reconstruction error, $\varepsilon(\omega)$, for both cases. (Matlab simulations.)

$m_b$ scales of the partial summation (2.32) varies on time. In this case the previous approach to compensate the band-pass behavior along the complete work frequency band does not apply.

In this section we present a pair of ideas as a guide for future research.

1. The first proposal is, as in the previous section, the presence of a pair of low-pass and high-pass lateral filters, but instead to cover the whole summation, in this case they have adaptability to cover only a smaller partial summation.

In Example 2.33 we employ the lateral filters to compensate the band-pass behavior of the summation $\sum_{m=1}^{16} H(r^m j\omega)$. In this case we need adaptive lateral filters to compensate the variable partial summation $\sum_{m=m_a}^{m_b}$, where $m_a$ and $m_b$ change now and then.

2. To explain the second proposal we first study the case of the Morlet wavelet used in the semidiscrete wavelet transform. We make the assumption that this wavelet covers all the requirements to be used in this type of wavelet transform, as actually it does, but here we will not make a formal review in this sense.
Figure 2.15: (a) Complex Morlet wavelet $\psi(t) = e^{-t^2/2} \cos 5t$. (b) Frequency spectrums of the complex Morlet wavelet scaled as a semidiscrete wavelet and the summation.

In Figure 2.15(a) is shown the Morlet wavelet $\psi(t) = e^{-t^2/2} \cos 5t$. The imaginary part of the frequency spectrum of this wavelet is zero. In Figure 2.15(b) are shown the spectrums of 24 Morlet wavelets scaled as semidiscrete wavelets. Note that the summation can not have spikes at the left and right edges, as in the case of a summation of biquadratic filters with high selectivity like the one shown in Figure 2.12. As a consequence, this summation can not have an important phase-shift.

Our second proposal is to find filters that join the best of the band-pass rational filters, zero-delay and easy implementation in analog circuits, and the best of the Morlet wavelet, zero imaginary part in its frequency spectrum.

In Example 2.31 and 2.32 could be seen how the reconstruction error due to the band-pass behavior of the system is much greater than the error bound due to the ripple (2.16).

On the other hand, the interior of the frequency response of the finite summation of Morlet filters, shown in Figure 2.15(b), practically is not affected by any band-pass behavior. Therefore, if the filters of the second proposal can be identified then the results of this chapter, concerning to the convergence to zero of the reconstruction error when the scale density is increased, will recover their complete usefulness.
2.5 Reconstruction examples

For the next examples we use the probe signals tone$2$ and chirp$2$ shown in Figure 2.3(a) and 2.5(a).

**Example 2.34.** For this reconstruction example we use the wavelet systems previously described in Examples 2.31 and 2.33. That is, a system with 16 scales using the filter biqu(1.414, 1) with a scale density of 2 scales per octave, and the same system plus a low-pass and a high-pass lateral filters that compensate the band-pass behavior due to the finite number of scales. Both systems operate in the range from 62.5mHz to 16Hz.

The probe signal tone$2$ is shown in Figure 2.16(a) together with the two reconstructed signals, which are very near to each other. The error $|f_{out} - f_{in}|$ is shown in Figure 2.16(b) for the system without compensation, and in Figure 2.16(c) for the system compensated with the lateral filters. Note the reduction in the error.

The probe signal chirp$2$ is shown in Figure 2.16(d) together with the two reconstructed signals. Note how in the center can be appreciated some phase-shift. The error $|f_{out} - f_{in}|$ is shown in Figure 2.16(e) for the system without compensation, and in Figure 2.16(f) for the system compensated with the lateral filters. Note how the error is lower and flatter in the second case.

**Example 2.35.** For this example we use the wavelet system described in Example 2.32. That is a system with 41 scales, using the prototype filter biqu(1.414,3), with a scale density of 5 scales per octave operating in the range 62.5mHz to 16Hz.

The probe signal tone$2$ is shown in Figure 2.17(a) together with the reconstructed signal, which is very close. The error $|f_{out} - f_{in}|$ is shown in Figure 2.17(b).

The probe signal chirp$2$ is shown in Figure 2.17(c) together with the reconstructed signal. Note how the amplitude of the reconstructed signal increases at low and high frequencies, as could be expected from the frequency response of the system shown in Figure 2.12(c). The error $|f_{out} - f_{in}|$ is shown in Figure 2.17(d).
Figure 2.16: (a) Probe signal tone2 plotted together with the two reconstructed signals obtained with the wavelet systems described in Example 2.34. (b) and (c) Error $|f_{\text{out}}(t) - f_{\text{in}}(t)|$ for each case. (d) Probe signal chirp2 plotted together with the two reconstructed signals obtained with the same wavelet systems. (e) and (f) Error $|f_{\text{out}}(t) - f_{\text{in}}(t)|$ for the two last cases. (Matlab simulations.)

Figure 2.17: (a) Probe signal tone2 plotted together with the reconstructed signal obtained with the wavelet system described in Example 2.35. (b) Error $|f_{\text{out}}(t) - f_{\text{in}}(t)|$. (c) Probe signal chirp2 plotted together with the reconstructed signal obtained with the same wavelet system. (d) Error $|f_{\text{out}}(t) - f_{\text{in}}(t)|$ for this last case. (Matlab simulations.)
Chapter 3

Principles of Analog Wavelet Denoising

The utility of the wavelet transform is in allow non-linear processing when the signal is in the wavelet domain. This is achieved putting some processing blocks between the decomposition and reconstruction filters of Figure 3.1.

Denoising has been a typical application of the discrete wavelet transform. It is a non-linear operation because the processing is developed over selected wavelet coefficients. The decision whether a particular coefficient is considered for the processing or not, depends on the comparison of its value against a threshold level.

In this chapter we take the well known shrinkage or thresholding denoising technique and explain how it can be applied together with the analog wavelet systems above described. The effect of the thresholding function and the effect of the reconstruction filters in the smoothness of the wavelet components is analyzed. The distribution of an input component of white noise into the different scales is reviewed. Finally, the denoising performance of the presented analog implementation is compared with the performance of discrete wavelet systems.
3.1 Wavelet denoising

The wavelet denoising, better known as *shrinkage* or *thresholding*, has been extensively applied with the discrete wavelet transform. From several published denoising operations we only consider the hard thresholding because of its simplicity of implementation in analog circuits. After a brief revision, we present a variation of the hard thresholding that overcomes an identified drawback.

3.1.1 Hard thresholding

The hard thresholding operation is covered in the most literature of the discrete wavelet transform.

**Definition 3.1.** The *hard thresholding operation* over the discrete wavelet transform $w_{m,n}$ is defined to be

$$
\tilde{w}_{m,n} = \begin{cases} 
\ w_{m,n} & \text{if } |w_{m,n}| \geq \lambda \\
\ 0 & \text{if } |w_{m,n}| < \lambda 
\end{cases}
$$

(3.1)

where the threshold level $\lambda > 0$ can be a constant or a function of the scale.

The main idea behind this operation is to keep the coefficients whose absolute value is larger than the threshold level, since they are considered to contain information
of the original noise-free signal, while the coefficients below the threshold level are removed because they are mainly considered as noise.

The last definition is adapted to the semidiscrete wavelet transform as follows.

**Definition 3.2.** Let \( w(t) \) be the input to the thresholding operation and \( \tilde{w}(t) \) the output. The hard thresholding is defined by

\[
\tilde{w}(t) = \begin{cases} 
  w(t) & \text{for } |w(t)| > \lambda \\
  0 & \text{for } |w(t)| \leq \lambda 
\end{cases}
\]  

(3.2)

where the threshold level \( \lambda > 0 \) can be a constant or a function of the scale.

This operation depends only on the current values of the wavelet component \( w(t) \) as is illustrated in Figure 3.2(a).

### 3.1.2 Delayed hard thresholding

Suppose that we apply the hard thresholding to a signal \( w(t) \) composed by a sinusoidal component plus a low level noise component. When the sinusoidal crosses the horizontal axis, in the proximities of the levels \( \lambda \) or \(-\lambda\), the higher frequency
oscillations of the noise component are able to introduce jump discontinuities of \( \lambda \)-amplitude in the output signal \( \tilde{w}(t) \). These discontinuities can be greater than the noise that we wish to eliminate, as is illustrated in the example of Figure 3.2(b).

To avoid this inconvenient we propose the use of the next variant of the hard thresholding.

**Definition 3.3.** Let \( w(t) \) be the input to the thresholding operation and \( \tilde{w}(t) \) the output. The *delayed hard thresholding* is defined by

\[
\tilde{w}(t) = \begin{cases} 
  w(t) & \text{if } |w(t)| \geq \lambda \text{ for some } t \in [t-T,t] \\
  0 & \text{if } |w(t)| < \lambda \text{ for all } t \in [t-T,t]
\end{cases}
\]

(3.3)

where the threshold level \( \lambda > 0 \) can be a constant or a function of the scale, and the *delay time* \( T \) is a function of the scale defined by

\[
T_m = r^m T_0
\]

(3.4)

where \( T_0 \) and \( T_m \) are the delays at the scales 0 and \( m \), and \( r \) is the scale ratio of the wavelet system.

The time delay included in this modified operation prevents that the signal being canceled when its level goes through the range \([-\lambda, \lambda]\), as is illustrated in Figure 3.2(c). As can be verified from the previous definition, at most two jump discontinuities can be introduced by the thresholding operation in any time interval of length \( T \).

### 3.2 Smoothness of the wavelet components

Some discontinuities in the wavelet components are introduced by the hard thresholding operation. This could be tolerated for some applications. But continuous signals, or signals many times continuously differentiable could be required for many practical applications.

To implement wavelet systems like the ones proposed in this work we have to choose the decomposition and the reconstruction filters. The decomposition filter can
be any band-pass rational function. While the reconstruction filter can be any kind of stable rational function. Nevertheless, the low-pass and the band-pass filters have the advantage of smoothing the input signal and its derivatives.

In this section we analyze the effect of the thresholding operation and the reconstruction filters in the smoothness of the wavelet component. Smoothness can be measured, between other ways, considering differences of the function and a translated version of itself [30]. But in this work, the smoothness of a signal is measured by counting the number of continuous derivatives it has [10]. This simplifies the design of the wavelet filters in the frequency-domain for two reasons:

1) The number of continuous derivatives that a function has, is directly related with the decline of its Fourier transform when the frequency tends to infinity, i.e., with the asymptotic slope that the function presents in its Bode plot.

2) The asymptotic slope of the rational filters can be directly determined from the orders of their numerator and denominator polynomials.

3.2.1 Effect of the thresholding operation in the smoothness

With the next definition [10] we can make precise statements about the smoothness of a function. It is just a short way to say how many continuous derivatives a function possess.

**Definition 3.4.** Given $n \in \mathbb{N}$, we say that a function $f(t)$ defined on $\mathbb{R}$ is $C^n$ if it is $n$-times continuously differentiable on $\mathbb{R}$. $C^0$ means that $f(t)$ is continuous on $\mathbb{R}$. $C^{-1}$ means that $f(t)$ defined on $\mathbb{R}$ is piecewise continuous on any finite subinterval except for at most finitely many points, with only jump discontinuities.

It is evident that any physically realizable signal is $L^1$ and $L^2$ on $\mathbb{R}$, and is $C^n$ with $n \geq -1$.

We already know that the hard thresholding operation introduces discontinuities in the wavelet component. This fact is formally established in the next proposition.
Proposition 3.5. If the output signal of a threshold operation is different to zero then it is $C^{-1}$.

This can be easily verified from Definition 3.3 with the help of Figure 3.2.

3.2.2 Smoothing effect of the reconstruction filter

The reconstruction filter have to be a realizable and stable rational function of any kind that fulfill the condition 1.11 within certain approximation degree. But the use of low-pass or band-pass filters has the effect of smoothing the signal, because for every $-20\,\text{dB}_{\text{dec}}$ in the slope of the right hand asymptote of the Bode plot of the reconstruction filter, the wavelet component wins one more continuous derivative. This result is formally stated in the next theorem.

Theorem 3.6. When a signal $f(t)$, $C^n$, $L^1$ and $L^2$ on $\mathbb{R}$, pass through a realizable and stable rational filter, the output of the filter $g(t)$ becomes $C^{n+q-p}$, where $p$ and $q$ are the orders of the numerator and denominator polynomials of the rational filter.

Proof: A realizable and stable rational filter has the form $H(s) = \frac{a_p s^p + a_{p-1} s^{p-1} + \cdots + a_0}{s^q + b_q s^{q-1} + \cdots + b_0}$, where $a_0, \ldots, a_p, b_0, \ldots, b_{q-1} \in \mathbb{R}$ and $a_p \neq 0$, with $p \leq q$, and the denominator is a strictly Hurwitz polynomial.

Suppose that the signal $f(t)$, $C^n$, $L^1$ and $L^2$ on $\mathbb{R}$, pass through the filter $H(s)$ producing the output $g(t)$, then $g(t) = f(t) * h(t)$, where $h(t)$ is the impulse response of $H(s)$.

We have that: If $f(t)$ is $C^n$ and $L^2$ on $\mathbb{R}$, then its Fourier transform $F(\omega)$ will satisfy $|F(\omega)| < K/|\omega|^{n+2}$ for some $K > 0$. [31, Eq. (3.57)]

Now, we define $s = 1/z$, where $s = \sigma + j\omega$ and $z = \nu + j\gamma$, then $j\omega = 1/j\gamma = -j/\gamma$.

We also define $\tilde{H}(z) = \frac{H(1/z)}{z^{q-p}} = \frac{a_p + a_{p-1} z + \cdots + a_0 z^p}{1 + b_q z + \cdots + b_0 z^q}$. Note that $|\tilde{H}(j\gamma)|$ is continuous in $\gamma \in [-1,0]$, and then $|\tilde{H}(j\gamma)|$ is bounded in this range, that is, $\left|\frac{H(1/j\gamma)}{(j\gamma)^{q-p}}\right| = |\tilde{H}(j\gamma)| \leq L$ for some $L > 0$. Then $|H(j\omega)| = |H(1/j\gamma)| = |\tilde{H}(j\gamma)| |(j\gamma)^{q-p}| \leq L |\gamma^{q-p}| = L/|\omega^{q-p}|$ for $\gamma \in [-1,0]$, that is, for $\omega \in [1, \infty)$.

Then convolution theorem says: If $f(t)$ and $g(t)$ are $L^1$ on $\mathbb{R}$ then $\mathcal{F}\{f \ast g(t)\} = F(\omega) G(\omega)$ [10, Theorem 3.21]. We can apply this result in $g(t) = f(t) \ast h(t)$, because the impulse response $h(t)$ of any stable function $H(s)$ is $L^1$ on $\mathbb{R}$. Then we obtain
Figure 3.3: (a) Piecewise differentiable signal. (b) Signal after being filtered by a biqu($\sqrt{0.5}, 1$) filter. (c) Signal after being filtered one more time by a biqu($\sqrt{0.5}, 1$) filter. In each stage the first derivatives of the signal are shown until the first discontinuous derivative.

\[ G(\omega) = F(\omega)H(j\omega), \text{ and then } |G(\omega)| = |F(\omega)| \cdot |H(j\omega)| < \frac{K}{|\omega|^{n+2}} \cdot \frac{L}{|\omega|^{q-p}} = \frac{KL}{|\omega|^{n+2+q-p}}. \]

Additionaly If \( f(x) \) is \( L^1 \) on \( \mathbb{R} \), then its Fourier transform is uniformly continuous on \( \mathbb{R} \) [10, Theorem 3.8]. Then \( F(\omega) \) together with the product \( |F(\omega)| \cdot |H(j\omega)| \) are bounded for \( \omega \in \mathbb{R} \), that is \( |F(\omega)| \cdot |H(j\omega)| \leq M \) for some \( M > 0 \).

Taking \( N = \max\{2KL, 2M\} \) and \( \epsilon = 1 \) we have that \[ |G(\omega)| = |F(\omega)| \cdot |H(j\omega)| \leq \frac{N}{1+|\omega|^n+p+q+2}. \]

Finally we just apply: If there exist a constant \( N \) and \( \epsilon > 0 \) such that \( |F(\omega)| \leq \frac{N}{1+|\omega|^n+p+q+2}, \) then \( f(t) \) is \( C^m \) [32, Proposition 2.1].

The importance of this result lies in the fact that it is applicable to any low-pass or band-pass rational filter and to any physically realizable signal.

An example of this property is shown in Figure 3.3, where the discontinuous signal shown in Figure 3.3(a) is filtered by two consecutive filters biqu($\sqrt{0.5}, 1$). The output signals in each state are shown in Figure 3.3(b) and 3.3(c), together with their firsts derivatives.
3.3 White noise behavior

Assuming that the input signal is contaminated with additive white noise, we are interested in the behavior of the rms noise value at each scale of the wavelet transform. This knowledge will indicate us the convenient values of the threshold levels $\lambda_m$ at each scale $m$.

Suppose that the input signal $f_{in}(t)$ is given by

$$f_{in}(t) = f_o(t) + n(t), \quad (3.5)$$

where $f_o(t)$ is the original signal, and $n(t)$ is a bandlimited noise source with bandwidth $f_{BW}$, which is far away of the work frequency band of the wavelet system. The noise source $n(t)$ has a constant spectral density for $\omega \in (0, f_{BW})$.

Applying the Fourier transform to the definitions of the continuous wavelet family (1.1) and the discrete wavelet family (1.12), we obtain

$$\sqrt{a} \Psi(a\omega), \quad (3.6)$$

where the translation $b$ is not considered, and $a = r^m$ for the discrete case.

Now, if we redefine the semidiscrete wavelet family (1.7) for the present analysis as

$$\psi_m(t) = \frac{1}{\sqrt{r^m}} \psi\left(\frac{t}{r^m}\right), \quad (3.7)$$

and taking $a = r^m$ we also obtain (3.6) as its Fourier transform.

Now suppose that $f_o(t) = 0$ in (3.5), i.e., the input signal is $f_{in}(t) = n(t)$. Then the rms noise value at the output of the filter $\sqrt{a}\Psi(a\omega)$ is given by

$$\bar{n} = \sqrt{\int_0^{\infty} |\sqrt{a}\Psi(a2\pi f)|^2 N^2(f) \, df}, \quad (3.8)$$

where $N^2(f)$ is the noise spectral density, which is a constant for the white noise case.

A brick-wall filter is defined as . . . the frequency span of a brick-wall filter that has the same output noise rms value that the given filter has when white noise is applied
to both filters... (peak gains are the same for the given and the brick wall filters)...” [33, pg. 192].

Now, suppose that $f_c$ is the central frequency of $\Psi(2\pi f)$ then $f_c/a$ is the central frequency of $\Psi(a 2\pi f)$. Now replacing $|\Psi(a 2\pi f)|$ by a brick-wall filter with an equivalent bandwidth $k_f f_c/a$, and with an amplitude $A_{max}$, for some $k_f > 0$, and if the noise spectral density is $N^2(f) = k_n$ then

$$\bar{n} = \sqrt{\int_0^{k_f f_c/a} a A_{max}^2 k_n df} = A_{max} \sqrt{k_n k_f f_c}, \quad (3.9)$$

where $A_{max}$, $k_n$, $k_f$, and $f_c$ are all constants, and then the rms noise value, $\bar{n}$, is independent of the scale of the filter. In this way, we can use the same threshold value for all the scales of the system.

If instead of (3.7), the normal definition of the semidiscrete wavelet family (1.7) is taken, then the relation between the rms noise and the scale $a = r^m$, given in the last relation, now takes the form

$$\bar{n} = \frac{A_{max}}{\sqrt{a}} \sqrt{k_n k_f f_c}. \quad (3.10)$$

And in this case $\bar{n}$ is a function of the scale $a$.

The amplitude of a signal of noise $n(t)$ is related with its rms value $\bar{n}$ by a Gaussian probability distribution. Meaning that the peak-to-peak value of $n(t)$ will be bellow $\pm \bar{n}$ during 68% of the time, bellow $\pm 2 \bar{n}$ for 95.4% of the time, and so on (see Table 3.1).

In this section we have analyzed the noise levels as a function of the scale when the source of noise has a gaussian distribution. But the threshold levels could also be adjusted to manage noise of any other distribution.
Table 3.1: Percent of time that the noise signal remains between a DC level.

<table>
<thead>
<tr>
<th>DC level</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>± (\bar{n})</td>
<td>68%</td>
</tr>
<tr>
<td>± 2.0 (\bar{n})</td>
<td>95.4%</td>
</tr>
<tr>
<td>± 3.0 (\bar{n})</td>
<td>99.7%</td>
</tr>
<tr>
<td>± 3.4 (\bar{n})</td>
<td>99.94%</td>
</tr>
</tbody>
</table>

### 3.4 Denoising examples

For comparative purposes, in this section the denoising technique with discrete and semidiscrete wavelet systems is implemented. For this purpose, the probe signals *tone2*, *sin2* and *chirp2* contaminated with additive white noise are used. The frequency range of all the implemented systems is approximately of 8 octaves. While the range of the original probe signals is less than 8 octaves, as can be seen in Figure 2.3(b), 2.4(b), and 2.5(b).

The examples are split in four parts. In the first part we present an example of denoising using a system designed with the method of custom built filters (Section 2.2.5). While in the last three cases the denoising is performed with systems designed with the method of increasing the scale density (Section 2.2.4). We focus in this last approach since it allows a higher overlap between the impulse responses of adjacent filters, helping to alleviate the mismatch effect of implementations in analog integrated circuits.

The quality of the achieved reconstructed signals is measured with the signal-to-noise ratio which is defined as follows.

**Definition 3.7.** The *signal-to-noise ratio* of the denoised signal, \(f_{out}(t)\), respect to the original signal free of noise, \(f_{in}(t)\), is given by

\[
\text{SNR} = 20 \log \frac{\|f_{in}\|}{\|f_{out} - f_{in}\|} \text{ (dB)}.
\]  

(3.11)

This definition is identical to the signal-to-error ratio, Definition 2.21. In fact, this
definition is just a measure of the ratio between the power in the signal $f_{out}(t) - f_{in}(t)$
to the power in the original signal $f_{in}(t)$, without concern whether the difference is
due to noise or due to the band-pass behavior of the wavelet system. Nevertheless,
in the context of the present discussion, it is more convenient to call such a measure
signal-to-noise ratio.

The next simulations have been implemented in Matlab. The noisy probe signals
were obtained adding bandlimited white noise to the original signals. The bandwidth
of the noise is approximately 3.5 times the bandwidth of the wavelet systems. As a
matter of reference, we have filtered the noisy signals with a band-pass Butterworth
filter working at approximately the same bandwidth as the wavelet systems. The
used Butterworth filter is given by

$$H_{Butt}(s) = \frac{(\omega_2 - \omega_1)s}{s^2 + (\omega_2 - \omega_1)s + 1},$$

(3.12)

where $\omega_1 = 2\pi(62.5\text{mHz})$ and $\omega_2 = 2\pi(16\text{Hz})$ are the lower and the upper edges of
the work frequency band of the wavelet systems.

In fact, the SNR that was reached with the Butterworth filter has more significance
than the level of white noise with which the probe signal was contaminated, because
the noise that resist the Butterworth filtering is approximately the same noise that
the wavelet systems are able to observe.

The criterion on all the graphical examples has been to use signal that were
contaminated near to the limit of the capacity of the wavelet systems to recover a
signal with a SNR greater than 5 dB. We consider this SNR the minimum required to
recover a still recognizable signal. Obviously, with lower levels of noise we can obtain
better reconstructions.

3.4.1 Denoising with custom built filters

The next example is useful to see all the steeps involved in the denoising tech-
nique: the wavelet transform, the thresholding operation, and the inverse wavelet
transform. Additionally, the effect of the reconstruction filter in the smoothness of the reconstructed signal is clearly illustrated.

**Example 3.8.** In Figure 3.4(a) is shown the noise-free probe signal *tone2*, while in Figure 3.4(b) is shown the same signal but contaminated with bandlimited white noise of rms = 0.8. In Figure 3.4(c) is shown the wavelet transform of this noisy signal plotted as an image of the absolute value. For this wavelet transform were used the biqu($\sqrt{2}, 1$) as decomposition filter and a scale ratio of 1 scale per octave, $D = 1$. In Figure 3.4(d) is shown the previous wavelet transform after applying the delayed hard thresholding operation.

In Figure 3.4(e) is shown, only for illustrative purposes, the sum of the eight wavelet components shown in the previous thresholded wavelet transform. As can be seen this signal presents a lot of little discontinuities due to the thresholding operation and the heights of the pulses do not maintain the correct proportions.

In Figure 3.4(f) is shown the reconstructed signal where for the application of the inverse wavelet transform has been used the custom built reconstruction filter $H_\chi(s) = \frac{6.468s^3 + 4.565s^2 + 4.648s}{s^4 + 4.5s^3 + 7s^2 + 4.5s + 1}$. Note the effect that this band-pass filter has in the smoothness of the signal.

In Figure 3.4(g) is shown, for reference, the Butterworth filtered signal.

The wavelet system of this example has the advantage of require only one scale per octave, but the reconstruction filter is more complex and posses a greater sensitivity compared with filters obtained with the increasing density approach.

### 3.4.2 Denoising with the probe signal *tone2*

For the next examples, it is convenient to remember the notation that we have used along this chapter and the previous one: $D$ indicates the scale density in octaves per scale; $H_\psi(s)$ and $H_\chi(s)$ specify the decomposition and the reconstruction filters, respectively. And we write $H_\chi(s) = k$ to mean that the reconstruction filter is only a normalization constant.
Figure 3.4: Wavelet denoising with an analog wavelet system with custom built filters.

**Example 3.9.** In Figure 3.5(a) is shown the noise-free probe signal *tone2*. This signal was contaminated with bandlimited additive white noise of rms = 0.8. The contaminated signal is not shown, but in Figure 3.5(b) is shown, for reference, this noisy signal after been filtered by the Butterworth filter (3.12). The noise in this filtered signal is the noise that lies in the frequency band of the used wavelet systems.

From Figure 3.5(c) to 3.5(e) are shown the results of applying the denoising technique over the contaminated signal using 3 different semidiscrete wavelet systems. The decomposition and reconstruction filters of all these systems were designed with
the increasing-scale-density approach.

For the system used in Figure 3.5(c), we use $H_\psi = \text{biqu}(\sqrt{2}, 1)$, $H_\chi(s) = k$, and $D = 2$. For the system used in Figure 3.5(d), we use $H_\psi = \text{biqu}(\sqrt{2}, 1)$, $H_\chi(s) = \text{biqu}(\sqrt{2}, 1)$, and $D = 4$. And for the system used in Figure 3.5(e), we use $H_\psi = \text{biqu}(\sqrt{2}, 2)$, $H_\chi(s) = k$, and $D = 4$.

In Figure 3.5(f) is shown, for comparative purposes, a signal denoised with the discrete wavelet transform. This transform has been implemented by the fast wavelet transform algorithm using the Daubechies wavelet db6.

The emphasis in this example has been to obtain denoised signals of good appearance, meaning that signals with a slightly better SNR could be obtained but with the cost of more spikes in the output signal.

**Example 3.10.** In Figure 3.6 is shown the SNR of the denoised *tone2* signal when it had been contaminated with several rms levels of bandlimited additive white noise. The denoising was performed using three different wavelet systems and the Butterworth filtered signal has been included as SNR reference.

The first system is a discrete wavelet system where the wavelet transform has been implemented using the fast wavelet transform algorithm with the Daubechies wavelet db6. The second system is a semidiscrete wavelet system for which we have used $H_\psi = \text{biqu}(\sqrt{2}, 2)$, $H_\chi(s) = k$, and $D = 4$. The third system is another semidiscrete wavelet system for which we have used $H_\psi = \text{biqu}(\sqrt{2}, 1)$, $H_\chi(s) = k$, and $D = 2$. While the used Butterworth filter is given by (3.12). The noise that remains in the signal after the Butterworth filtering is approximately the noise that lies in the frequency band of all the wavelet systems.

The criterion in these simulations has been to obtain the points of maximum SNR. For the Figure 3.6 only the points with a SNR $> 5$ have been considered. Note that with the discrete wavelet system were reached the best results for noise levels of $\text{rms} \leq 0.4$. While for a noise level of $\text{rms} = 0.8$ the results were similar with the three systems. But only the system of the filter $\text{biqu}(\sqrt{2}, 2)$ was able to work with a noise
Figure 3.5: Comparison of the denoising of the signal tone2 contaminated with bandlimited additive white noise of rms = 0.8, using a Butterworth filter (b), 3 different semidiscrete wavelet systems (c)-(e), and the discrete wavelet transform (f). The five systems work over approximately the same frequency band.

level of rms = 1.2.

**Remark 3.11.** The delay time $T$, required for the delayed hard thresholding operation, is proportional to the scale $a = r^m$ as can be seen in (3.4). But in Section 3.3 we saw that the central frequency $f_c$, of a scaled filter, is inversely proportional to the scale. Then we can express the delay time at the scale $m$ by

$$T_m = r^m T_o = \frac{k_D}{f_{c,m}} = k_D T_{c,m},$$  

(3.13)
for some constant $k_D > 0$, where $f_{c,m}$ is the central frequency of the filter at scale $m$, and $T_{c,m}$ is the period of this central frequency. Therefore, the delay time can be expressed just specifying the quantity $k_D T_c$.

In the previous example the delay time used with the filters biqu($\sqrt{2}, 2$) and biqu($\sqrt{2}, 1$) was of $1.5T_c$ and $1.9T_c$, respectively.

**Remark 3.12.** For the simulations of the previous example a gain factor had to be introduced in the reconstructed signal in order to maximize the SNR. As can be seen in Figure 3.7(a), this factor depends on the used threshold level. The use of this gain factor was needed even in the case of the discrete wavelet transform (not shown), but the effect is more accentuated in the case of the semidiscrete wavelet systems.

As can be seen the relation between the factor and the threshold level is non-linear, but the relation becomes more linear when the threshold level is in logarithmic scale, as can be seen in Figure 3.7(b). However, this behavior is applicable to this special case only and not to the general case. Then, the dependence between the factor and the threshold level has to be analyzed for each particular implementation.
Figure 3.7: Dependence between the threshold level and the gain factor needed to maximize the SNR during the wavelet denoising.

### 3.4.3 Denoising with the probe signal \( \text{sin} \, 2 \)

**Example 3.13.** In Figure 3.8(a) is shown the noise-free probe signal \( \text{sin} \, 2 \). This signal was contaminated with bandlimited additive white noise of rms = 1.0. The contaminated signal is not shown, but in Figure 3.8(b) is shown, for reference, this noisy signal after being filtered by the Butterworth filter (3.12). The noise in this filtered signal is the noise that approximately lies in the frequency band of the used wavelet systems.

From Figure 3.8(c) to 3.8(e) are shown the results of applying the denoising technique over the contaminated signal with 3 different semidiscrete wavelet systems. The decomposition and reconstruction filters of all these systems were obtained with the increasing-scale-density method.

For the system used in Figure 3.8(c), we use \( H_\psi = \text{biqu}(\sqrt{2}, 1) \), \( H_\chi(s) = k \), and \( D = 2 \). For the system used in Figure 3.8(d), we use \( H_\psi = \text{biqu}(\sqrt{2}, 1) \), \( H_\chi(s) = \text{biqu}(\sqrt{2}, 1) \), and \( D = 4 \). And for the system used in Figure 3.8(e), we use \( H_\psi = \text{biqu}(\sqrt{2}, 2) \), \( H_\chi(s) = k \), and \( D = 4 \).

In Figure 3.8(f) is shown, for comparative purposes, a signal denoised with the discrete wavelet transform. This transform has been implemented by the fast wavelet transform algorithm using the Daubechies wavelet db12.
Figure 3.8: Comparison of the denoising of the signal $\sin^2$ contaminated with bandlimited additive white noise of rms = 1.0, using a Butterworth filter (b), 3 different semidiscrete wavelet systems (c)-(e), and the discrete wavelet transform (f). The five systems work over approximately the same frequency band.

The emphasis in this example has been to obtain denoised signals of good appearance. This means that signals with a slightly better SNR could be obtained but the output signals would have more spikes.

Example 3.14. In Figure 3.9 is shown the SNR of the denoised $\sin^2$ signal when it had been contaminated with several rms levels of bandlimited additive white noise. The denoising was performed using three different wavelet systems and the Butterworth filtering has been included as SNR reference.
The first system is a discrete wavelet system where the wavelet transform has been implemented using the fast wavelet transform algorithm with the Daubechies wavelet db12. The second system is a semidiscrete wavelet system for which we have used $H_\psi = \text{biqu}(\sqrt{2}, 2)$, $H_\chi(s) = k$, and $D = 4$, where we used a delay time of $2.6T_c$. The third system is another semidiscrete wavelet system for which we have used $H_\psi = \text{biqu}(\sqrt{2}, 1)$, $H_\chi(s) = k$, $D = 2$, where we used a delay time of $1.3T_c$. While the used Butterworth filter is given by (3.12). The noise that remains in the signal after the Butterworth filtering is approximately the noise that lies in the frequency band of all the wavelet systems.

The criterion in these simulations has been to obtain the points of maximum SNR. For the Figure 3.9 only the points with a SNR $> 5$ have been considered. Note that with the discrete wavelet system were achieved the best results for noise levels of $\text{rms} \leq 0.8$, but it did not work for greater noise levels. In the case of a noise level of $\text{rms} = 1.2$ the results were similar for the two analog wavelet systems. But only the system of the filter $\text{biqu}(\sqrt{2}, 2)$ was able to work with a noise level of $\text{rms} = 1.6$.

If we continue the simulations with lower levels of noise, we would find with an upper limit for the SNR that any of the systems is able to reach, including the Butterworth filter. These upper plateaus are due to the band pass behavior of the systems. That is, we can make the noise level lower and lower but the error introduced by the phase-shift or the magnitude variation due to the band-pass behavior of the system will remain constant.

### 3.4.4 Denoising with the probe signal \textit{chirp2}

**Example 3.15.** In Figure 3.10(a) is shown the noise-free probe signal \textit{chirp2}. This signal was contaminated with bandlimited additive white noise of $\text{rms} = 1.0$. The contaminated signal is not shown, but in Figure 3.10(b) is shown, for reference, this noisy signal after been filtered by the Butterworth filter (3.12). The noise in this filtered signal is the noise that lies in the frequency band of the used wavelet systems.
Figure 3.9: SNR obtained with different wavelet systems when the probe signal \( \sin^2 \) is contaminated with different rms levels of white noise.

From Figure 3.10(c) to 3.10(e) are shown the results of applying the denoising technique over the contaminated signal with 3 different semidiscrete wavelet systems. The decomposition and reconstruction filters of all these wavelet systems were obtained with the increasing-scale-density method.

For the system used in Figure 3.10(c), we use \( H_\psi = \text{biqu}(\sqrt{2}, 1) \), \( H_\chi(s) = k \), and \( D = 2 \). For the system used in Figure 3.10(d), we use \( H_\psi = \text{biqu}(\sqrt{8}, 1) \), \( H_\chi(s) = k \), and \( D = 4 \). And for the system used in Figure 3.10(e), we use \( H_\psi = \text{biqu}(4, 2) \), \( H_\chi(s) = k \), and \( D = 8 \).

In spite of the fact that these three systems do not possess reconstruction filters to smooth the discontinuities introduced by the thresholding operation, the amplitude of the discontinuities in the denoised signal are reduced as the scale density is increased.

In Figure 3.10(f) is shown, for comparative purposes, a signal denoised with the discrete wavelet transform. This transform has been implemented by the fast wavelet transform algorithm using the Daubechies wavelet db12.

The emphasis in this example has been to obtain denoised signals of good appearance, meaning that signals with a slightly better SNR could be obtained but with the cost of more spikes in the output signal.

**Example 3.16.** In Figure 3.11 is shown the SNR of the denoised \( \text{chirp}^2 \) signal when
Figure 3.10: Comparison of the denoising of the signal $\text{chirp2}$ contaminated with bandlimited additive white noise of $\text{rms} = 1.0$, using a Butterworth filter (b), 3 different semidiscrete wavelet systems (c)-(e), and the discrete wavelet transform (f). The five systems work over approximately the same frequency band.

it had been contaminated with several $\text{rms}$ levels of bandlimited additive white noise. The denoising was performed using two different wavelet systems and the Butterworth filtering has been included as SNR reference.

The first system is a discrete wavelet system where the wavelet transform has been implemented using the fast wavelet transform algorithm with the Daubechies wavelet db12. The second system is a semidiscrete wavelet system for which we have used $H_\psi = \text{biqu}(\sqrt{2}, 1)$, $H_\chi(s) = k$, and $D = 2$, where we used a delay time of $1.8T_c$. While the used Butterworth filter is given by (3.12). The noise that remains in the signal
after the Butterworth filtering is approximately the noise that lies in the frequency band of all the wavelet systems.

The criterion in these simulations has been to obtain the points of maximum SNR. For the Figure 3.11 only the points with a SNR > 5 have been considered. Note that with the discrete wavelet system were reached better results for all the simulated noise levels, except for a rms = 1.2 where both systems offered the same SNR.

The effect of the upper plateau of the SNR is more accentuated here in the case of the analog wavelet system and the Butterworth filter than in the previous examples, since the used probe signal covers a wider frequency range, making the systems work nearer of the edges of their frequency range.

### 3.4.5 Remarks

Comparing the denoising performance of the discrete and semidiscrete systems of the previous examples we can conclude that, in general, better SNR measures can be obtained using the discrete wavelet transform. With the exception of large noise levels where the the semidiscrete wavelet systems have better performance, and as a consequence, the semidiscrete wavelet systems have a slightly greater denoising range.

It is important to remember that the processing with the semidiscrete wavelet
transform is performed in real time, with zero-delay. While, in the case of discrete
wavelet transform, a delay proportional to the maximum used scale is needed as was
shown in Section 2.3.

Nevertheless, in the delayed hard thresholding operation, the system has to wait
until the level of the wavelet component exceeds the threshold level to enable the pass
of the component up to the reconstruction filter. For this reason, a small part of the
signal is lost at the beginning of the pulses. In fact, this behavior agrees with the
real-time, non-anticipatory, nature of the system.

In each of the previous examples we knew \textit{a-priori} the original noise-free signal
that we were looking to reach. But in actual applications we know neither the exact
wave shape nor the exact level of noise. Therefore, a method to estimate the noise
level has to be developed, similar to those used in the case of the discrete wavelet
transform, but in this case the estimation should be performed \textit{in real-time}. On the
other hand, the previous examples suggest that, as in the case of the discrete wavelet
transform, \textit{a-priori} information about the hoped signals is needed in order to select
the best wavelets to fit with the signal shapes.

Finally, in relation with the poor improve in the SNR reached for the semidiscrete
systems respect to the Butterworth filtered signal, that was observed in the cases of
the probe signals \textit{sin2} and \textit{chirp2}, we have two comments:

- Better SNR were obtained in the case of the signal \textit{tone2}, since it has larger
  intervals of zero value. While the signals \textit{sin2} and \textit{chirp2} are more spread
  over time. This just corroborates that the wavelet is specially helpful in the
  processing of noisy, intermittent, or transitory signals[1].

- Nevertheless, it is clear that the better results in almost all the noise level
  range were reached with the discrete wavelet transform. This is because the
  upper limit to the SNR of the semidiscrete wavelet systems is very low, because
this limit is determined by the phase-shift and magnitude variation due to the band-pas behavior of the whole wavelet system.

At this point we can understand that the main drawback of the wavelet denoising with semidiscrete wavelet systems is the band-pass behavior, which was explained in Section 2.4.2. For this reason, we believe that the present research should be continued in the directions depicted in the proposals of Section 2.4.4, that is, we should look for methods or continuous filters with which the phase-shift or the imaginary part of the filter frequency response could be reduced or compensated.
Chapter 4

IC Implementation of a Wavelet Transform System

Some design criterions

We already know that the reconstruction error comes from the ripple present in the summation of the frequency response of all the scaled filters. In the method of custom filters the ripple is reduced by the use of filters of higher degree, whereas in the method of increasing the scale density the ripple is reduced putting a greater amount of filters per octave. The last approach allows the use of simpler filters and helps to alleviate the effect of the random process variations, since higher scale densities produce a higher overlap between the spectrums of adjacent filters.

For the implemented system we use a biquadratic filter for decomposition, while the reconstruction filter is just a normalization constant. The system covers 8 octaves with a scale density of 2 scales per octave. Just for demonstrative purposes, we have selected that the system works in the frequency range (62.5 Hz, 16 kHz).

The selected technology for the fabrication of the system was the AMIS-C5 process. This is a 0.5 μm CMOS technology, enough for the purposes of the chip: to show that the results theoretically obtained facilitate the implementation of wavelet systems in analog circuits.
For the mathematical design of a semidiscrete wavelet system, we first have to select a prototype filter and then we produce versions of it exponentially scaled in frequency. For the electrical design of the implemented system we have proceeded in a similar way. First, we designed a block with all the needed structures for one single scale. Then, we produced 16 copies of the block, scaling exponentially the work frequency band of the constituted structures.

Since the goal during the chip design was to increase the possibilities of proper functioning of the manufactured chip, rather than propose new structures or innovative design techniques, we have focused in the use of well known design approaches. For example, we decided to use OTAs with transistors working in subthreshold region since they allow a very simple way of frequency scaling, in spite of the disadvantage that we are restricted to work with low amplitude signals.

The most important figure of any decomposition-reconstruction wavelet system is to reach an exact, or nearly exact, reconstruction. For this reason, during the design of the most structures of the chip, Monte Carlo analysis has been applied to identify the effect that the process variations have in the reconstruction error of the output signal, being the main criterion for determining the final size of transistors, capacitors and resistors. In fact, the original size of some of the used OTAs and Op-Amps had to be adjusted many times in order to achieve the desired reconstruction error.

Owing to the amount of repeated blocks and the required large size of the components of the system, we have tried to select structures that require few transistors and capacitors.

On the other hand, structures with low amount of internal nodes are desirable in order to reduce the internally produced noise, which is critical when we work with signals of low level.
Table 4.1: Central frequency of the 16 band-pass biquadratic filters.

<table>
<thead>
<tr>
<th>Scale</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>74.33 Hz</td>
</tr>
<tr>
<td>2</td>
<td>105.2 Hz</td>
</tr>
<tr>
<td>3</td>
<td>148.7 Hz</td>
</tr>
<tr>
<td>4</td>
<td>210.2 Hz</td>
</tr>
<tr>
<td>5</td>
<td>297.3 Hz</td>
</tr>
<tr>
<td>6</td>
<td>420.4 Hz</td>
</tr>
<tr>
<td>7</td>
<td>594.6 Hz</td>
</tr>
<tr>
<td>8</td>
<td>840.9 Hz</td>
</tr>
<tr>
<td>9</td>
<td>1.189 kHz</td>
</tr>
<tr>
<td>10</td>
<td>1.682 kHz</td>
</tr>
<tr>
<td>11</td>
<td>2.378 kHz</td>
</tr>
<tr>
<td>12</td>
<td>3.363 kHz</td>
</tr>
<tr>
<td>13</td>
<td>4.757 kHz</td>
</tr>
<tr>
<td>14</td>
<td>6.727 kHz</td>
</tr>
<tr>
<td>15</td>
<td>9.514 kHz</td>
</tr>
<tr>
<td>16</td>
<td>13.454 kHz</td>
</tr>
</tbody>
</table>
Figure 4.1: Three wavelets $\psi_m(t)$ of the implemented wavelet transform system.

While, in Figure 2.11(c) is shown the magnitude and phase of the summation, and in Figure 2.11(d) the reconstruction error due to the band-pass behavior of the system.

### 4.2 AMIS-C5 process

The selected technology to fabricate the wavelet transform system was the AMI Semiconductors C5 process. This is a 0.5 $\mu$m CMOS technology intended for mixed-signal applications. In Table 4.2 are listed the main characteristics of this technology including some typical parameters for transistors, resistors and capacitors, and a list of the available worst-case simulation models.

The design kit provided by the vendor includes typical and worst case models for N-MOS and P-MOS devices (Spice BSIM3 Version 3.1 Parameters), as well as Parameters for a Generalized Pelgrom Model to estimate the effect of the process variations in transistors, resistors and capacitors via Monte Carlo simulations.

**Output load capacitance**

Information about the Chip, bounding pads, and package DIP40 also is provided by the vendor. The pad for analog signals has an approximated capacitance of 5 pF, and the pins of the DIP40 package have a capacitance lower than 5 pF.

Additionally, the printed circuit boards present a typical capacitance of 0.5 pf/cm, and in the market can be found Op-Amps with input capacitances lower than 3 pF.
Table 4.2: AMIS-C5 Process.

<table>
<thead>
<tr>
<th>Main Characteristics</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Drawn Transistor Length</td>
<td>0.6 µm</td>
<td>Metal Layers</td>
<td>3</td>
</tr>
<tr>
<td>Poly to Poly Capacitors</td>
<td></td>
<td>Operating Voltage</td>
<td>5 V</td>
</tr>
<tr>
<td>Substrate Material</td>
<td>P-Type</td>
<td>Gate Oxide Thickness</td>
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<tr>
<td>High-Resistance Poly</td>
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<table>
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<tr>
<th>CMOS Transistors</th>
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</thead>
<tbody>
<tr>
<td>N-Ch (N)</td>
<td>P-Ch (P)</td>
<td></td>
</tr>
<tr>
<td>Vt</td>
<td>0.7 V</td>
<td>-0.9 V</td>
</tr>
<tr>
<td>Idsat</td>
<td>450 µA/µm</td>
<td>-260 µA/µm</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Resistors</th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Poly</td>
<td>Hi-R Poly</td>
<td></td>
</tr>
<tr>
<td>25 ω/square</td>
<td>1000 ω/square</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Capacitors</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Poly-Poly</td>
<td>Poly-N+active</td>
<td>Poly-P+active</td>
<td></td>
</tr>
<tr>
<td>Area</td>
<td>0.9 fF/µm²</td>
<td>2.5 fF/µm²</td>
<td>2.4 fF/µm²</td>
</tr>
<tr>
<td>Periphery</td>
<td>0.065 fF/µm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Available Worst-Case Models</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>typical NMOS – typical PMOS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>slow NMOS – fast PMOS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>fast NMOS – slow PMOS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>slow NMOS – slow PMOS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>fast NMOS – fast PMOS</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: Load capacitance estimated for the outputs of the chip.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Analog input-output PAD</td>
<td>5 pF</td>
</tr>
<tr>
<td>DIP40 pin</td>
<td>5 pF</td>
</tr>
<tr>
<td>4 cm line in PCB</td>
<td>2 pF</td>
</tr>
<tr>
<td>OpAmp input</td>
<td>3 pF</td>
</tr>
<tr>
<td>Oscilloscope probe</td>
<td>5 pF</td>
</tr>
<tr>
<td>TOTAL</td>
<td>20 pF</td>
</tr>
</tbody>
</table>
and oscilloscope probes with input capacitances lower than 5 pF.

All these load capacitance sources have been listed in Table 4.3, and a total of 20 pF was obtained. But for precaution, in many cases the designed system has been tested with load capacitances of 20 pF and 100 pF.

4.3 Designing in subthreshold

We decided to design the decomposition filters with CMOS transistors working in subthreshold, since in this region the drain current of the transistors has an exponential behavior respecto to the gate voltage. Behavior that simplifies the obtention of $g_m-C$ filters with exponential frequency scaling.

4.3.1 CMOS subthreshold model

A drain current model for PMOS transistors of long channel is given by [34]

$$I_D = I_0 \frac{W}{L} e^{\frac{\rho V_{BG}}{U_t}} \left( e^{-\frac{V_{BS}}{U_t}} - e^{-\frac{V_{BD}}{U_t}} \right),$$

where $V_{BG}$, $V_{BS}$ and $V_{BD}$ are the bulk-gate, bulk-source and bulk-drain voltages, $W$ and $L$ are the width and length of the transistor channel, $I_0$ is the zero-bias current, $\rho$ is the electrostatic coupling coefficient between channel and gate, and $U_t = kT/q$ is the thermal voltage. In work [35], $I_0$ and $\rho$ are considered as empirical parameters with $\rho \leq 1$ and typically ranging around $1.5^{-1}$. When $V_{BD} > V_{BS} + 5U_t$ we said that the transistor is in saturation [34].

In Figure 4.2 is plotted the drain current $I_D$ versus the source-gate voltage $V_{SG}$ for PMOS transistors that have the same area but different aspect-ratios. Note that in subthreshold the currents are of the order of nA and pA. The selected aspect ratio for the implemented system is $60\lambda/15\lambda$, since it has a wide exponential range and at the same time is not so far of the optimum aspect ratio of $30\lambda/30\lambda$, which allows the maximum reduction of the mismatch effect.
Figure 4.2: $I_D$ vs. $V_{SG}$ for PMOS transistors of different aspect-ratios, simulated in Hspice with the shown circuit. The aspect ratio selected for the implemented system is $60\lambda/15\lambda$ (bold line).

### 4.3.2 Differential pairs in subthreshold

#### Basic differential pair

The basic differential pair shown in Figure 4.3(a) comprises two identical transistors and a current source. In this circuit $V_{DM} = V_1 - V_2$ is the input differential voltage, $I_{DM} = I_1 - I_2$ is the output differential current, and $V_{BCM} = \frac{V_{BG1} + V_{BG2}}{2}$ is the input common voltage referenced to bulk. From these relations follows that

$$V_{BG1} = V_{BCM} + \frac{1}{2} V_{DM},$$

$$V_{BG2} = V_{BCM} - \frac{1}{2} V_{DM}. \tag{4.4}$$

If both transistors of the basic differential pair are in saturation then $e^{-\frac{V_{BD}}{U_t}} \approx 0$, and from (4.3) we have that

$$\frac{I_1 - I_2}{I_1 + I_2} = \frac{I_{DM}}{I_b} = \frac{I_0 W}{L} e^{-\frac{\rho V_{BCM}}{U_t}} e^{-\frac{V_{BG}}{U_t}} \left( e^{\frac{\rho V_{DM}}{U_t} - e^{-\frac{\rho V_{DM}}{U_t}} = e^{-\frac{\rho V_{DM}}{2U_t}} \right), \tag{4.5}$$

and finally

$$I_{DM} = I_b \tanh \left( \frac{\rho V_{DM}}{2U_t} \right). \tag{4.6}$$

The transconductance of the differential pair, defined as $G = \frac{\partial I_{DM}}{\partial V_{DM}}$, is given by

$$G = \frac{\rho I_b}{2U_t} \text{sech}^2 \left( \frac{\rho V_{DM}}{2U_t} \right). \tag{4.7}$$
The function \( \text{sech} \, x \) has a maximum at \( x = 0 \), where \( \text{sech} \, 0 = 1 \). Then the normalized transconductance is given by
\[
G_{\text{norm}} = \text{sech}^2 \left( \frac{\rho V_{DM}}{2U_t} \right).
\] (4.8)

In [34] the distortion is defined as the greater deviation of \( G_{\text{norm}} \) from the unit. Using the values \( U_t = 25.7 \, \text{mV} \) and \( \rho = 1.5^{-1} \), a distortion of 1\% is reached in the interval \( \pm 7.7 \, \text{mV} \) of \( V_{DM} \). This usable range is very poor, considering an expected peak-to-peak amplitude of the noise voltage at the output of the system of about 1 mV.

**Linearized differential pair**

With the purpose to increase the linear range of the simple differential pair, in [34] is suggested the use of the differential pair of Figure 4.3(b), which has degeneration in the source via symmetric diffusors. An expression for the differential output current for this linearized differential pair is given by
\[
I_{DM} = I_b \tanh \left( \frac{\rho V_{DM}}{2U_t} \right) - \tanh^{-1} \left( \frac{1}{4p + 1} \tan \left( \frac{\rho V_{DM}}{2U_t} \right) \right).
\] (4.9)

In paper [34] is found that with a value of \( p = 0.5 \) is obtained the range of maximally-flat response. In this case, a distortion of 1\% is reached in the interval \( \pm 30.7 \, \text{mV} \) of \( V_{DM} \), which is 4 times the range of the basic differential pair.
4.3.3 Frequency scaled filters in subthreshold

Substituting the values $U_t = 25.7 \text{ mV}$, $\rho = 1.5^{-1}$ and $m = 0.5$ in (4.9) can be numerically proved that the maximum transconductance occurs at $V_{DM} = 0$. Then, for these values an expression of the maximum transconductance is given by

$$G_{max} = \left. \frac{\partial I_{DM}}{\partial V_{DM}} \right|_{V_{DM}=0} = 18.2 I_b .$$  \hfill (4.10)

If each current source $I_b$ of the linearized differential pair of Figure 4.3(b) is a long-channel PMOS transistor that is in saturation, like the one shown in Figure 4.4(a), then $V_{BD} = V_{SD} > 5 U_t$, and from the model (4.3) we obtain the expression

$$I_b = I_0 \frac{W}{L} e^{\frac{\rho V_{BG}}{U_t}} .$$ \hfill (4.11)

And if the capacitance of a $g_m - C$ filter is constant, then the central frequency of the filter, $\omega_c$, varies linearly with the bias current $I_b$, and exponentially with the bias voltage $V_b$, that is

$$\omega_c \propto I_b ,$$ \hfill (4.12)

$$\omega_c \propto e^{V_{BG}} .$$
This exponential dependence of the current $I_D$ respect to lineal changes in $V_{BG}$ can be used in systems with a lot of identical filters but scaled in frequency [36].

In Figure 4.4(b) is shown the biasing of the 16 identical $g_m - C$ filters of the implemented wavelet system. The 16 filters take the bias voltages at regular intervals of the resistor ladder, which can be fabricated as a single line of polysilicon [36].

The range in which the current bias of the MOS transistor have an exponential behavior covers 4 decades as can be seen in Figure 4.2. This is sufficient for our application whose frequency range of $(74.32\text{Hz}, 13.45\text{kHz})$ is slightly greater than two decades.

4.4 System structures

In this section, we present the structures of the decomposition-reconstruction part of the implemented wavelet system together with some of their performance characteristics, while in the next chapter are presented the structures corresponding to the denoising function with which the system is provided.

The simulations for the fine adjustment of the electrical design have been done in Hspice using mainly the typical MOS model provided in the design kit of the vendor. The schematics of the complete system are shown in the Appendix A.

4.4.1 Power supply

The implemented system is intended to work with a voltage source of 2.5 V. In Figure 4.5(a) are shown the symbols of the 3 power lines and 2 ground lines that have been included in the system together with the needed pads. Depending on the noise sensitivity, the structures of the system had separated in 3 types, each one with dedicated power lines:

1. Lines $V_{pp} = 2.5$ V and $G_{nn} = 0$ V. Intended to bias the analog circuits of low sensitivity and the static digital circuits.
2. Line $V_{ppl} = 2.5$ V working with $G_{nn} = 0$ V. Intended to bias the most critical analog devices.

3. Lines $DV_{p} = 2.5$ V and $DG_{n} = 0$ V. Intended to bias the sequential digital circuits.

The separation of the circuits helps to reduce the influence that the digital noise has in the sensitive analog devices.

The selected technology is manufactured in P-type substrate. Then, the digital and analog PMOS transistors can be mutually isolated putting them in different N-wells. On the other hand, all the NMOS transistors share the same bulk. Nevertheless, using the configuration shown in Fig. 4.5(b) for the digital NMOS transistors, the digital ground $DG_{n}$, is isolated from the analog ground $G_{pp}$, due the depletion region around the source of the transistor. In this configuration, during digital commutations, the most of the current peaks are sunk through the digital ground instead of through the common bulk.

### 4.4.2 Bias voltages

For each of the 16 decomposition filters the system has a block of identical structures. The biasing of the filters was made using a resistor ladder similar to the one
Figure 4.6: Resistor ladder, fabricated as a single line of polysilicon, and voltages generated to bias the 16 identical blocks of the system.

As we saw earlier for filters working in subthreshold, if the resistors of the ladder have the same value is produced an exponential frequency scaling of the filters. Nevertheless, due to the presence of second order effects, we had to make a fine adjustment in the resistor values, by an iterative process involving Hspice and Matlab, in order to adjust the central frequencies of the filters and to flat the magnitude frequency response of the summation. The resistive ladder finally used in the implemented system is shown in Figure 4.6.

With the same voltage with which the decomposition filter of each block is biased are also biased an analog buffer, two operational amplifiers, and two comparators needed for the processing of the corresponding wavelet component. In this way the gain-bandwidth of these structures is limited only to the one necessary for each scale. Nevertheless, to produce the adequate amount of bias current, the bias voltage that comes from the resistor ladder has to pass through one or both of the current amplifiers $B_{\text{Amp}}$ or $B_{\text{X Amp}}$ that are shown in Figure 4.7.

### 4.4.3 Operational transconductance amplifier $\text{TL}$

The 16 decomposition filters of the wavelet system were implemented using the $g_m\cdot C$ technique. The used operational transconductance amplifier is shown in Figure 4.8. This OTA works in subthreshold with a linear range improved by means

<table>
<thead>
<tr>
<th>Resistance ($\Omega$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{15}$ 158.89</td>
</tr>
<tr>
<td>$R_{14}$ 155.30</td>
</tr>
<tr>
<td>$R_{13}$ 154.27</td>
</tr>
<tr>
<td>$R_{12}$ 153.76</td>
</tr>
<tr>
<td>$R_{11}$ 154.79</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PMOS Source–Gate Bias Voltages (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16}$ 0.6309</td>
</tr>
<tr>
<td>$V_{15}$ 0.6445</td>
</tr>
</tbody>
</table>
Figure 4.7: Schematic circuits of the current amplifiers $BAmp$ and $BXAmp$.

Figure 4.8: Operational transconductance amplifier $TL$ and main performance features obtained by Hspice simulations.

of the technique explained in Section 4.3.2. The Hspice simulations of the transconductance and the normalized transconductance of this OTA are shown in Figure 4.9 and 4.10, respectively. As can be appreciated, the normalized transconductance has a slightly different behavior from one scale to another due to second order effects. Despite that this difference could be reduced by adjusting the widths of transistors $m_3$ and $m_4$ of $TL$ at each scale, they were kept identical to simplify the layout design, since the obtained difference was not excessive.

In Figure 4.11 are shown the CMRR and the PSRR of $TL$ at the scale 8. At the central frequency of the corresponding decomposition filter, 1189 Hz, the CMRR presents a healthy value of 66dB, the $PSRR_-$ is 39 dB, and the worst case corresponds
to the $PSRR_+$ with only 25 dB. In spite of this fact, this OTA was selected for the implemented system, due to its compactness and the easy way to scale it in frequency; but some precautions like the inclusion of dedicated power lines have been taken.

The transistors of the differential pair were selected to be PMOS since, in the selected N-well process, they can be put inside individual wells.

The bias currents $I_b$ are generated by means of the transistors $m_5$ and $m_6$ which are working in subthreshold region. $I_b$ is set by the voltage $V_b$ which is taken from the resistor ladder shown in Figure 4.6.

The power line $V_{ppl}$ is used exclusively to power the $TL$ transconductors, while the line $V_{pp}$ is used to power all the other analog components. In this way is avoided that voltage drops that are present along the line $V_{pp}$ affect the $V_{SG}$ biasing voltage
Figure 4.11: CMRR, PSRR⁻ and PSRR₊ of the operational transconductor $T_L$ of the scale 8.

of transistors $m_5$ and $m_6$.

**Other OTA implementations**

At the moment we have already made an overview of the main characteristics of the MOS transistors working in subthreshold region, and of the linearized OTA presented in this section.

The main goal during the electrical design of the decomposition filters was just to obtain a simple and feasible implementation. For this purpose we made a review of some other available implementations of OTAs, Op-Amps and complete filters working in weak or strong inversion.

The criterions for the selection of a good OTA, were the number of needed transistors, the size of the reported implementation, the linear range, and the power consumption.

Some of the reviewed OTAs were all the presented in [34], [36], and [37]. The selected OTA was preferred because it has a good balance between sensitivity, linear range and number of components. Other options have a slightly larger linear range but with a higher sensitivity and more required transistors.

The main disadvantage of the most reviewed Op-Amps was the large area consumption. While among the reviewed complete filters we proved the one presented in [38], which is a bandpass current mode filter working in saturation with only 7
transistors and 3 capacitors. It allows a good scaling in frequency but only along 1
decade of frequency, but we require a frequency scaling of the filters along more than
2 decades.

4.4.4 Monitoring buffer \textit{TXBfer}

To simplify the monitoring of some important internal signals, each of the 16 blocks
comprises one voltage buffer \textit{TXBfer} like the one shown in Figure 4.12. To achieve
the monitoring task, this buffer is only used in the voltage follower configuration, as
is shown in Figure 4.13. The minimum charge that this buffer has to drive, which is
due to the the PAD and DIP40 pin parasitics, is of about 10 pF. Nevertheless, the
buffer was intended to drive a nominal load of 20 pF, and for precaution also has
been tested with a load of 100 pF via Hspice simulations.

4.4.5 Band-pass biquadratic filter

In this section, we present the circuit of the implemented biquadratic filter, to-
gether with large signal and linear range analysis. Additionally, the way to send the
wavelet component to the next summing stage is explained, likewise the circuit used

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\textbf{Input comm. mode range} & \textbf{Output swing} & \textbf{Current consumpt. (uA)} & \textbf{GBW (kHz)} & \textbf{Phase Margin} & \textbf{Gain} & \textbf{Phase Margin} \\
\hline
(0, 0.9V) & (0, 1.1V) & & & & & \\
\hline
1 & 101.2 & 5640 & 138° & 0.976 & -0.24° & 0.975 & -1.19° \\
4 & 36.6 & 3430 & 135° & 0.981 & -0.15° & 0.981 & -0.76° \\
8 & 9.75 & 1280 & 136° & 0.981 & -0.04° & 0.981 & -0.19° \\
12 & 2.60 & 427 & 134° & 0.981 & -0.01° & 0.981 & -0.05° \\
16 & 0.689 & 123 & 135° & 0.981 & -0.00° & 0.981 & -0.01° \\
\hline
\end{tabular}
\end{table}
to send the monitored signals outside the chip.

Circuital implementation

The circuital implementation of the biquadratic filter (4.1) is made with the circuit of Figure 4.13 which is composed by the transconductors TL1 and TL2, and the two capacitors $C_1$ and $C_2$, where $C_2 = C_{2a} + C_{2b}$.

The voltages $v_x$ and $v_y$ are the input and the output signals of the filter, respectively.

The filter, like other analog circuits in the system, requires of the external reference voltage $A_{Gnd} = 0.7V$.

If $G_1$ and $G_2$ are the transconductances of TL1 and TL2, and if $G_1 = G_2 = G$, then the transfer function of the biquad is given by

$$\frac{V_w(s)}{V_x(s)} = \frac{q \omega_c s}{s^2 + \frac{2}{\omega_c} s + \omega_c^2},$$

where

$$q = \sqrt{\frac{G_1 C_2}{G_2 C_1}} = \sqrt{2},$$

$$\omega_c = \sqrt{\frac{G_1 G_2}{C_1 C_2}} = \sqrt{2} \times 10^{11} G.$$ (4.13, 4.14, 4.15)

It means that the central frequency of the biquad $\omega_c$, is proportional to the transconductance $G$, which in turn depends exponentially on the bias voltage $V_b$, as was previously explained. With this, the exponential frequency scaling of the filters is guaranteed, as is required by (1.7). This band-pass biquadratic filter has unit gain at the central frequency $\omega_c$. The response of the filter at each of the 16 scales is shown in Figure 4.14.

Wavelet components

The node $V_{Gnd} \approx 0.7$ V is a virtual ground at the input of the operational amplifier connected in an inverting summing configuration that is shown in Figure 4.19.
Now, in Figure 4.13 the voltage $v_w$ through the capacitor $C_{2a}$ produces the current $i_w$. All the $i_w$ currents of the 16 blocks are summed at the node of virtual ground, $V_{Gnd}$. In this way is implemented the summation involved in the inverse semidiscrete wavelet transform (1.9).

**Large signal stability**

As can be seen in Figure 4.13, the input signal is AC coupled. In Figure 4.15 is shown the transient response of the biquadratic filters at scales 4 and 12, in the presence of an input signal composed by large amplitude square pulses plus a sinusoidal component of the same frequency that the central frequency of the filter. At the beginning, the DC level of the input signal is equal to $A_{Gnd} = 0.7 \, V$. After each pulse the filter requires only 2 or 3 cycles until the capacitor $C_1$ is charged by the current provided by transconductor $TL1$, after that the filter continues working normally.
Linear range of $TL1$ and $TL2$

We want to know the maximum range of the amplitude of the input signal $v_x$ that ensures that $TL1$ and $TL2$ work within the 1% of linearity.

From the circuit of Figure 4.13, can be easily verified that the operation point at the node $v_w$ is around the level $A_{Gnd}$. And we already know that the maximum gain of $v_w$ respect to $v_x$ is of 1. Hence $v_x$ can oscillate as much as the input linear range within 1% of linearity of the linearized differential pair, that is, $\pm 30.7 \text{mV}$ as was saw earlier.

On the other hand, from the circuit of Figure 4.13 can be easily shown that

$$\frac{V_a(s) - V_w(s)}{V_x(s)} = jq,$$  \hfill (4.16)

where $q = \sqrt{2}$ for the present case. Now, considering only signals in steady state to simplify the analysis, this relation implies that the input signal, $v_x$, must be restricted to the range $\pm \frac{30.7 \text{mV}}{\sqrt{2}}$ in order to avoid that the differential input of $TL2$ work beyond the range of 1% of linearity.

Other biquadratic implementations

The selected $g_m-C$ structure to implement the biquadratic filter was preferred since the low number of components that it needs: only two OTAs and two capacitors. There are some other $g_m-C$ structures to implement band-pass biquadratic filters,
using 2 capacitors but with 2, 3 or 4 OTAs. The filters with 3 or 4 OTAs allow more freedom degrees to adjust the central frequency, the selectivity and the gain of the filter. But due to the characteristics of the present system, in which the filters have a fixed unit gain and a fixed selectivity, we have chosen the implementation of only 2 OTAs that is shown in Figure 4.13.

**Signal monitoring**

The amplifier TXBfer that is able to drive loads up to 100 pF, is used to send the wavelet component $v_w$ to the output pad of the chip. The voltage $v_b$, used to bias $TL1$ and $TL2$, is also used to bias the amplifier TXBfer, after pass through the current amplifiers BAmp and TXAmp, as can be seen in Figure 4.13.

### 4.4.6 Summing circuit

**Ideal summing configuration**

The summation of the 16 wavelet components, implied in the inverse wavelet transform equation (1.9), is achieved by a circuit like the one shown in Figure 4.16 [39]. Suppose that $R_f$ is as high that at the frequency range of interest we have that $Z_{R_f} \gg Z_{C_f}$. Then in the Laplace domain is verified that

$$V_g(s) = \frac{Z_{C_f}}{Z_C} \sum_m V_{wm}(s) = \frac{C}{C_f} \sum_m V_{wm}(s),$$

(4.17)
where $Z_{C_f} = \frac{1}{sC_f}$ and $Z_C = \frac{1}{sC}$. And applying the inverse Laplace transform

$$v_y(t) = \frac{C}{C_f} \sum_m v_{w_m}(t). \tag{4.18}$$

Each of the $i_{w_k}$ currents shown in Figure 4.16 corresponds to the current $i_w$, shown in the circuit of Figure 4.13.

**Amplifiers TSum and AInv**

In the chip, we have included two different implementations of the amplifier that appears in Figure 4.16. These implementations correspond to the amplifier $TSum$ and the inverter amplifier $AInv$ whose schematics are shown in Figure 4.17 and 4.18.

By means of an external control signal $TS$, can be selected the amplifier with which the summation is performed.

We decided to include the usual OTA in the design and, at the same time, to probe the inverter as amplifier, since during the simulations the inverter produces lower noise levels that can be critical in the current application.
Figure 4.18: Inverter amplifier $A_{Inv}$ and main performance features obtained by Hspice simulations.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>EN</th>
<th>Vd</th>
<th>$A_{Inv}$</th>
<th>Vy</th>
<th>Vx</th>
</tr>
</thead>
<tbody>
<tr>
<td>LC = 0</td>
<td>v x</td>
<td>$E_{N}$</td>
<td>V d</td>
<td>m 1</td>
<td>m 2</td>
</tr>
</tbody>
</table>

Gain | GBW | Phase Margin |
---|-----|--------------|
40.1 dB | 843 MHz | 63.2° |

Figure 4.19: (a) Inverting summing circuit using $T_{Sum}$, with only one input signal for testing purposes. (b) Frequency response of the of the summing circuit simulated in Hspice.

### Implemented summing circuits

The first implementation of the inverting summing circuit, which makes use of the amplifier $T_{Sum}$, is shown in Figure 4.19(a). The circuit includes only one input signal because this is sufficient to probe its performance. The $C_{in}$ and $C_f$ capacitor values are the actually used in the implemented system. They provide an amplification, or more precisely, an attenuation of $0.5 \text{ pF} / 0.9 \text{ pF} = 0.56$. It follows that an amplifier of high gain is not required. A resistor of high value $R_f$, is needed to maintain the DC level of the output near the reference voltage $A_{Gnd}$.

The selected structure for the summing circuit that uses only capacitors, suggested
in [39], has been preferred over the normal configuration with resistors, since a better matching between capacitors can be achieved and the due to the simplicity of the obtained summing circuit.

Using the amplifier $A_{Inv}$ has been implemented the alternative summing circuit shown in Figure 4.20, which from the AC point of view is identical to the previous one, but its output is not kept at $A_{Gnd}$ level. Note that the frequency response is almost identical to that obtained in the previous case.

The complete summing circuit can be seen in appendix A-7. It includes the control input $TS$. When $TS$ is high the summation is performed by $TSum$ and its bias current is set by the voltage $v$ (amplified by $BAmpEn$). When $TS$ is low the summation is performed by $A_{Inv}$ and its power supply is provided through the node $v$. Finally, as is suggested in [39], the high value resistor ($R_f$ in fig. 4.19(a), is made by the PMOS transistor $m_f$ working in subthreshold region inside a dedicated N-well, while its gate voltage is controlled by the input $vftv$.

As can be seen in the frequency response of Figure 4.19(b), or alternatively in Figure 4.20(b), the summing circuit acts like a wide band-pass filter. But, if $R_f$ was lower than 100 GΩ or the load capacitance $C_L$ was larger, then the bandwidth of the filter would be shrunk.
4.5 Sensitivity of the wavelet system

In this section is explained why the classical approach of sensitivity analysis, which is made in function of the parameter of greater variation, does not apply to the implemented wavelet system due to its complexity. Instead, a simple Monte Carlo based method can be useful to compare the sensitivity of different wavelet systems.

For the purpose of this work, we are interested in the sensitivity of the reconstruction error in the presence of Gaussian variations in the constituent parts of the wavelet system. To quantify adequately this kind of error, here we define the *rms reconstruction error*.

The proposed method is explained by means of an example, in which is shown that the sensitivity of a wavelet system is inversely proportional to the square root of the scale density, as could be expected.

**Sensitivity analysis with the classical approach**

For the present analysis we consider the semidiscrete wavelet system described by the infinite summation

\[
\sum_{m=-\infty}^{\infty} \psi(r^m j\omega) \chi(r^m j\omega) = \sum_{m=-\infty}^{\infty} H_m(j\omega),
\]

(4.19)

for some scale ratio \( r > 1 \), where \( H_m(j\omega) = \psi(r^m j\omega) \chi(r^m j\omega) \).

Now, from equation (1.11) can be seen that the wavelet system has the transference function \( F_{out}(j\omega)/F_{in}(j\omega) = \sum_{m=-\infty}^{\infty} H_m(j\omega) = 1 \), acting as an all-pass filter of unit gain and zero phase shift at any frequency.

Classical approaches to analyze the sensitivity, like [40, Moschytz] can not be applied here because they make the analysis respect to only one parameter, the component of greater variation, over a restricted frequency range near the central frequency of the filter, and without consider the gain.

The variations in the individual filters \( H_m(j\omega) \), must be analyzed over all the frequency range of the spectrum, not only near their respective central frequencies,
since each filter acts together with some of its overlapped neighbor filters.

The goal in this work is to be able to implement wavelet system in analog integrated circuits. Then, we are interested in the effect that the Gaussian process variations have in the reconstruction error. Such process variations occur in all the components of the system, not only on a single element.

**Mean reconstruction error**

For the next definition, is convenient to remember that the reconstruction error is due to the ripple present in \( \sum_{m=-\infty}^{\infty} \Psi^*(r^m \omega)X(r^m \omega) \), as was explained in section 2.2.2.

**Definition 4.1.** The *rms reconstruction error* in some interval \((\omega_a, \omega_b)\) is given by

\[
\varepsilon_{\text{rms}} = \frac{1}{\omega_b - \omega_a} \sqrt{\int_{\omega_a}^{\omega_b} \left| \sum_{m=-\infty}^{\infty} H'_m(j \omega) - 1 \right|^2 d\omega},
\]  

(4.20)

were \( H'_m(j \omega) \) is the simulation (or the measure) of the filter \( K_m(\omega) \) with Gaussian variations introduced into its constituent parts.

In this definition was not said anything about how the Gaussian variations are introduced or about which these constituent parts are.

**Example 4.2.** In this example we compare the reconstruction error sensitivity of different wavelet systems with infinite scales, which make use of the same decomposition and reconstruction filters than the implemented wavelet system, but with different scale density.

Applying the Monte Carlo method, here we show that for a given Gaussian variation in the constituent parameters of the filters, the rms reconstruction error is inversely proportional to the square root of the scale density. Where the constituent parameters are \( G_1, G_2, C_1, \) and \( C_{2b} \) of equations (4.14) and (4.15).

The standard deviation of the Gaussian variations introduced in such parameters are of 10% the nominal value of each parameter.
The result of the 5 sets of Monte Carlo simulations corresponding to the scale densities \( D = 2, 4, 8, 16, 32 \) are shown in Figure 4.21, where also the average \( \bar{\epsilon}_{\text{mthr rms}} \) has been indicated for each case. Each set of Monte Carlo simulations was compound of 100 runs, but for clarity only 32 simulations are shown in the plots. As can be seen, when the scale density is increased 4 times, the noise is reduced to the half.

4.6 Response of the complete implemented wavelet system

In this section, *complete implemented wavelet system* stands for the 16 biquadratic band-pass filters, that perform the direct wavelet transform, and the inverse summing circuit, that performs the inverse wavelet transform, working together.

4.6.1 Frequency response of wavelety system

In Figure 4.22 is shown the frequency response of the complete system obtained by Hspice simulations. For the simulations of Figure 4.22(a) the typical-NMOS/typical-PMOS models have been used. Note the similitude between this graphic and the
Figure 4.22: Frequency response of the complete system obtained by Hspice simulations made with (a) typ-NMOS/typ-PMOS models, (b) fast-NMOS/fast-PMOS models, and (c) slow-NMOS/slow-PMOS models. In all cases the bias voltages $V_{16}$ and $V_{01}$ have been calibrated.

Matlab simulation shown in Figure 2.11(c), except that the phase is shifted 180°.

For Figure 4.22(b) and Figure 4.22(c) the fast-NMOS/fast-PMOS and the slow-NMOS/slow-PMOS models have been used, respectively.

For the three cases, the bias voltages $V_{01}$ and $V_{16}$ were calibrated to move the central frequencies of the first and last filters to the ends of the range (74.3 Hz-13.45 kHz), and automatically all the other 14 biquads were also set at the correct frequencies. Therefore, the frequency work band of the system can be calibrated in spite of the process variations between manufacturing runs.

### 4.6.2 Hspice mismatch simulation

The vendor, AMIS, supplies the process parameters to simulate the effect of random process variations using a Pelgrom model. The details of the model are intellectual property of AMIS.
Figure 4.23: Frequency response of the complete system obtained by Monte Carlo simulations in Hspice applying the Pelgrom model.

Basically, in the Pelgrom model [41] [42] [43] the central idea is the presence of random variations in the mobility, gate capacitance, threshold voltage, or in the transistor gain factor. The model assumes that any of these variations has a Gaussian distribution with a standard deviation inversely proportional to the square root of the device area. This is expressed by

\[
\sigma_X = \frac{A_X}{\sqrt{WL}},
\]  

(4.21)

where \(X\) is any of the mentioned parameters, and \(A_X\) is a process dependent constant. Additionally, a slightly modified formula is applied to simulate random variations in the \(W/L\) dimensions of the transistor channel.

The results of the Monte Carlo simulations over the 16 biquad filters and the summing circuit are shown in Figure 4.23. The resulting rms reconstruction error was of 2.46%. For this Monte Carlo simulations, a total of 100 runs were made but, for clarity, only 32 simulations were plotted. In this case we have used only the magnitude of the frequency spectrum to calculate the reconstruction error in the range (100 Hz, 10.0 kHz). With this, we have isolated the reconstruction error due to the process variations, from the error due to the band-pass behavior of the system that was explained in section 2.4.2.
4.6.3 Noise generated in the wavelet system

We are interested in the effect that the intrinsic noise of the complete system has in the amplitude of the output signal. This can be found by an *AC sweep analysis* in the Hspice simulator.

Hspice can model the thermal noise of resistors, and the flicker and thermal noise of CMOS transistors. We have selected the *level-3 flicker model* because it offers a more consistent simulation at the interface between the linear and the saturation regions [44].

The flicker noise exponent and the coefficient of the thermal noise in the channel have been left at their default values: \( AF=1 \) and \( GDSNOI=1 \). Whereas the flicker noise coefficient was set to \( KF=1 \times 10^{-24} \). We have made the simulations with these assumptions because we do not have the exact values for the used manufacturing process.

On the other hand, thermal noise has a Gaussian distribution but flicker noise does not. However, just for the purpose to compare the rms noise values, we consider the flicker noise also as Gaussian because we do not know what is its exact distribution.

The simulator provides the *noise voltage spectral density* \( V_n(f) \), in \( V/\sqrt{Hz} \), or the *noise current spectral density* \( I_n(f) \), in \( A/\sqrt{Hz} \), at the frequency points indicated in the AC sweep. To obtain the rms noise value at the output due to the noise that is present in the frequency range \( (f_a, f_b) \), the noise spectral density is integrated [45] [33] as follows

\[
V_{n(rms)}^2 = \int_{f_a}^{f_b} V_n^2(f) df, \tag{4.22}
\]

where \( f \) stands for frequency. Finally, the peak-to-peak voltage of the noise component, \( V_{n(p-p)} \), will be less than \( \pm 2 V_{n(rms)} \) for 95.4\% of the time, less than \( \pm 3V_{n(rms)} \) for 99.7\%, and less than \( \pm 3.8V_{n(rms)} \) for 99.94\% of the time [45] [33].

In Figure 4.24 is shown a pair of Hspice AC sweep simulations to obtain the intrinsic noise of the complete system, composed by the 16 biquad band-pass filters and the summing circuit, for both cases, when *TSum* is used and when *AInv* is used.
Integrating the noise over the range (1Hz,1MHz), and multiplying it by 6, a peak-to-peak amplitude of 1.35 mV for TSum and 0.702 mV for AInv is obtained. This result was the main reason why we decided to include an additional summing circuit with the amplifier AInv in the implemented wavelet system.

### 4.6.4 Decomposition-reconstruction transistor level simulations

Below we present 3 decomposition-reconstruction examples obtained with the designed system by Hspice transistor level simulations. The used probe signals are *tone2*, *sin2* and *chirp2* which have been used in the previous chapters, but for the present case their frequency spectrums have been frequency scaled by a factor of 1000, in order to ensure that they lie inside of the work frequency band of the implemented system. In the 3 cases the maximum amplitude of the signal is of ±30 mV, and a load of 20 pF has been put at the output. All these simulations have been made without consider process variations. If they were considered then we could expect a greater reconstruction error.

**Example 4.3.** In Figure 4.25(a) is shown the original probe signal *tone2*.

In Figure 4.25(b) is shown the wavelet transform of the probe signal, where the absolute value of the 16 wavelet components have been plotted as an image. In each
Figure 4.25: Decomposition-reconstruction of the probe signal *tone2* using the designed system obtained by Hspice transistor level simulations.

wavelet component the DC level has been subtracted, since these components are AC coupled. For a better visualization of the wavelet transform each wavelet component $v_w$, has been multiplied by a factor of $\sqrt{r^m}$.

In Figure 4.25(c) is shown the output signal whose DC level has been subtracted. Note that the signal is inverted, and has an amplitude slightly larger than the amplitude of the input signal.

In Figure 4.25(d) are shown the original probe signal, and the reconstructed signal with a suppressed DC level and multiplied by the factor $k = -0.897$. The resulting signal to error ratio between these signals is SER=17.11 dB.
**Example 4.4.** In Figure 4.26(a) is shown the original probe signal $\sin 2$.

In Figure 4.26(b) is shown the wavelet transform of the probe signal, where the absolute value of the 16 wavelet components have been plotted as an image. In each wavelet component the DC level has been subtracted.

In Figure 4.26(c) is shown the output signal whose DC level has been subtracted. Note that the signal is inverted, and has an amplitude slightly larger than the amplitude of the input signal.

In Figure 4.26(d) are shown the original probe signal, and the reconstructed signal with a suppressed DC level and multiplied by the factor $k = -0.874$. The resulting signal to error ratio between these signals is SER=21.27 dB.
Example 4.5. In Figure 4.27(a) is shown the original probe signal \textit{chirp2}.

In Figure 4.27(b) is shown the wavelet transform of the probe signal, where the absolute value of the 16 wavelet components have been plotted as an image. In each wavelet component the DC level has been subtracted.

In Figure 4.27(c) is shown the output signal whose DC level has been subtracted. Note that the signal is inverted, and has an amplitude slightly larger than the amplitude of the input signal.

In Figure 4.27(d) are shown the original probe signal, and the reconstructed signal with a suppressed DC level and multiplied by the factor $k = -0.872$. The resulting signal to error ratio between these signals is SER=15.15 dB.
Remark 4.6. In each of the previous examples has been selected the factor $k$ that produces the optimum SNR. As can be seen, the coefficient is very similar in the three cases. The gain of the complete system, which could be adjusted by the capacitors $C_{in}$ or $C_f$ of Figure 4.20, was intentionally left slightly greater than one.


Chapter 5

IC Implementation of a Denoising Wavelet System

In this chapter is described how could be incorporated the denoising capacity to the implemented wavelet system that was described in the previous chapter. In that system the reconstruction filter is just a normalization constant, for this reason the thresholding operation introduces discontinuities in the output signal, as was shown in Chapter 3.

In this chapter is presented a structure with an extra filter with which the signal can be smoothed without the necessity of a low-pass or a band-pass reconstruction filter. This new structure is the selected one for the implementation of the denoising system, whose circuital realization is presented in this chapter. In the design of the circuits, we make an extensive use of the Monte Carlo simulations to analyze the effect of the process variations over the accuracy of the denoising operation.

The complete denoising system is simulated at transistor level and the results are presented at the end of the chapter.
5.1 Description of the implemented denoising system

In this section is explained an alternative to the denoising operation presented in Chapter 3. This alternative configuration, that avoids the discontinuities produced by the thresholding function, is the one actually implemented in the fabricated IC.

5.1.1 Implemented wavelet system

The implemented wavelet system, over which we have incorporated the denoising capacity, is the one described in Section 4.1. It has 16 scales, with a scale density of 2 scales per octave. The decomposition filter is a band-pass biquadratic filter with $q = \sqrt{2}$, while the reconstruction filter is just a normalization constant.

5.1.2 Simplification of the denoising function

The delayed hard thresholding operation, described in Definition 3.3, can be represented by the block of Figure 5.1(a). But for the analog implementation purpose, it is better to represent this denoising operation as is indicated in Figure 5.1(b), i.e., using the Level and Delay blocks and a simple switch. The operation of these two blocks is described as follows.

**Definition 5.1.** Let $w(t)$ and $L(t)$ be the input and the output signals of the Level block. The Level operation is defined by

$$L(t) = \begin{cases} 
1 & \text{if } |w(t)| \geq \lambda \\
0 & \text{if } |w(t)| < \lambda
\end{cases}. \quad (5.1)$$

**Definition 5.2.** Let $L(t)$ and $\tilde{L}(t)$ be the input and the output signals of the Delay block. The Delay operation is defined by

$$\tilde{L}(t) = \begin{cases} 
1 & \text{if } L(t) = 1 \text{ for some } t \in [t - T, t] \\
0 & \text{if } L(t) = 0 \text{ for all } t \in [t - T, t]
\end{cases}. \quad (5.2)$$
Finally, the action of the switch is simply expressed by

\[ \tilde{w}(t) = \tilde{L}(t) w(t) . \]  

(5.3)

It can be easily shown that the Level operation can be implemented using two comparators and an AND gate, whereas the Delay operation can be implemented with a ramp function with a reset control and a comparator.

In Figure 5.1(c) is shown the conventional configuration of the denoising function in a system in which the reconstruction filter is equal to a normalization constant \( k \). Note that in this configuration the reconstructed signal will contain discontinuities introduced by the thresholding operation.

### 5.1.3 Alternative denoising configuration

In Figure 5.2(a) is shown an alternative denoising configuration that avoids the inconvenient of the discontinuities in the output by means of an extra biquadratic filter, without the necessity of using a low-pass or a band-pass reconstruction filter.
Suppose that $H_\psi(s)$ is the original decomposition filter that has a factor $H_\phi(s)$ which is a band-pass biquadratic filter. The system shown in Figure 5.2(a) contains three filters per scale. Two of these filters, $H_\psi(s)$ and $H_\phi(s)$, are obtained by the factorization of the original decomposition filter $H_\psi(s)$, while the extra filter $H'_\phi(s)$ is identical to $H_\phi(s)$.

In this configuration the level detection is made over the signal $w(t)$ that is continuously present at the output of the extra filter $H'_\phi(s)$, while the switch is placed before the input of the main filter $H_\phi(s)$.

With the use of this extra filter, a continuous reconstructed signal is produced without modifications in the filters of the original system. The behavior of this configuration is illustrated by some examples in the next section.

When the original decomposition filter $H_\psi(s)$ is just a biquadratic filter of second degree, the lower possible degree, then the factor function $H_\phi(s)$ is the decomposition filter itself, and the resulting configuration is like the one shown in Figure 5.2(b). This is the case of the system that we have implemented in IC.
5.1.4 Performance of the alternative configuration

In this section is compared the performance of the conventional denoising configuration with the alternative denoising configurations by means of two examples. For this purpose we use the probe signal \textit{tone2} shown in Figure 3.4(a), which we have repeated in Figure 5.3(a) but only for some selected interval in order to highlight the denoising effect.

In Figure 5.3(b) is shown the probe signal contaminated with bandlimited white noise of rms = 0.4. For comparative purposes, in Figure 5.3(c) the noisy signal filtered with the Butterworth filter (3.12) is shown, which works at approximately the same bandwidth that the wavelet systems of the next examples.

In spite of the fact that in the figures related with the next two examples are only shown the plots for the selected interval, all the simulations, including the SNR values, have been obtained over the full length of the probe signal \textit{tone2}.

\textbf{Example 5.3.} The wavelet system of this example has 32 scales, with a scale density of 4 scales per octave, a decomposition filter $H_\psi(s) = \text{biqu}(\sqrt{2}, 2)$, and a reconstruction filter equal to a normalization constant. The image shown in Figure 5.3(d) is the
wavelet transform of the probe signal of Figure 5.3(b) obtained with this system.

We have made simulations in Matlab using the conventional configuration shown in Figure 5.1(c), and the alternative configuration shown in Figure 5.2(a). For the last case we have taken $H_\phi(s) = H'_\phi(s) = \text{biqu}(\sqrt{2}, 1)$, and $\frac{H_\psi(s)}{H_\phi(s)} = \text{biqu}(\sqrt{2}, 1)$.

In Figure 5.4 are shown the results of the denoising operation with both configurations, for three different values of the delay parameter, $T_c = 0.64, 0.11, 0$. In this case the delay is expressed as periods of the central frequency of the corresponding scale, as was explained in Remark 3.11.

At the top of the Figure 5.4 are shown as images the outputs of the 32 delay blocks, $\tilde{L}_m(t)$, where black stands for 1 and white for 0. For the 6 cases the same threshold level $\lambda$ has been used, and the output has been multiplied by a different factor in each case in order to compensate some gain variations.

Note that as the delay time is decreased, the output of the $\tilde{L}_m(t)$ signals contain more and more discontinuities, which at the same time introduce discontinuities in the signal after the switch. These discontinuities pass up to the output signal in the conventional configuration, but they can not reach the output when the alternative configuration is used.

**Example 5.4.** This example is similar to the previous one, except that in this case the wavelet system has 16 scales, with a scale density of 2 scales per octave, a decomposition filter $H_\psi(s) = \text{biqu}(\sqrt{2}, 1)$, and a reconstruction filter equal to a normalization constant. The image shown in Figure 5.3(e) is the wavelet transform of the probe signal of Figure 5.3(b) obtained with this system.

We have made simulations in Matlab using the conventional configuration shown in Figure 5.1(c), and the alternative configuration shown in Figure 5.2(b).

In Figure 5.5 are shown the results of the denoising operation with both configurations, for three different values of the delay parameter, $T_c = 0.64, 0.11, 0$. In this case the delay is expressed as periods of the central frequency of the corresponding scale, as was explained in Remark 3.11.
Figure 5.4: Performance comparison between the conventional and alternative denoising configurations for a system with $H_\psi(s) = \text{biqu}(\sqrt{2}, 2)$.

At the top of the Figure 5.5 are shown as images the outputs of the 16 delay blocks, $\tilde{L}_m(t)$, where black stands for 1 and white for 0. For the 6 cases the same threshold level $\lambda$ has been used, and the output has been multiplied by a different factor in each case in order to compensate some gain variations.

**Remark 5.5.** From the previous examples can be seen that when the delay time is equal to zero, i.e., when the delay block is not already used, the conventional denoising configuration introduce the greatest possible number of discontinuities in the output signal, whereas using the alternative denoising configuration the output maintains a similar aspect for the three delay cases.

### 5.1.5 Implemented denoising configuration

As has been already mentioned, the implemented denoising system is the alternative configuration shown in Figure 5.2(b). At each scale this configuration has a *main* filter $H_\psi(s)$ and an *extra* or *duplicated* filter $H'_\psi(s)$. Both filters are identical and equal to biqu$(\sqrt{2}, 1)$. 
Figure 5.5: Performance comparison between the conventional and alternative de-noising configurations for a system with $H_\psi(s) = \text{biqu}(\sqrt{2}, 1)$.

Figure 5.6: Duplicated decomposition filter.

With this denoising system we can test the chip for a wide range of delay values ensuring the good appearance of the output signal.

The main filter is implemented with the circuit shown in Figure 4.13, while the duplicated filter is implemented with the circuit shown in Figure 5.6.

5.2 Level detection circuit

As can be seen in Figure 5.2, for the achievement of the denoising operation is required the application of the Level operation followed by the Delay operation.
The Level function have to be applied over the signal $v_{wn}(t)$ at the output of the biquad filter at each scale $m$. The designed system has been intended to filter signals contaminated with white noise. As was shown in section 3.2, the rms noise voltage at the scale $m$ of a semidiscrete WT system follows the relation

$$V_{n(rms),m} = \frac{V_{n(rms),1}}{\sqrt{r^{m-1}}},$$

(5.4)

where $r$ is the scale ratio of the system and $V_{n(rms),1}$ is the rms noise voltage at the scale 1. An equivalent expression can be obtained using peak-to-peak noise levels instead of rms values

$$V_{pp,m} = \frac{V_{pp,1}}{\sqrt{r^{m-1}}},$$

(5.5)

For example if the noise voltage component at the scale 1 has 30mV peak-to-peak, then the noise component at the scale 16 has only $30\text{mV}/\sqrt{15} = 2.23\text{mV}$. We are forced to work with these low level signals because the implemented system can only cope with signals up to 43.5 mV peak-to-peak to keep the non-linearity below 1%. (But we accept a slightly greater non-linearity to be able to work with signals up to 60 mV peak-to-peak.)

The effect that the process variations have over the offset voltage at the output of the biquad filter is shown on Table 5.1. It was obtained with 100 Monte Carlo simulations using Hspice. The uncertainty of $\pm 7$ mV at scale 16 is 6 times greater than the 2.23mV peak-to-peak of the previous example. Therefore, a direct comparison between the signal $v_{wn}(t)$ and some threshold level $V_{th} (\lambda)$ can not be done. To reduce the mismatch effects we decided to amplify the wavelets components $v_{w,m}$.

This amplification must be AC coupled to allow the elimination of the offset voltage present at the biquad outputs.

5.2.1 Amplification of the wavelet components

The amplification of the wavelet component $v_{wn}(t)$ is accomplished by the amplifier $TAmp1$ shown in Figure 5.7. It is in non-inverting amplifier configuration using
Table 5.1: Offset voltage of the DC level at the output of the biquadratic filter obtained by 100 Monte Carlo simulations using Hspice.

<table>
<thead>
<tr>
<th>scale</th>
<th>offset (mV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.2 \pm 7.4$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.3 \pm 7.2$</td>
</tr>
<tr>
<td>8</td>
<td>$-0.4 \pm 7.1$</td>
</tr>
<tr>
<td>12</td>
<td>$-0.6 \pm 7.1$</td>
</tr>
<tr>
<td>16</td>
<td>$-1.2 \pm 7.0$</td>
</tr>
</tbody>
</table>

$\pm 3\sigma$ uncertainties

capacitors instead of resistors as input and feedback impedances, as is suggested in [39]. The high value feedback resistor, implemented with transistor $m_{f1}$, is needed to hold the mean voltage value of $C_{f1}$ near to zero. The transistor $m_{f1}$ is a minimum size PMOS working in subthreshold region and placed within a dedicated N-well. Its resistance is controlled by the external voltage $v_{f1}$ [39].

At each scale band all the components of this circuit are identical except the capacitor $C_{ext}$. Hence different amplification levels are obtained at each scale. Our purpose has been to obtain equal rsm noise levels at all the scales and to stretch the range of the signal from $\pm 30\text{mV}$ up to $\pm 200\text{mV}$. To reach it, at each scale $m$ the amplification must be given by

$$A_m = \frac{200}{30} \sqrt{m^{-1}}. \quad (5.6)$$

The original operation range of $\pm 30\text{mV}$ has been amplified only up to $\pm 200\text{mV}$ because beyond of this limit the distortion introduced due to the source-gate dependence of the resistance of $m_{f1}$ grows quickly.

5.2.2 Amplifier TAmp

The schematic circuit of the amplifier $T Amp$ is shown in Figure 5.8. It has only one stage and its open loop gain $A_o$ in each scale is in the range $(215\text{V/V}, 395\text{V/V})$. 
From the circuit in figure 5.7 can be seen that the amplification of $v_{wa}$ is given by

\[ A_m = \frac{v_{wa}}{v_w} = \frac{C_{ext,m} + C_{min} + C_{f1}}{C_{f1} - (C_{ext,m} + C_{min} + C_{f1})/A_o,m}. \] (5.7)

The $A_o$ values of $T$Amp shown in figure 5.7 were obtained simulating with the typical MOS model. This gain can vary a lot between manufacturing runs. But if the system bias voltages $V_{01}$ and $V_{16}$ are calibrated to adjust the central frequencies of the first and the last biquadratic filters to their correct values, then the open loop gain varies as is shown in Table 5.2. The values in this table were obtained by simulation using the typical and the four worst case models.

The worst $A_o$ variation is of $-9.8\%$ corresponding to the scale 1 with the fast-NMOS/fast-PMOS model. But when this uncertainty is used in (5.7) the effect over the closed loop gain is of only $0.4\%$. The worst case of this final uncertainty of $A_m$ corresponds to the scale 16 with the fast-PMOS/fast-NMOS model with $0.6\%$ of variation. It follows that an amplifier with higher gain is not necessary since these two errors are bellow $1\%$. 

Figure 5.7: Level detection circuit.
Table 5.2: Worst case values of the open loop gain, $A_o (V/V)$, of the $T Amp$ amplifier. Simulations by Hspice.

<table>
<thead>
<tr>
<th>Scale</th>
<th>typ NMOS</th>
<th>fast NMOS</th>
<th>slow NMOS</th>
<th>fast PMOS</th>
<th>slow PMOS</th>
<th>fast PMOS</th>
<th>slow PMOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>215</td>
<td>194</td>
<td>220</td>
<td>206</td>
<td>211</td>
<td>206</td>
<td>211</td>
</tr>
<tr>
<td>4</td>
<td>292</td>
<td>271</td>
<td>288</td>
<td>281</td>
<td>286</td>
<td>281</td>
<td>286</td>
</tr>
<tr>
<td>8</td>
<td>347</td>
<td>331</td>
<td>333</td>
<td>334</td>
<td>340</td>
<td>334</td>
<td>340</td>
</tr>
<tr>
<td>12</td>
<td>369</td>
<td>355</td>
<td>353</td>
<td>357</td>
<td>361</td>
<td>357</td>
<td>361</td>
</tr>
<tr>
<td>16</td>
<td>395</td>
<td>379</td>
<td>379</td>
<td>382</td>
<td>386</td>
<td>382</td>
<td>386</td>
</tr>
</tbody>
</table>

The $C_{ext,m}$, $C_{min}$ and $C_{f1}$ values shown in Figure 5.7 have been obtained with fine adjustment using Hspice and the circuit extracted from the designed layout considering all the parasitic components. For this reason there are slight discrepancies between the gain $A_m$ obtained with (5.7) using these capacitor values and the ideal gain given by (5.6). Nevertheless, the previous analysis of uncertainties even holds.

**Inverting amplifier**

The Level operation implies the comparison of the wavelet component $w_m$ against the levels $\lambda$ and $-\lambda$. But the same task can be achieved comparing any of these levels against the two signals $w_m$ and $-w_m$. In this work the second implementation is preferred because with an inverting amplifier using capacitors as impedances only
Dynamic signals can be amplified while the DC components see the capacitors inputs as open circuits.

The amplifier $TAmp2$ of figure 5.7, together with capacitors $C_{in2}$ and $C_{f2}$ have the purpose of invert the signal $v_{wa}$. Ideally, these capacitors should provide an amplification of $-1$. But the actual amplification of the circuit is given by

$$A'_m = \frac{v_{wb}}{v_{wa}} = -\frac{C_{in2}}{C_{f2} + (C_{in1} + C_{f2})/A_{o,m}}. \quad (5.8)$$

If $C_{in2}$ and $C_{f2}$ were equal, the gain $A'_m$ calculated with this formulae and using the worst cases of $A_o$ shown in Table 5.2 will fluctuate between $-0.9900$ and $-0.9950$.

Nevertheless, the implemented values of these two capacitors were obtained with fine adjustment using Hspice and the circuit extracted from the designed layout considering all the parasitic components. The gain really obtained $A'_m$ is in the range $-1 \pm 0.25\%$. In this case the error is also below 1%.

**Offset voltage at $TAmp1$ and $TAmp2$ outputs**

Another error considered in the level detection is the offset voltage at the nodes $v_{wa}$ and $v_{wb}$ due to the mismatch. 100 Monte Carlo DC sweeps were done to estimate them. The results are shown in Table 5.3 for various scales. The worst case media value of the offset voltage was $-1.3mV$ for scale 16 and the worse case uncertainty was $\pm10.8mV$ corresponding to scale 1. Both values in node $v_{w1}$. Comparing this last error with $\pm200mV$ is obtained $\pm10.8/\pm200 = \pm5.4\%$. An important error.

**5.2.3 Effect of the internal noise in the level detection**

Now we are interested in the noise present at the input of the comparators. Specifically we want to know how the amplitude of the noise component affect the level detection.

The noise internally produced by the system at the nodes $v_{wa}$ and $v_{wb}$ is shown in Figure 5.9(a) for the scale 8. It was obtained by Hspice simulations.
Table 5.3: Offset voltage of the DC level at the nodes \( v_{w1} \) and \( v_{w2} \) obtained by 100 Monte Carlo simulations using Hspice.

<table>
<thead>
<tr>
<th>scale</th>
<th>node ( v_{w1} ) (mV)</th>
<th>node ( v_{w2} ) (mV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.1 ± 10.8</td>
<td>0.5 ± 7.3</td>
</tr>
<tr>
<td>4</td>
<td>0.4 ± 10.5</td>
<td>-0.1 ± 6.8</td>
</tr>
<tr>
<td>8</td>
<td>-0.1 ± 10.3</td>
<td>-0.5 ± 6.7</td>
</tr>
<tr>
<td>12</td>
<td>-0.5 ± 10.2</td>
<td>-0.6 ± 6.6</td>
</tr>
<tr>
<td>16</td>
<td>-1.3 ± 10.2</td>
<td>-0.8 ± 6.5</td>
</tr>
</tbody>
</table>

\( \pm 3\sigma \) uncertainties

in Figure 5.9(b) are shown the rms noise values, \( V_{n(rms)} \), at nodes \( v_{wa} \) and \( v_{wb} \) for various scales. They were obtained integrating the noise spectrum values over the frequency ranges (1Hz,1MHz) and (1Hz,1GHz). The worst case rms noise value at the two nodes was 1.23 mV corresponding to scale 16. This rms noise level produce a noise component with amplitude smaller than 1.23*3.4*2 = 8.36mV the 99.94% of the time. Greater levels of noise would be obtained if the amplifier \( TAmpl \) had two stages instead of one.

Comparing the 8.36mV peak-to-peak level with the amplitude of \( \pm 200mV \) is obtained the value of \( (8.36mV/2)/\pm 200mV = 2.1\% \).

### 5.2.4 Comparator \( TComp \)

The schematic diagram of the comparator \( TComp \) is shown in Figure 5.10. To simulate the effect of the process variations over the level detection, the circuit of Figure 5.11(a) has been used. 100 Monte Carlo DC sweeps were made using Hspice. The input \( v_{in} \) was linearly varied in the range (-30mV,30mV). The level detection was taken from the output \( LEV \). The results have been plotted in Figure 5.11(b) for the scale 8. Whereas in Figure 5.11(c) are shown the offset voltages for various scales using the format mean \( \pm 3\sigma \).

The obtained average values are not important because they are almost equal and
therefore there is not variation between the different scales. By another hand, the worst uncertainty corresponds to the scale 1 which is equal to ±5.9mV. comparing this value with an amplitude of ±200mV is obtained a value of ±5.9mV/±200mV = 3.0%.

5.2.5 Total error in the level detection

The accuracy of the level detection is affected mainly by 3 sources of uncertainty:

1. ±5.5% of voltage offset at the nodes $v_{wa}$ and $v_{wb}$ due to the mismatch in $TAmp1$ and $TAmp2$.

2. ±2.1% of the noise component at the nodes $v_{wa}$ and $v_{wb}$ due to the internal noise produced in the transistors of the transconductors $TL$ and the amplifiers $TAmp$.

3. ±3.0% of uncertainty at the input of the comparators due to the mismatch in $TComp$. 
These uncertainties are non-correlated errors that can be combined as orthogonal vectors. The result is $\pm \sqrt{(5.5\%)^2 + (2.1\%)^2 + (3.0\%)^2} = \pm 6.6\%$ compared with $\pm 200\text{mV}$, or $\pm 6.6\% \times \pm 200\text{mV} = \pm 13.2\text{mV}$ expressed as an absolute error.

### 5.2.6 Level detection error as an angular uncertainty

A discrepancy between the instant at which the level of the signal should be ideally detected and the moment of actual detection is produced by the sources of uncertainty above explained. The input signal to the Level circuit can have any shape that the
previous biquadratic filter allows to pass through itself. If a pure sinusoidal signal is used then the time uncertainty can be traduced to an angular uncertainty.

In Figure 5.12(a) a sinusoidal signal $v_w$ have been used as the stimulus of the Level circuit of the scale 8. The sinusoidal has a frequency of $f_s = f_0/r^7 = 1189$Hz and an amplitude of $30mV/r^7/2 = 8.92mV$. 100 Monte Carlo Hspice simulations have been done to simulate the mismatch effect in $T_{Amp}$ and $T_{Comp}$ over the detection time. The time uncertainty is measured with the central point of the transitions of the signal $L$. This time uncertainty is converted to angular uncertainty multiplying it by the factor $360^\circ/T_s$, where $T_s = 1/f_s$. In this way the uncertainty is expressed in degrees relative to the period of the sinusoidal with a frequency equal to the central frequency of the biquadratic filter at such scale.

In fig. 5.12 are shown the angular uncertainties measured in this way for various scales. There are four different kind of transitions corresponding to the detection instants $A$, $B$, $C$, $D$. 

<table>
<thead>
<tr>
<th>Scale</th>
<th>$A' - A$</th>
<th>$B' - B$</th>
<th>$C' - C$</th>
<th>$D' - D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.7^\circ \pm 4.0^\circ$</td>
<td>$2.6^\circ \pm 4.2^\circ$</td>
<td>$1.8^\circ \pm 3.3^\circ$</td>
<td>$1.7^\circ \pm 3.5^\circ$</td>
</tr>
<tr>
<td>4</td>
<td>$2.0^\circ \pm 3.4^\circ$</td>
<td>$2.4^\circ \pm 3.3^\circ$</td>
<td>$1.9^\circ \pm 3.4^\circ$</td>
<td>$0.2^\circ \pm 3.5^\circ$</td>
</tr>
<tr>
<td>8</td>
<td>$1.8^\circ \pm 3.3^\circ$</td>
<td>$1.7^\circ \pm 3.5^\circ$</td>
<td>$1.9^\circ \pm 3.4^\circ$</td>
<td>$0.2^\circ \pm 3.5^\circ$</td>
</tr>
<tr>
<td>12</td>
<td>$1.9^\circ \pm 3.4^\circ$</td>
<td>$0.2^\circ \pm 3.5^\circ$</td>
<td>$1.9^\circ \pm 3.4^\circ$</td>
<td>$0.2^\circ \pm 3.5^\circ$</td>
</tr>
<tr>
<td>16</td>
<td>$-2.1^\circ \pm 3.7^\circ$</td>
<td>$-1.3^\circ \pm 3.8^\circ$</td>
<td>$1.9^\circ \pm 3.4^\circ$</td>
<td>$0.2^\circ \pm 3.5^\circ$</td>
</tr>
</tbody>
</table>

Figure 5.12: Angular detection uncertainty of the Level circuit
5.3 Delay time circuit

In the Level circuit shown in Figure 5.7, when any of the signals \( v_{wa} \) or \( v_{wb} \) is lower than the threshold level \( V_{th} \), the output \( DlyE \) is set high.

In the Delay operation, performed by the circuit of Figure 5.13, the signal \( Sw \) is enabled when the signal \( DlyE \) is enabled. But \( Sw \) is disabled with a delay time respect to the \( DlyE \) disabling, as is required by Definition 5.2.

The transistor \( m_{D7} \) is used as a current source. Its bias voltage \( V_{Dly} \) is determined by the circuit composed by the transistors \( m_{D1} - m_{D6} \). When the control signal \( DlyS \) is active the bias current is duplicated. The PMOS transistor \( m_{C} \) is used as a gate-active capacitor.

Supposing that the denoising enable \( DE \) is active, if \( L \) has been low for a long time, then the voltage \( V_{rmp} \) is set to \( DVp \). When \( L \) is again set high, the capacitor \( m_{C} \) is quickly charged trough the now open transistors \( m_{D8} \) and \( m_{D9} \) until the voltage \( V_{rmp} \) is equal to \( DGn \). This activates the signal \( D \).

If \( L \) is now set low, the transistors \( m_{D8} \) and \( m_{D9} \) become an open circuit. Then the bias current provided by \( m_{D7} \) charges the gate of \( m_{C} \) and the voltage \( V_{rmp} \) has a ramp behavior until it reaches \( DVp \). If \( L \) remain in low state enough time to allow that voltage \( V_{rmp} \) crosses the threshold voltage of the gate \( Nand1 \), then the output \( D \) is set low. But if the signal \( L \) goes active again before that \( V_{rmp} \) reaches the threshold.

---

**Gate Area (\( \mu m^2 \))**

| \( m_{D1} \) | 37.5/10 |
| \( m_{D2} \) | 10/2 |
| \( m_{D3}-m_{D7} \) | 10/10 |
| \( m_{D8},m_{D9} \) | 5/5 |
| \( \lambda = 0.3\mu m \) |
| \( m_{C} \) | 4718 (=11.05pF) |
Figure 5.14: Implementation of the analog switch required for the denoising operation.

gate voltage then the signal $D$ remains active.

The dimensions of transistors $m_{D8}$ and $m_{D9}$ and their series configuration reduce the current when the voltage $V_{rmp}$ is suddenly pull down to $DGn$.

The bias current through $m_{D7}$ and the size of the capacitor $m_C$ have been designed to allow a delay time of $T_c = 10$ when $DlyS$ is low, and $T_c = 20$ when $DlyS$ is high, where $T_c$ has the meaning given in Remark 3.11. The uncertainty on the delay time is of ±3.7% and does not have an important effect in the performance of the denoising function. Additional fine adjustment in the delay time can be achieved if we introduce a little difference between the power supplies $DVp$ and $Vpp$.

### 5.4 Analog switch

In Figure 5.14 is shown the circuitual implementation of the analog switch required in the denoising implementations of Figure 5.2(b). The transistors $m_{x1}$ and $m_{x2}$ control the pass of the signal $v_x$ to the point $v_{xsw}$. The transistors $m_{g1}$ and $m_{g2}$ control the connection between $V_{AGnd}$ and $v_{xsw}$.

When $V_{AGnd} = 700mV$ and the input signal $v_x$ is kept inside the range $700mV \pm 30mV$ then the analog switches are never simultaneously in conduction when $Sw$ changes its logical value. This result was found making DC sweep simulations with Hspice. Nevertheless some voltage peaks were found in the node $v_{xsw}$ due the charge
injection via the gates of the transistors $m_{x1}$, $m_{x2}$, $m_{g1}$ and $m_{g2}$ during the transitions of $D$. Each of these transistors have a gate-active capacitance of $C_{gb} = 4.4nF$.

The simulations to explore the clock feed through have been made connecting the node $v_{xsw}$ to the input capacitor $C_1$ of the main biquadratic filter shown in Figure 4.13. Then a change of 2.5V at the gate of one of the transistors produce a change in the node $v_{xsw}$ determined by the divisor made up with $C_{gb}$ and $C_1$, which is given by

$$\Delta v_{xsw} = 2.5V \times \frac{C_{gb}}{C_{gb} + C_1} = 2.2mV.$$ 

This can be the maximum contribution of each transistor.

Only NMOS transistors could be used to make the analog switch since the signals are very near of the negative rail, at only 0.7V. But PMOS transistors of similar dimensions also were included to compensate the charge injection through their gates.

Simulations at scale 1, the scale with faster commutations, have shown a maximum voltage peak of $8mV$ in the node $v_{xsw}$ during the transitions of $D$. These peaks have a duration of 2ns. The frequency of this glitch is totally out of the range of the biquad.

The resistors $R_x$ and $R_g$ were kept to prevent exesives currents due to external missconections of the chip. They have no effect on the bandwidth of the biquad. They can have any value between 1kΩ and 10kΩ.

\section{Additional circuits}

Complementary circuitry non particularly related with the denoising operation is presented in this section including all the digital circuitry used in the system and a structure to monitor some important analog signals.

\subsection{Digital circuitry}

Two different types of digital circuits have been designed for this system.

1. Non-commuting gates intended to allow programmability in the circuit. In these
gates the source and bulk of the PMOS transistors are connected to $V_{pp}$ and the source and bulk of the NMOS transistors are both connected to $G_{nn}$.

In Figure 5.15 are shown the 4 gates of this type: $Inv$, $Nand$, $Nor$ and $Nor_3$. In these gates all the transistors have a channel of minimum length and an aspect ratio $W/L = 10\lambda/2\lambda = 5$.

2. Commuting gates used for the dynamic digital signals of the denoising system. The PMOS of these circuits are set in dedicated N-wells with the bulk and source connected to $DV_p$. The bulk of the NMOS is connected to $G_{nn}$ while the source of the NMOS is connected to the digital ground $DG_n$. Both grounds remain isolated by the depletion region around the source, and the most of the current peaks that are present during the digital transitions are sunked through digital ground.

In Figure 5.16 are shown the 4 gates of this type: $Inv_D$, $Inv_{DS}$, $Nand_{DS}$, $Nor_{DS}$. All their transistors have an aspect ratio of $W/L = 5\lambda/5\lambda = 1$. This
lower value produce slower commutation rates and slower current peaks during the transitions.

The only exception is the inverter InvD. Having minimum length channel and with the PMOS wider than the NMOS allowing a better balance of the threshold voltage of the gate. This inverter is intended to act as a buffer to manage external capacitive loads up to 100pF.

### 5.5.2 Analog monitoring

The analog signals $v_x s w$, $v_w$, $v_w a$ and $v_w b$ are multiplexed using the circuit shown in figure 5.17. The follower amplifier TXBfer has been already presented in section 4.4.4 and it can manage capacitive charges up to 100pF. The analog transmission gates $T G$ and $S W$ are shown in Figure 5.18(a) and 5.18(b).

Note that when the analog output is not selected, $A E$ high, the TXBfer input is connected to $V_{A G n d}$.

The selection of a particular signal by the control signals $S 0$, $S 1$, and $A E$ is
Figure 5.17: Analog monitoring circuit.

Table 5.4: Programming the Analog Multiplexor.

<table>
<thead>
<tr>
<th>_AE</th>
<th>_S1</th>
<th>_S0</th>
<th>output signal</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>L</td>
<td>L</td>
<td>$v_{wb}$</td>
</tr>
<tr>
<td>L</td>
<td>L</td>
<td>H</td>
<td>$v_{wa}$</td>
</tr>
<tr>
<td>L</td>
<td>H</td>
<td>L</td>
<td>$v_{xsw}$</td>
</tr>
<tr>
<td>L</td>
<td>H</td>
<td>H</td>
<td>$v_w$</td>
</tr>
</tbody>
</table>

explained in Table 5.4.

5.6 Denoising transistor level simulations

In this section we present a denoising simulation at transistor level of the probe signal \textit{tone2} that is shown in Figure 5.19(a). This signal has been conveniently frequency scaled by a factor of 1000, in order to ensure that its frequency spectrum lies inside of the work frequency band of the implemented system.

The maximum amplitude of the probe signal at the high frequency pulse has been
Figure 5.18: Analog switches required for the analog monitoring circuit.

set to ±10 mV, in order to avoid that the outputs of the amplifiers \textit{Amp1} and \textit{Amp2} exceed the range of ±200 mV to ensure that the resistance of transistors \textit{mf1} and \textit{mf2} remains at a very high value.

In Figure 5.19(b) is shown the probe signal contaminated with bandlimited white noise of rms = 4.0 mV.

In Figure 5.19(c) are plotted as an image the 16 components \(v_{wa}\) at the output of the amplifiers \textit{Amp1}. In each component has been subtracted the level \(AGnd = 700\) mV, since these signals are DC coupled with the comparators. This image is equivalent to the wavelet transform of the noisy signal but with each component multiplied by a factor of \(\sqrt{r^{m/2}}\).

In Figure 5.19(d) are shown the 16 outputs of the delay function \(\tilde{L}_m\) plotted as an image, where black stands for 1 and white for 0.

In Figure 5.19(e) is shown the denoised output signal whose DC level has been subtracted. Note that the signal is inverted, and has an amplitude slightly larger than the amplitude of the original probe signal.
Figure 5.19: Denoising of the probe signal \textit{tone2} simulated at transistor level by Hspice.
In Figure 5.19(f) are shown the original probe signal \textit{tone2}, and the reconstructed signal with a suppressed DC level and multiplied by the factor $k = -0.834$. The resulting signal to noise ratio between these signals is $\text{SNR}=6.85$ dB.

This simulation has been made without consider process variations. If they were considered then we could expect a greater reconstruction error.
Chapter 6

Layout Design

Through MOSIS-organization, the designed chip has been manufactured in the AMI Semiconductors C5 process, which is a 0.5 µm CMOS technology intended for mixed-signal applications.

The layout of the system has been designed using the Tanner tools: the schematic editor, the layout editor, the extraction tool and the layout-versus-schematic verification tool.

In this chapter are presented the particularities of the layout design.

6.1 Floor planning

The floor planning of the chip is shown in Figure 6.1(a). The 8 blocks at the right side of the chip are mirror images of the 8 blocks at the left side.

As can be seen in the inner-view of the blocks shown in Figure 6.1(b), the section nearest to the PAD is the switching digital circuitry, which is composed by the Delay circuit and a digital multiplexor used to monitor some internal digital signals. The MOS transistors acting like switches of the input signal are in the next section followed by a biquadratic filter, here called main biquad. The level circuit and the analog multiplexor are in the next section. Finally, in the farther section was put the second biquadratic filter, here named duplicated biquad in order to protect the
wavelet components from digital noise, since the denoising operation is controlled by the comparison of such components against the threshold level.

**Signals and power distribution**

The floor planning of the power distribution nets is shown in Figure 6.2(a) for analog circuits and in Figure 6.2(b) for digital circuits.

The distribution of the analog and digital input signals used in the programming of the system is shown in Figure 6.2(c). While in Figure 6.2(d) is shown the flow of the processed signal $v_x$ passing through the main and the duplicated biquadratic filters, followed by the flow of the currents $i_w$ up to the sum point and the outputs $y_t$ and $y_v$. In this figure are also shown the $y$ outputs of each of the 16 blocks.

The digital and analog programming signals in some cases run inside dedicated channels but in other cases pass over poly-poly capacitors. In this last case only metal-3 has been used. All these lines are kept at constant voltage levels. Each of
these poly-poly capacitors is affected only by a little fraction of its value and in all the blocks the effect is the same. The unique fluctuating input signal is \( v_x \), which has not been anywhere lied over poly-poly capacitors. The low frequency at which the system works allows the fast recovery of all the input programming lines when the upper capacitor plate manifest voltage changes. This consideration has been carefully verified by manual calculations.

Allowing the pass of some lines over the poly-poly capacitors has been avoided an unnecessary increment in the chip size.

6.2 Design rules

The layout design was done using the tool Tanner L-Edit. The design was completely full-custom, but it was simplified following a modular and hierarchical design.

The vendor independent scalable design rules provided by MOSIS were used for the majority of the layout design process. Specifically, the set of design rules \( SCN3ME\_SUBM \) has been utilized. Only in few cases the native design rules were
used for fine adjustments.

The *SCN3ME_SUBM* design rule set is freely available via Internet in the MOSIS web page [46]. The native design rules of the AMIS-C5 process are intellectual property of AMI Semiconductor, Inc., and is only available for customers previous confidential agreement.

**Electromigration**

In conditions of high current density the metal in interconnections or contacts can migrate over time, getting thinner in some places and thicker in others, at the point that open or short circuits can be produced. This electromigration phenomenon grows at higher temperatures and higher current densities.

Nevertheless, pay much attention for electromigration rules for prototype designs is not recommended for MOSIS [46], since a prototype is not intended to work for a long period of time. Even working in extreme conditions the electromigration can take several years to produce a failure in the chip.

Anyway, the native electromigration rules were verified for the present design. The result was that with the low current consumption of the chip, less than 1mA, there will not be any electromigration problem.

**Minimum density rules**

Chemical-Mechanical Polishing is used in the selected MOS process to achieve planarity, which implies some restrictions in the density of the metal and poly layers [46]. In the case of the AMIS C5 process, the next rules must be verified:

- Minimum poly layer density: 15%
- Minimum metal-1 layer density: 30%
- Minimum metal-2 layer density: 30%
MOSIS indicates that the smallest unit of applicability of these rules is a 1mm x 1mm square.

**Antenna rules**

“The Antenna Rules deal with process induced gate oxide damage caused when exposed polysilicon and metal structures, connected to a thin oxide transistor, collect charge from the processing environment (e.g., reactive ion etch) and develop potentials sufficiently large to cause Fowler Nordheim current to flow through the thin oxide” [46].

The antenna rules establishes upper limits to the ratio between the poly or metal area to the area of the connected gates.

When the poly or metal is also connected to an active junction area then the rule does not apply.

A careful review of the antenna rules were made for the present design. Dummy transistors that do not interfere with the system functionality were put in the few cases where these rules were being violated.

**Layer color code**

For the design of the layouts shown in this chapter has been used the color code shown in Figure 6.3. The shown layers are *N-well, active, poly, poly-2, contact, high-res-implant, metal-1, metal-2, via-1, via-2* and *metal-3*. The layers corresponding to the *p-select* and the *n-select* areas were hidden for clarity of the layout images.

**6.3 Rules of good layout design**

**Reduction of the effect of systematic fluctuations**

There are eight well known rules to reduce the effect of the mismatch during the layout design:
The purpose of the first rule is to reduce the effect of local random fluctuations. This rule is implicit in the Generalized Pelgrom model used in the previous two chapters for the Monte Carlo simulations.

The most of these rules have the purpose of reduce the effect of global systematic gradients. They are not considered in the used Pelgrom model. Hence, the mismatch results obtained by Monte Carlo simulations will grow in the manufactured chip if these guidelines are not followed.
For the layout design of the wavelet chip, an agreement between the fulfillment of these rules and the area consumption had to be made. Nevertheless, the rules of minimum distances, common centroid structures, same orientation and same structures always were fulfilled during the design of the matched transistor pairs.

The transistors in the mirrored blocks at the right side and the left side of the chip maintain the same orientation because the current flows vertically.

Using the power dissipation in the chip, it is estimated a temperature increase in the package of less than 1° C respect to the air. Then, the effects due to temperature gradients can be ignored.

The rule of same surrounds could be only partially satisfied due to area considerations.

Substrate noise

To reduce the substrate noise, coupling guard rings have been put around each set of matched transistor pair in the amplifiers $TL$, $TAMP$, $TCOMP$, $TXBfer$, $TSum$, likewise around each other sensible analog structure.

6.4 Basic blocks

In this section the basic blocks have been grouped in accordance with the used design methodology.

6.4.1 Standard cells

The layout of the next digital circuits were designed with the same height to fit together as in the standard cell methodology. But in this case all the connections were manually traced.

- The gates $Inv$, $Nor3$ and the analog switches $ASw$ and $TG4xE$ shown in Figure 6.4, have been designed in cells with a height of $53\lambda$, where $\lambda = 0.3 \mu m$. 
• The gates $InvD$, $InvDS$, $NandDS$, $NorDS$, the multiplexor $Mux2\_DS$ and the transmission gate $TG4xE_D$ shown in Figure 6.5, have been designed in cells with a height of $46\lambda$.

In both cases the N-well and the substrate were connected to the rails with as many contacts as possible. With this caution, the substrate and N-well resistivity is reduced which, at the same time, reduces the possibility of latch-up.
6.4.2 Analog switches and delay selector

In Figure 6.6 are shown the layouts of \textit{TGsimple} which is the switch that controls the pass of the input signal to the \textit{main biquadratic filter}, \textit{CSwitch} that sets the selectivity of this filter, and \textit{EDly} used to select between two different delay times.

6.4.3 OTAs and Amplifiers

The matched transistor pairs of all the OTAs have been designed using common centroid arrangements and guard rings placed around each matched pairs:

- The layout of a pair of \textit{TL} OTAs is shown in Figure 6.7.
- The layouts of the amplifiers \textit{TXBfer} and \textit{TAmpl} are shown in Figure 6.8.
- The layout of two amplifiers \textit{TComp} is shown in Figure 6.9.
- The layouts of the amplifier \textit{TSum} and the inverter amplifier \textit{AInv} are shown in Figure 6.10.
Figure 6.7: Two TL OTAs.

Figure 6.8: Amplifiers TXBfer and TAmp.
Figure 6.9: Two TComp amplifiers.

Figure 6.10: Amplifiers TSum and AInv.
6.4.4 Current amplifiers

The layouts of the three current amplifiers used in the system, $B_{Amp}$, $BX_{Amp}$ and $B_{Amp EN}$ are shown in Figure 6.11. Guard rings have been placed around each type of CMOS transistors.

6.4.5 Resistors

The layouts of the resistors named $R_{2k}$ and $R_{7k}$ are shown in Figure 6.12. These resistors were made in poly-2 with high resistivity implant.

In the same figure is shown the layout of a minimum size PMOS transistors inside a dedicated N-well. This transistor is used to implement the high value resistors shown in Figure 4.19 and 5.7.
6.4.6 Capacitors

The most capacitors of the system have been made using poly-2 over poly. The behavior of these capacitors is very linear respect to the applied voltage. Only the capacitors of the Delay circuit were implemented using the capacitance of PMOS gates. The exact value of these capacitors is less critical and a lot of area has been saved using this type of capacitors.

The majority of the needed capacitors are in the order of pF, requiring a lot of area and reducing the importance of the fringe capacitance. To make the layout of the capacitors only was needed to fill the free areas between the placed blocks. The capacitors designed in this way have no much regularity and their values can present deviations from the ideal values. But these deviations will be low in relation with the high values of the used capacitances. Moreover, these low deviations will be systematically present in all the 16 blocks and a low effect in the ripple of the final summation can be expected.

In all the poly-poly capacitors, the lower plate (poly-1 plate) has been connected to the node with constant value. In this way the effect of the parasitic capacitance with the substrate is reduced. In all the cases the substrate below the capacitor has been connected to $V_{pp}$ or Gnd and a guard ring around the capacitor has been placed.

The poly-poly capacitors can be observed as big red areas in almost all the layout images of the rest of the chapter. The 34.6% of the core surface is covered by this type of capacitors.

On the other hand, in Figure 6.13 is shown the layout of the capacitors $C_{f1}$, $C_{in2}$ and $C_{f2}$. The effects of the fringe capacitance, the parasitic capacitance of the interconnections and the rule of same surrounds were carefully considered for the design of this layout in order to ensure the desired ratios between these small capacitors.

Finally, the PMOS capacitors used for the Delay circuit can be seen in Figure 6.16.
Figure 6.13: Capacitors $C_{f1}$, $C_{in2}$ and $C_{f2}$ with carefully matched ratios.

6.5 Layouts of functional modules

In this section are shown the layouts of the blocks of medium size constituted with the basic blocks already presented.

6.5.1 Main and duplicated biquadratic filters

The duplicated biquadratic filter shown at Figure 6.14 contains almost the same elements than the schematic circuit of Figure 5.6. While the layout of the main biquadratic filter shown in Figure 6.14 contains the $TL$ OTAs and the capacitors of Figure 4.13, the transmission gates shown in Figure 5.14 and the transistors $m_{D1} - m_{D6}$ of Figure 5.13.

6.5.2 Level circuit and analog multiplexor

The layout of Figure 6.15 includes the Level circuit, whose schematic diagram is shown in Figure 5.7, and the analog multiplexor shown in Figure 5.17.
6.5.3 Delay circuit and digital Multiplexor

The layout of Figure 6.16 contains the Delay circuit, whose schematic diagram is shown in Figure 5.13 (without the $m_{D1} - m_{D6}$ transistors), and also contains the components of a digital multiplexor.

6.5.4 Summing circuit block

In Figure 6.17 is shown the layout of the block ModSum corresponding to the summing circuits shown in Figure 4.19 and 4.20.

The other two modules shown in Figure 6.17 correspond to a low-pass and a high-pass biquadratic filters that had been considered in the initial design (to compensate the band-pass behavior of the complete wavelet system).
Figure 6.15: Level circuit and analog multiplexor.

Figure 6.16: Delay circuit and digital multiplexor.

6.5.5 Resistor ladder and external capacitors

The three layouts of Figure 6.18 are rotated 90° counterclockwise. As can be seen in the floor-planning they are vertically placed and cover almost all the height of the core. Exactly at the middle was placed the Resistor ladder. It provides the adequate bias voltages to the 16 identical blocks. The resistors of the ladder has been made with a continuous strip of poly-1.

In the floor planning can be seen that the external capacitors are placed along two vertical strips that are lied between the Level circuit and the duplicated biquad layouts. These capacitors were not included as internal parts of the 16 identical blocks because the value of each external capacitor is different at each scale.
Figure 6.17: Summing circuit, low-pass and high-pass filter blocks.

All blocks have a height of 242.0 λ

\[ \lambda = 0.3 \mu m \]

Figure 6.18: Resistor ladder and external capacitors rotated 90° counterclockwise.

6.6 Complete system

6.6.1 Core

The layout of the complete Core is shown at Figure 6.19. It is constituted by 16 identical blocks. 8 blocks in the left half of the core and 8 mirrored blocks in the right half. The blocks are split by the strips of external capacitors, as also can be seen in the floor planning diagram. Almost at the middle, along a vertical strip, is the resistor ladder. Above of the 16 identical blocks, placed over an horizontal band, are at the left the high-pass filter layout, in the middle the Summing circuits, and at the right of this band is the layout of the low-pass filter. Interconnections from and to the PADs were placed over two horizontal strips at the bottom and at the top of the
core. Some vertical interconnections go through dedicated rails or over the poly-poly capacitors. In this last case the lines go exclusively through metal-3 to reduce the parasitic capacitance with poly-2.

6.6.2 Pads

The layouts of the used bounding pads are shown in Figure 6.20. These layouts intended for the AMIS C5 process are provided with the Tanner L-edit tool.

The pads PADGND, PADVDD and PADREF have NMOS and PMOS transistors in diode configuration as ESD protection. These pads have a low resistance since the interconnections from the bounding pad down to the bottom is only metal.

When the Pad Frame is completed it forms a ring of $V_{dd}$ and $Gnd$ potentials. The SPACESM layout has been designed from the SPACE layout in order to adjust the internal width of the pad frame to the core width.

6.6.3 Chip

The layout of the complete chip is shown at Figure 6.21. Its size is $3000\mu m \times 1500\mu m$. The resulting area is $4.5mm^2$. The 48% of this area corresponds to the core and the 52% corresponds to the pad frame.
Figure 6.20: Pad layouts.

Figure 6.21: Chip layout.
Density rules verification

In Figure 6.22 are shown the poly-1, metal-1 and metal-2 masks with their respective percent of filled area. As can be seen, the density rules of 15% for the poly and 30% for the metals have been satisfied.
Chapter 7

On-Chip Measurements

The designed chip has been fabricated in the run AMIS C5 on Feb/4/08. In this chapter the first task is to prove, by Hspice simulations, how well the Spice parameters of this manufacturing run fit into the worst-case models provided by the vendor, with which the system had been previously designed.

We also explain the experimental setup prepared to test the fabricated chip, constituted by a PCB and some protection against conducted and radiated EMIs.

After that we present the experimental results of the decomposition-reconstruction tests as well as the measures of the denoising function.

7.1 Manufacturing run AMIS C5 - Feb/4/08

The chip was fabricated in the run AMIS C5, lot T7CU at Feb/4/2008, whose corresponding wafer electrical test data and Spice model parameters can be found in the MOSIS web page [46].

7.1.1 Bias voltages

For the measurements have been used the bias voltages $V_{01}$ and $V_{16}$ shown in Table 7.1. They were obtained simulating the system using the Feb/4/08 models,
Table 7.1: Manufacturing run and bias voltages of the implemented wavelet system. These bias voltages are source-gate voltages referenced to $V_{pp}$.

<table>
<thead>
<tr>
<th>Vendor</th>
<th>Technology</th>
<th>Run</th>
<th>$V_{01}$</th>
<th>$V_{16}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMIS</td>
<td>SCN05</td>
<td>T7CU (Feb 4/08)</td>
<td>0.8493</td>
<td>0.6396</td>
</tr>
</tbody>
</table>

Figure 7.1: Frequency response of the designed wavelet system, obtained by Hspice simulations using the typical model, the 4 worst-case models and the Feb/4/08 run model.

such that the central frequencies of the first and the last filters were adjusted to the correct frequencies, i.e., 74.3Hz and 13.45kHz.

7.1.2 Comparison between the worst case models and the Feb/4/08 run model

We are interested in know how the manufacturing run fits into the typical and worst case models used during the design of the chip. In Figure 7.1 are shown the Hspice simulations corresponding to the typical model, the 4 worst case models, and the Feb/4/08 run model. In each case the bias voltages $V_{01}$ and $V_{16}$ have been adjusted to make that the central frequency of the first and the last filters coincide with the frequencies 74.3Hz and 13.45kHz that are indicated with vertical lines at each plot.
As can be noted, when the typical or any of the 4 worst-case models are used for the simulations the system has a flat magnitude response, but when the Feb/4/08 run model is used the response of the system presents a slope respect to the frequency. From the observation of this and other simulations can be concluded that the parameters of this particular run are not inside the worst-case models provided by the vendor with which the system has been designed.

7.2 Die pictures

In Figure 7.2(a) is shown a photograph of the complete manufactured chip, while the picture shown in Figure 7.2(b) is a zoom of the right hand half of the core.

7.3 Test setup

Since the system is intended to operate with voltage signals lower than ±30 mV, the main consideration during the chip measures has been to control the environmental noise, i.e., to keep low the radiated and conducted EMI.
Figure 7.3: Noise registered by an oscilloscope when the input is connected to its calibration output and the bandwidth of the oscilloscope is limited to 20 MHz.

7.3.1 Noise in the laboratory

In the photograph of Figure 7.3 can be seen, in the oscilloscope screen, the typical noise present in the laboratory. This noise seems to be constituted by white noise of an amplitude of about ±8 mV, and impulsive noise (probably due to switching voltage sources) of amplitude up to ±20 mV. This environmental noise is excessively high to make measures in the manufactured chip.

In order to reduce the environmental noise, it has been necessary to work at the night, with all the near computer equipment disconnected from the line. A special line filter has been used to reduce the high frequency noise coupled through the 60 Hz AC power line. The PCB has been placed inside a metallic box fabricated with ferromagnetic sheet. The bandwidth of the oscilloscope has been limited up to 20 MHz.

Finally, the testing PCB, the ferromagnetic box, the signal generator and the oscilloscope have been connected to physical ground. In these conditions has been possible to reduce the noise down to ±1.0 mV or less,
In Figure 7.4 is shown a photograph of the PCB specially designed to develop the test of the manufactured chip. The best results have been obtained when the system is powered exclusively with batteries, in order to reduce the line conducted noise.

7.4 Frequency response test

For this test, a sinusoidal probe signal in steady state produced with a Wave Generator has been used as input of the chip. In order to reduce the effects of the environmental noise in the lectures, the probe signal and the reconstructed output signal have been averaged over 128 samples using a Digital Oscilloscope. In Figure 7.5
Figure 7.6: Chip measures of the frequency response of the system. Sinusoidal probe signals of five different frequencies. Reconstruction with and without the activation of the lateral filters.

are shown, in the display of the used oscilloscope, both signals when the probe signal has a frequency of 100 Hz, while in Figure 7.6 are shown the results of the tests for 100 Hz, 316 Hz, 1 kHz, 3.16 kHz, and 10 kHz. To elaborate these plots the tested signals have been saved in binary format using the Digital Oscilloscope. Then the files have been passed from the oscilloscope to a computer via USB port, and the final plots have been obtained using Matlab. Using this software, a precise analysis of the characteristics of the signals can be easily achieved.

In the plots shown in Figure 7.6, the reconstruction has been performed with and without the activation of the low-pass and high-pass lateral filters that have been included in the chip to compensate the band-pass behavior of the designed system (section 2.4.3). Note the compensation of the phase-shift when the lateral filters are used. In all the plots of Figure 7.6 the output sinusoidal has been scaled to present the same amplitude than the input signal, but the actual gains have been included in
Figure 7.7: Frequency response of the fabricated wavelet system measured point by point.

Figure 7.8: Oscilloscope displays of the probe signal *tone3* and the corresponding reconstructed signal for two different intervals and amplitudes.

In Figure 7.7 is shown the frequency response of one of the manufactured chips. The frequency response has been obtained using a Wave Generator to inject sinusoidal probe signals of logarithmically spaced frequencies, completing 16 points per decade, from 10 Hz up to 100kHz.

## 7.5 Decomposition-reconstruction test

For the decomposition-reconstruction tests has been used the probe signal *tone3* shown in Figure 7.8, which is composed of three pulses of 500 Hz, 1 kHz and 2 kHz. Averages over 128 samples were obtained to reduce the effects of the environmental
In Figure 7.8(a) can be seen an impulse included to activate the oscilloscope trigger. For clarity, these probe and reconstructed signals have been plotted in Figure 7.9(a) and 7.9(b) respectively. In Figure 7.9(c) both signals are plotted together, but the reconstructed signal has been multiplied by a normalization factor $k = -0.913$. The DC level of these signals has been subtracted.

For the test of Figure 7.10 also has been used the probe signal *tone3*, but in a wider interval and with a lower amplitude as is shown in the oscilloscope display of Figure 7.8(b). A second chip has been used for these measures. The binary files of the measured signals have been copied to a PC via USB port. In this way the tested signals could be processed by Matlab. The complete results are plotted in Figure 7.10.
Figure 7.10: Reconstruction test measures. (a) Probe signal \textit{tone3} composed of modulated pulses of 2kHz, 1Hz, and 500Hz. (b) Images of the wavelet transform of the probe signal obtained with the wavelet components $v_{wga}$. (c) Reconstructed signal. (d) Original signal together with the reconstructed signal scaled by a factor $k = -0.878$. 
Figure 7.11: Oscilloscope display of the probe signal tone3 contaminated with Gaussian noise and the corresponding reconstructed denoised signal.

In Figure 7.10(a) is shown the input probe signal tone3. In Figure 7.10(b) are plotted as images the wavelet components $v_{wa}$ (used in the level comparison stage when the denoising is enabled). This image corresponds to the wavelet transform of the probe signal when each wavelet component is multiplied by a factor of $\sqrt{r_m/2}$.

In Figure 7.10(c) is shown the reconstructed output signal. Note that this signal is inverted respect to the probe signal. In Figure 7.10(d) are shown the input probe signal together with the reconstructed signal when it has been multiplied by a normalization factor $k = -0.878$. In all the cases the DC level of each individual signal has been subtracted, except for the case of the DC coupled wavelets components, where a global voltage mean has been subtracted.

7.6 Denoising test

For the denoising test has been used the probe signal tone3 contaminated with bandlimited Gaussian noise that is shown in the oscilloscope display of Figure 7.11. This noisy signal has been generated in Matlab and copied to the Arbitrary Wave Generator. In this way the probe signal and the same sequence of noise could be periodically repeated, allowing that averages over 128 samples could be obtained to reduce the effects of the environmental noise.

The measured signals were saved in binary files using a Digital Oscilloscope. The
files were copied to a PC via USB port, and these data were analyzed in Matlab.

In Figure 7.12(a) and 7.12(b) are shown the original and the contaminated signals respectively. In Figure 7.12(c) are shown as images the 16 wavelet components $v_{wa}$ for both cases, AC coupled, i.e., with the DC level subtracted at each component, and DC coupled, with a global voltage level subtracted. These images constitute the wavelet transform of the noisy signal.

In Figure 7.12(d) is shown the reconstructed noisy signal without the activation of the denoising function. This signal with SNR = $-1.52$ dB contains the noise that is actually detected by the wavelet system.

Finally, in Figure 7.12(e) is shown the original noise-free probe signal together with the denoised signal when it has been multiplied by a normalization factor $k = -0.573$. A SNR = $4.79$ dB has been obtained.

Higher denoising levels were expected, but the performance of the system is seriously affected by the presence of different DC levels in the $v_{wa}$ wavelet components, since these components are DC coupled with the comparators in the Level detection circuit.

Another cause of the poor performance in the denoising function could be that the system was designed with an excessively long delay time, equivalent to 10 cycles of the central frequency of each filter. After that the design had been sent to the foundry, was found that a delay value of around 1 cycle was the optimum for the denoising operation.

In the denoised output also can be noted a high level of distortion. The commutations in the digital circuitry can be affecting the sensible analog low level signals when the denoising function is selected. Whereas good results can be obtained when the denoising function is not activated, that is, when no digital commutations occur.
Figure 7.12: Denoising test measures. (a) Original noise-free probe signal \textit{tone3}. (b) Probe signal \textit{tone3} with bandlimited additive Gaussian noise. (c) Images of the wavelet transform of the noisy signal obtained with the wavelet components $v_{wa}$. (d) Reconstructed signal without the activation of the denoising function. (e) Original noise-free signal together with the reconstructed denoised signal scaled by a factor $k = -0.573$.  

\[ SNR = -1.52 \text{ dB} \]
\[ SNR = 4.79 \text{ dB} \]

Reconstructed signal without denoising function

\[ 10 \]
\[ -10 \]
\[ 0 \]

Amplitude (mV)

\[ 10 \]
\[ -10 \]
\[ 0 \]

Amplitude (mV)

\[ 10 \]
\[ -10 \]
\[ 0 \]

Amplitude (mV)
7.7 Limitations of the resistance implemented by a PMOS transistor

When the system performs the summation of the 16 wavelet components, it eliminates the DC components because this operation is AC coupled. But when the system performs the denoising operation it needs to compare the level of each wavelet component against a common threshold level. In this case, high DC levels present in the wavelet components will produce a malfunction on the level detection.

In Figure 7.13 is shown the behavior of the DC level at each wavelet component for the case of the noisy probe signal of Figure 7.12(b). In the same plot is also shown the rms voltage of each component. In the most scales can be seen a relation between the rms level and the DC level. Then the selected implementation for the resistor $R_f$ of Figure 5.7, made with a minimum size PMOS transistor in a dedicated N-well (suggested in [39]), introduce a DC level in the capacitor $C_f$ which is in function of the amplitude of the wavelet component $v_{wa}$. 

Figure 7.13: Plot of the common mode voltage $V_{CM}$, and the rms voltage $V_{rms}$, present at each component $v_{wa}$. 

<table>
<thead>
<tr>
<th>$V_{rms}$</th>
<th>$V_{CM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Referenced to AGnd</td>
<td></td>
</tr>
<tr>
<td>$v_{ral}$</td>
<td></td>
</tr>
<tr>
<td>0.4109V</td>
<td></td>
</tr>
<tr>
<td>Source–gate voltage</td>
<td></td>
</tr>
</tbody>
</table>

Scale index

0.00 0.02 0.04 0.06
<table>
<thead>
<tr>
<th>Voltage − (mV)</th>
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Chapter 8

Conclusions

In this chapter we enumerate the results obtained during the development of this work. We have classified them into theoretical and applied outcomes.

It is worth mentioning that the original goals that we had set at the beginning of this thesis have been satisfied and exceeded. Specifically, the identification of rational filters to perform the direct and the inverse wavelet transform and the implementation of one of these systems in an analog IC, have been reached.

As an extra goal, we have adapted the hard thresholding technique, which is used in the discrete wavelet transform, to make non-linear filtering in noisy signals with the proposed analog wavelet systems.

8.1 Theoretical results

The main results derived of the theoretical research of this work are the following.

1. Any band-pass rational filter performs the semidiscrete wavelet transform of the entering signal respect to the wavelet $\psi(t) = h(-t)$, where $h(t)$ is the impulse response of the system.

2. For the design of a wavelet system we have to select a scale density value $D > 0$, which establishes the number of scales (or decomposition frequency bands) per
octave.

3. The reconstruction filter can be any realizable rational filter, and we have identified that the origin of the reconstruction error is the ripple present in the frequency spectrum of the infinite summation $\sum_{m=-\infty}^{\infty} H_m(j\omega)$, where $H_m(j\omega)$ is the product between the decomposition and the reconstruction filters at the scale $m$.

4. We have proposed a numerical estimation of the worst case reconstruction error due to the ripple.

With the purpose of design wavelet systems with a reconstruction error as low as be required we have proposed two methods.

5. In the first method we start from the selection of a pair of decomposition and reconstruction filters, then we proceed to increase the scale density until the ripple of the summation reaches the desired level.

6. In the second method we start from the selection of a decomposition filter and a fixed scale density, then we proceed to build the reconstruction filter as a linear combination of simpler rational functions.

We have found that the semidiscrete wavelet systems here developed, satisfy an important property that the classical discrete and continuous wavelet transforms do not posses.

7. The semidiscrete wavelet systems here described have the zero-delay property, since the decomposition and reconstruction wavelets are completely causal. As a consequence a real-time processing can be achieved.

The mentioned maximum error due to the ripple of the frequency response of the wavelet system stands only for infinite summations.
8. For practical systems with a finite number of scales, the more important error is due to the band-pass behavior of the complete wavelet system. This band-pass effect produces a phase-shift and a gain variation near the spectrum edges.

9. To remedy the problem mentioned in the previous point, we propose as future research the study of filters whose frequency spectrum have an imaginary part of low or null value, or to compensate such a band-pass behavior with adaptable low-pass and high-pass lateral filters.

10. As an example of wavelet processing, we propose a modification of the hard thresholding function, used with the discrete wavelet transform, to reduce the noise of continuous time signals with the proposed analog wavelet systems.

11. To cope with the discontinuities generated by the hard thresholding function we propose two different approaches: the introduction of a delay function and the use of a structure with a duplicated biquadratic filter.

12. We have analyzed the rms noise values that are present at each scale when a source of Gaussian noise is at the input of the wavelet system. This result helps to select the adequate threshold levels needed for the denoising operation.

13. A specific subset of decomposition filters, the powers of the biquadratic band-pass filter, has been studied and a family of curves have been generated as a help to choose the filters during the design stage of the analog wavelet system.

14. The decomposition-reconstruction and the denoising tasks performed with the proposed systems have been profusely illustrated with examples and figures, and a careful quantification of the results of the simulations has been done with the purpose to compare between different semidiscrete and discrete wavelet systems.

Some of these theoretical results has been included in the published works [47] and [48].
8.2 Applied results

Considering that the purpose of the IC implementation was to show the applicability of the principles theoretically obtained, more than outcomes, we present the next key points of the design and characterization stages.

1. The designed analog system is highly parallel processing needing an elevated number of repeated structures. The redundancy of the wavelet decomposition implied by the overlapped frequency spectrums of the decomposition filters helps to alleviate the effect of the process variations. For these reasons the design approach of the needed amplifiers and OTAs has been the simplicity. In fact, high gains were unnecessary, while the low amount of nodes and transistors of the designed structures helped to reduce the internally produced noise.

2. The implemented system performs the wavelet decomposition in 16 wavelet scales, with a density of two scales per octave. The system works in the range [74.3 Hz-13.5 kHz], and reconstruct the input signal with a gain near to 1. The system also includes the denoising capacity and has been designed in a 0.5 μm CMOS process.

3. The decomposition filters of the system are band-pass biquadratic filters and have been implemented using the $g_m-C$ technique with OTAs working in sub-threshold, limiting the maximum amplitude of the input signal to ±30 mV.

4. For the design of the chip, including the layout stage, the commonly recommended cautions for the analog design have been taken: non-minimum size, same size, minimum distances, common centroid structures, same orientation, same structure, same temperature and same surrounds.

5. Other considerations have been: separated power lines for analog and digital circuits, guard rings around sensitive analog structures.
6. During the design stage has been extensively used the Monte Carlo simulations using a generalized Pelgrom model.

7. During the test of the chip, we had difficulties to control the EMIs conducted and radiated in the laboratory.

8. We have obtained good results in the decomposition-reconstruction tests, in spite of the fact that the Spice parameters of the specific run with which the chip was fabricated, do not lie between the corners established by the worst-case models used during the design of the system.

9. We have obtained only modest results in the test of the denoising function in the fabricated chip. We have identified two probable sources of this poor performance: a) The noise introduced by the digital commutations and b) the non-linearity of the minimum size PMOS transistor used as high value resistor. In future designs more attention must be paid to these issues.

8.3 Final remarks

The implemented system was intended to work into the audio-frequency range. But the selection of this range was only incidental, it could be selected any other because the developed principles are applicable at any frequency. Our purpose was only to show that the outcomes theoretically obtained actually allow simple and direct analog implementations of wavelet transform systems.

The mathematical level used throughout the firsts chapters of this work, has been necessary in order to establish a theoretical base oriented to the analog implementation of the wavelet transform. But if we put aside the formality of the mathematical proofs, we obtain a set of simple and concrete results which can be easily understood and applied to obtain analog wavelet systems as simple as the implemented one, which has required only 16 biquadratic filters and an analog adder. In part, the simplicity
of the needed circuits can be seen as a consequence of the high redundancy of the parallel processing of the proposed systems.

At the beginning of this work, was mentioned that some previous authors had tried to adapt the well developed numerical and algorithmic techniques to the analog circuits, obtaining complex systems that needed complicated structures.

We conclude this work with the premise initially presented: *The hard work in the theoretical basis have made possible very simple analog implementations that fit to the design philosophy of analog circuits.*
Appendix A

Complete Schematics
Analog Wavelet Transform System
Module: ModBlock - pg 2/2
Author: Marco Antonio Gurrola Navarro
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Analog Wavelet Transform System
Module: PadIO / PadVdd / PadGnd
Author: Marco Antonio Gurrola Navarro
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Bibliography


Resumen en Extenso

En esta tesis se investigan las implementaciones analógicas de la transformada wavelet. Cada sección de este resumen corresponde a un capítulo de la tesis.

En la sección I se exponen algunas bases teóricas existentes de la transformada wavelet. Los resultados teóricos originales referentes a la implementación de sistemas analógicos de transformada wavelet y de un sistema de filtrado no lineal se exponen en las secciones II y III. La exposición de estos principios teóricos incluye sus demostraciones en el cuerpo de la tesis, pero en este resumen sólo se incluyen los resultados más importantes.

En las secciones IV y V se explica el diseño eléctrico de uno de estos circuitos de transformada wavelet analógica que incluye la función de filtrado no lineal, para lo cual se emplea una tecnología CMOS de 0.5 μm.

En la sección VI se explican las estrategias empleadas para el diseño del layout del sistema implementado, y en la sección VII se muestran las mediciones en el chip una vez fabricado. Finalmente, en el sección VIII se comentan las conclusiones del presente trabajo de tesis.
I  Introducción

La transformada wavelet permite analizar una señal simultáneamente en tiempo y en frecuencia. Se han desarrollado técnicas especiales de compresión, identificación de patrones y filtrado no lineal para ser aplicadas en el dominio wavelet.

A pesar de que se cuenta con un gran número de trabajos publicados en los cuales se establecen las bases teóricas para la implementación algorítmica de la transformada wavelet, su implementación analógica ha sido muy poco estudiada.

Tipos de transformada wavelet

La transformada wavelet de una función \( f(t) \) es una función \( w(a, b) \), dónde las variables de escala \( a \) y de posición o traslación \( b \) pueden ser continuas o discretas. En el caso de la transformada wavelet discreta ambas variables son discretas. En el caso de la transformada wavelet continua ambas variables son continuas. Para ambos tipos de transformadas se han desarrollado implementaciones algorítmicas.

En este trabajo se demuestra que la transformada wavelet semidiscreta, hasta ahora la menos estudiada, es la que más se adapta a las implementaciones analógicas.

Antecedentes teóricos

En la transformada wavelet semidiscreta \([8][11]\) la posición \( b \) es una variable continua, mientras que la escala es una variable discreta restringida a los valores exponencialmente espaciados

\[
\begin{align*}
a &= r^m, \\
\end{align*}
\]

para \( m \in \mathbb{Z} \), donde \( r \) es dado por

\[
\begin{align*}
r &= \sqrt[2]{D},
\end{align*}
\]

para algún \( D > 0 \), implicando que \( r > 1 \).

Al parámetro \( r \) le llamamos razón de escala pues nos proporciona la razón entre el tamaño de las wavelets de escalas adyacentes, y al parámetro \( D \) le llamamos densidad


de escalas ya que nos proporciona el número de escalas por octava de frecuencia.

A partir de la wavelet prototipo seleccionada \( \psi(t) \), obtenemos la siguiente familia de wavelets semidiscretas

\[
\psi_m(t) = \frac{1}{r^m} \psi \left( \frac{t}{r^m} \right). \tag{I.3}
\]

La transformada wavelet semidiscreta de la función \( f(t) \) respecto a la wavelet prototipo \( \psi(t) \) se define mediante la siguiente familia de funciones continuas

\[
w_m(b) = \int_{-\infty}^{\infty} f(t) \psi^*_m(t-b) \, dt, \tag{I.4}
\]

donde \( w_m(t) \) es la \( m \)-ésima componente wavelet.

La transformada wavelet semidiscreta inversa está dada por

\[
f(t) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} w_m(b) \chi_m(t-b) \, db, \tag{I.5}
\]

donde \( \chi(t) \) es dual de la wavelet semidiscreta \( \psi(t) \), según se explica más adelante.

Para un valor dado \( r > 1 \), la función \( \psi(t) \) puede ser usado como wavelet en la transformada wavelet semidiscreta si satisface la condición de admisibilidad \[7\][8]

\[
C_\psi = 2 \int_0^{\infty} \frac{|\Psi(\omega)|^2}{\omega} \, d\omega = 2 \int_0^{\infty} \frac{|\Psi(-\omega)|^2}{\omega} \, d\omega < \infty, \tag{I.6}
\]

además de la condición de estabilidad de la transformada wavelet semidiscreta la cual requiere que para \( \omega > 0 \)

\[
A \leq \sum_{m=-\infty}^{\infty} |\Psi(r^m \omega)|^2 \leq B, \tag{I.7}
\]

para algunos \( A, B > 0 \).

Adicionalmente, si \( \chi(t) \) satisface

\[
\sum_{m=-\infty}^{\infty} \Psi^*(r^m \omega) X(r^m \omega) = 1, \tag{I.8}
\]

decimos que \( \chi(t) \) es una wavelet dual de \( \psi(t) \), y la reconstrucción perfecta puede lograrse.
Filosofía de diseño seguida para la implementación analógica de la transformada wavelet

En algunos trabajos [25] [26] [24] [25] se ha tratado de adaptar a circuitos analógicos la teoría expresamente creada para implementaciones algorítmicas o numéricas de la transformada wavelet.

Pero en (I.4) se puede observar que cada componente wavelet, \( w_m(t) \), es la convolución entre \( f(t) \) y la función \( \psi_m^*(-t) \). Entonces \( w_m(t) \) puede obtenerse haciendo pasar la señal \( f(t) \) a través de un sistema lineal invariante en el tiempo con respuesta al impulso \( h(t) = \psi_m^*(-t) \). Este método se emplea en [20][21][22][23][24][25][26], pero únicamente para la descomposición. Pero en (I.5) podemos observar que el principio de convolución también se puede emplear en la reconstrucción, pues para cada escala debe realizarse la convolución entre la componente wavelet \( w_m(t) \) y la función \( \chi_m(t) \).

Por tanto, con un sistema como el mostrado en la Figura I.1, formado únicamente por filtros continuos escalados y un sumador, puede llevarse a cabo la transformada wavelet semidiscreta directa e inversa, lo cual resulta menos complejo que los sistemas con sumadores, multiplicadores, integradores, estructuras sample-hold, y retardos analógicos requeridos en otras publicaciones reportadas, sobre implementaciones analógicas de la transformada wavelet discreta.
Objetivos de la tesis

El principal objetivo del presente trabajo de tesis es el identificar pares de filtros racionales \( H_\psi(s) \) y \( H_\chi(s) \) los cuales realicen mediante convolución la trasformada wavelet semidiscreta directa e inversa, y cuya implementación en circuitos analógicos pueda hacerse aplicando las conocidas técnicas para implementar filtros racionales.

Como segundo objetivo, un sistema de transformada semidiscreta se implementa en un circuito integrado analógico mediante la aplicación de los resultados teóricamente obtenidos.

Cómo un objetivo adicional, presentamos una técnica de filtrado no lineal para ser aplicada en conjunto con los sistemas de transformada wavelet analógica aquí desarrollados.
II Principios de la transformada wavelet analógica

Filtros de descomposición en el dominio de Laplace

En este trabajo llamamos función racional pasabanda a la función

\[ H(s) = \frac{p_{d-1}s^{d-1} + p_{d-2}s^{d-2} + \cdots + p_1 s}{s^d + q_{d-1}s^{d-1} + \cdots + q_1 s + q_0}, \]  

(II.1)

la cual es el cociente de dos polinomios de la variable compleja \( s \), con coeficientes reales \( p_1, \ldots, p_{d-1}, q_0, \ldots, q_{d-1} \), donde al menos un \( p \) es diferente de cero, y el denominador es un polinomio estrictamente Hurwitz de grado \( d \geq 2 \). Decimos que un polinomio \( Q(s) \) es estrictamente Hurwitz si todos sus ceros están a la izquierda del eje imaginario, implicando que \( Q(j\omega) \neq 0 \) para \( \omega \in \mathbb{R} \).

Uno de los resultados más importantes del presente trabajo es (Teorema 2.4):

*Para cualquier razón de escala \( r > 1 \), cualquier función racional pasabanda \( H(s) \) posee una respuesta al impulso \( h(t) \), tal que \( \psi(t) = h(-t) \) satisface la condición de admitibilidad (I.6), y la condición de estabilidad (I.7).* Ello significa que la respuesta al impulso de cualquier filtro pasabanda puede usarse como wavelet de descomposición prototipo, y que obteniendo un conjunto de filtros escalados en frecuencia obtendremos las componentes wavelet de la señal entrante.

Resolución del análisis tiempo-frecuencia

La resolución del análisis tiempo-frecuencia que se puede lograr mediante la transformada wavelet es una propiedad de la wavelet prototipo empleada. Para poder hacer comparaciones entre diferentes wavelets se requieren los siguientes conceptos:

El centro \( \bar{t} \) y el radio \( \Delta_t \) de una wavelet \( \psi(t) \), tal que \( \psi(t) \) y \( t\psi(t) \) son \( L^2 \) en \( \mathbb{R} \), están definidos por [8]

\[
\bar{t} = \frac{1}{\|\psi\|^2} \int_{-\infty}^{\infty} t |\psi(t)|^2 dt, \\
\Delta_t = \frac{1}{\|\psi\|} \sqrt{\int_{-\infty}^{\infty} (t - \bar{t})^2 |\psi(t)|^2 dt}.
\]  

(II.2) (II.3)
De manera similar, si $\Psi(\omega)$ es la transformada de Fourier de $\psi(t)$, y $\Psi(\omega)$ y $\omega \Psi(\omega)$ son $L^2$ en $\mathbb{R}$, definimos el centro $\bar{\omega}$ y el radio $\Delta_\omega$ de $\Psi(\omega)$, pero en el caso de wavelets reales consideramos solo frecuencias positivas.

El producto $\Delta_t \Delta_\omega$ puede considerarse como una medida de la resolución del análisis tiempo-frecuencia que puede alcanzarse con una wavelet específica. Entre menor sea el producto mayor será la resolución. Por el principio de incertidumbre [8, Teorema 3.5] sabemos que éste producto satisface la relación

$$\Delta_t \Delta_\omega \geq \frac{1}{2}. \quad (II.4)$$

Otro concepto de utilidad para comparar el desempeño de diferentes wavelets es la selectividad en frecuencia de la función $\Psi(\omega)$ definida por

$$Q = \frac{\bar{\omega}}{\Delta_\omega}, \quad (II.5)$$

donde $\Psi(\omega)$ y $\omega \Psi(\omega)$ son $L^2$ en $\mathbb{R}$.

** Ejemplos de filtros de descomposición **

Como ejemplo de filtros de descomposición, consideramos las potencias de los filtros bicuadráticos pasabanda mediante la siguiente notación

$$\text{biqu}(q, n) = \left( \frac{s/q}{s^2 + s/q + 1} \right)^n. \quad (II.6)$$

Con la expresión wavelet biqu$(q, n)$ nos referiremos a la wavelet $\psi(t) = h_\psi(-t)$, donde $h_\psi(t)$ es la respuesta al impulso del filtro correspondiente.

En la Figura II.1(a) y II.1(b) se muestran el espectro en frecuencia y la wavelet correspondiente a los filtros biqu$(q, n)$ para $n = 2, 3, \ldots, 8$ y $Q = 2.5$. Estos espectros y wavelets están normalizados para tener la misma energía. Estas wavelets son muy similares entre sí pues comparten el mismo valor $Q$, pero difieren en su capacidad de resolución y en el número de derivadas continuas que poseen. Una de estas wavelets, biqu$(0.698, 3)$, se muestra nuevamente en la Figura II.1(c) donde puede compararse con la wavelet Daubechies db4, y la wavelet Morlet, $\psi(t) = e^{-t^2/2} \cos 5t$, mostradas en la Figura II.1(d) y II.1(e).
Figura II.1: (a)-(b) Espectros en frecuencia y wavelets normalizados de los filtros biqui$(q, n)$ para $n = 2, 3, \ldots, 8$ y selectividad $Q = 2.5$. (c) Wavelet biqui$(0.698, 3)$. (d) Wavelet Daubechies, db4. (e) Wavelet Morlet, $\psi(t) = e^{-t^2/2} \cos 5t$.

**Ejemplo de descomposición**

Para este ejemplo usamos la señal de prueba $chirp^2$ mostrada en la Figura II.2(a) la cual es una señal chirp doble que va de altas frecuencias a bajas frecuencias y viceversa. Esta señal permite probar el sistema a lo largo de todo su rango de frecuencia. El espectro de frecuencia de esta señal se muestra en la Figura II.2(b). Dos transformadas wavelet fueron calculadas en este caso.

En la Figura II.2(c) se muestra el valor absoluto de la transformada wavelet continua de la señal de prueba $chirp^2$ graficada como una imagen. En esta transformada se empleo la wavelet Morlet $\psi(t) = e^{-t^2/2} \cos 5t$.

En la Figura II.2(d) se muestra el valor absoluto de la transformada wavelet semidiscreta de la señal $chirp^2$ graficada como una imagen. En este caso, se empleo la wavelet biqui$(1.414, 2)$ y una densidad de escalas $D = 3$.

Obsérvese que de la transformada wavelet continua se obtiene una imagen simétrica, mientras que en el caso semidiscreto se obtiene una imagen asimétrica estirada hacia la derecha.
Figura II.2: Dos transformadas wavelet de la señal de prueba chirp2 (a) cuyo espectro en frecuencia se muestra en (b). (c) Transformada wavelet continua empleando la wavelet Morlet. (e) Transformada wavelet semidiscreta empleando la wavelet biqu(1.414, 2) and $D = 3$.

**Error relativo de reconstrucción**

Si se emplea un par de filtros de descomposición y reconstrucción con los cuales no es posible alcanzar una reconstrucción perfecta, entonces habrá un error de reconstrucción. El error relativo de reconstrucción de la señal reconstruida $f_{out}(t)$ respecto a la señal original $f_{in}(t)$, donde $f_{in}(t)$ y $f_{out}(t)$ son $L^2$ en R, está definido por

$$
\varepsilon = \frac{\|f_{out} - f_{in}\|}{\|f_{in}\|},
$$

(II.7)

Este error depende del espectro de la señal de entrada y del par de wavelets empleadas. En tanto que el error máximo de reconstrucción depende solamente del par de wavelets empleadas y satisface la siguiente relación (Teorema 2.20):

$$
\varepsilon \leq \varepsilon_{max} = \sup \left| \sum_{m=\infty}^{\infty} H(jr^m \omega) - 1 \right| \quad \text{para} \quad \omega \in [\omega_c, r\omega_c],
$$

(II.8)

para todo $\omega_c > 0$, donde $H(j\omega) = \Psi^*(\omega)X(\omega)$.

Otra métrica para el error de reconstrucción es la razón de señal a error, la cual
está dada por

\[ \text{SER} = 20 \log \frac{\|f_{\text{in}}\|}{\|f_{\text{out}} - f_{\text{in}}\|} \text{ (dB)} \]  

(II.9)

1er método de reconstrucción: Incremento de la densidad de escalas

Afirmamos que diseñamos un sistema de transformada wavelet semidiscreta cuando encontramos una función racional pasabanda que actúa como filtro de descomposición, una función realizable para usarse como filtro de reconstrucción, y un valor de densidad de escalas, mediante los cuales podemos lograr que el error de reconstrucción se encuentre por debajo de un nivel requerido de antemano.

En el primer método para diseñar sistemas wavelet semidiscretos, seleccionamos los filtros de descomposición y reconstrucción, \( H_\psi(s) \) y \( H_\chi(s) \), manteniéndolos idénticos durante todo el proceso. Ahora, partiendo de un valor inicial \( D > 0 \), incrementamos \( D \) introduciendo un factor de normalización \( k_n \) tal que

\[ \text{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} k_n H_\psi (jr^{x/2\pi}) \ H_\chi (jr^{x/2\pi}) \ dx = 1, \]  

(II.10)
y continuamos incrementando \( D \) hasta obtener el error de reconstrucción máximo, calculado con (II.8), que satisfaga nuestros requerimientos.

Este primer método se fundamenta en el Teorema 2.25 el cual nos garantiza que el error de reconstrucción tiende a cero conforme la densidad de escalas aumenta, sin importar qué filtros se hayan escogido, con la condición necesaria de que el producto \( H_\psi(s) \ H_\chi(s) \) sea una función racional pasabanda.

En la Figura II.3 se muestra el error de reconstrucción \( \varepsilon_{\text{max}} \) de un sistema de transformada wavelet, dónde el filtro \( H(s) = H_\psi(s) \ H_\chi(s) \) es un filtro biqu(e, n), para diferentes densidades de escala \( D \). Notese la rápida convergencia del error conforme \( D \) se incrementa.

2o método de reconstrucción: Construcción de filtros a la medida

En este segundo método partimos de un filtro de descomposición dado \( \psi(t) \) y de una densidad de escalas \( D \), los cuales se mantienen constantes durante todo el diseño.
Figura II.3: Reconstruction error, $\varepsilon_{\text{max}}$, as a function of the scale density, $D$, of a wavelet system, where the product $H(s) = H_\psi(s) H_\chi(s)$ is a filter biqu(q, n), for several $q$-values and $n = 1, 2, 3, 4$.

del sistema wavelet. En este caso, siguiendo los pasos listados en la Sección 2.2.5, construimos un filtro de reconstrucción a la medida como una combinación lineal de funciones racionales más simples.

Retardo en el procesamiento mediante la transformada wavelet

Supongamos que el soporte efectivo de la wavelet $\psi(t)$ es el intervalo $t \in (-c, d)$, donde $c, d > 0$. En la Sección 2.3 se demuestra que para poder realizar la descomposición en tiempo real, es decir, con retardo cero, la wavelet de descomposición debe satisfacer $d = 0$, o bien

$$\psi(t) = 0 \quad \text{para} \quad t > 0. \quad (II.11)$$

Las wavelets de descomposición correspondientes a cualquier filtro racional, las cuales son el inverso en el tiempo de la respuesta al impulso, satisfacen esta condición. La wavelet Morlet no satisface esta condición debido a que es simétrica respecto al origen. De ahí que necesitemos esperar un tiempo $D = ad$ para tener toda la información necesaria para calcular la componente wavelet en la escala $a$. 
Ahora supongamos que el soporte efectivo de la wavelet \( \chi(t) \) es el intervalo \( t \in (-c', d') \), donde \( c', d' > 0 \). En la Sección 2.3 se demuestra que para poder realizar la reconstrucción en tiempo real, es decir con retardo cero, la wavelet de reconstrucción debe satisfacer \( c' = 0 \) o bien

\[
\chi(t) = 0 \quad \text{para} \quad t < 0. \tag{II.12}
\]

Las wavelets de reconstrucción correspondientes a cualquier filtro racional, las cuales son igual a la respuesta al impulso, satisfacen siempre esta condición. La wavelet Morlet tampoco satisface esta condición debido a que es simétrica respecto al origen. De ahí que necesitemos esperar un tiempo \( D = ac' \) para tener toda la información necesaria para calcular la componente wavelet de la escala \( a \).

Resultados similares pueden obtenerse también para la transformada wavelet discreta, donde el retardo en la descomposición-reconstrucción es también dependiente de la escala.

También en la sección 2.3, se demuestra que el tiempo de retardo máximo en el cálculo de descomposición-reconstrucción siguiendo el algoritmo rápido de la transformada wavelet (fast wavelet transform) está dado por

\[
D_L = b_0 \sum_{M=1}^{L} (N - 1)2^{M-1} = b_0(N - 1)(2^L - 1),
\]

donde \( N \) es el número de coeficientes del filtro FIR, \( b_0 \) es el periodo de muestreo de la secuencia a la escala menor, y \( L \) es el nivel de descomposición máximo. Cómo puede observarse, en este caso el retardo de descomposición-reconstrucción depende del número de niveles de descomposición calculados y del número de coeficientes del filtro FIR.

En definitiva, la propiedad de retardo-cero, que poseen los sistemas de transformada wavelet desarrollados en este trabajo, no es compartida por ningún tipo de wavelets hasta ahora descritas en la literatura. Y como el procesamiento con todas esas wavelets implica un retardo, esta propiedad rara vez ha sido analizada, llegando
Figura II.4: Frequency response of the system described in Example 2.31. Real (a) and imaginary (b) parts of the individual filters and their sum. (c) Magnitude and phase of the whole system. (d) Reconstruction error of a sinusoidal input in steady state.

a considerarse normal un retardo implícito en cualquier procesamiento mediante la transformada wavelet.

Número de escalas finito

Los dos métodos para el diseño de sistemas wavelet arriba explicados, mediante los cuales se puede diseñar un sistema con un error de reconstrucción tan bajo como sea requerido, funcionan siempre y cuando se tome un número infinito de escalas.

Pero en un sistema práctico sólo se puede tomar un número finito de escalas a lo largo de una banda de frecuencia limitada. La suma infinita de la condición (I.8) se convierte ahora en una suma finita causando un comportamiento pasabanda del sistema completo.

En el cuerpo de la tesis se muestra cómo, partiendo de la magnitud y la fase del espectro en frecuencia del sistema completo, podemos obtener el error de reconstrucción relativo en función de la frecuencia. Esto se muestra en la Figura II.4 para el caso de un sistema wavelet específico.

Una forma muy sencilla de compensar este indeseado comportamiento pasabanda se describe en la Sección 2.4.3, donde se explica el empleo de un filtro pasabajas y
de un filtro pasaaltas para aproximar la respuesta de las escalas no consideradas en el sistema wavelet. Estos dos filtros pueden ser muy simples, pues sólo tienen que ajustarse en el rango de frecuencias donde el sistema wavelet opera.

**Ejemplos de reconstrucción**

Para este ejemplo empleamos dos sistemas wavelet. El primero consta de 16 escalas y emplea el filtro biqu(1.414, 1) con una densidad de 2 escalas por octava. El segundo es casi idéntico, salvo que emplea un par de filtros laterales pasa-baja y pasa-alta para compensar el comportamiento pasabanda debido al número finito de escalas. Ambos sistemas operan en el rango de 62.5mHz a 16Hz.

La señal de prueba chirp2 se muestra en la Figura II.5(a) junto con las dos señales reconstruidas. Puede observarse un corrimiento de fase en las bajas frecuencias. El error de reconstrucción $|f_{out} - f_{in}|$ se muestra en la Figura II.5(b) para el sistema sin compensación, y en la Figura II.5(c) para el sistema compensado con filtros laterales. Observe como el error es menor en el segundo caso.
Figura II.5: (a) Señal de prueba chirp2 graficada junto con dos señales reconstruidas usando un sistema wavelet con y sin filtros laterales de compensación. (b) y (c) error $|f_{out}(t) - f_{in}(t)|$ para cada caso.
III Principios de filtrado no lineal con la transformada wavelet analógica

La utilidad de la transformada wavelet radica en permitir el procesamiento no lineal de señales cuando estas se encuentran en el dominio wavelet. Esto se logra poniendo algunos bloques de procesamiento entre los filtros de descomposición y de reconstrucción de la Figura III.1.

El filtrado no lineal, mejor conocido en inglés como wavelet denoising, shrinkage o thresholding, es una aplicación típica de la transformada wavelet discreta. Se trata de una operación no lineal porque el procesamiento se realiza sobre algunos coeficientes wavelet. La decisión de si un coeficiente en particular es considerado en el procesamiento o no, depende de la comparación de su valor contra un nivel de umbral.

De entre varias operaciones de filtrado wavelet publicadas para la transformada wavelet discreta, aquí sólo consideramos la llamada hard thresholding debido a la facilidad con que ésta puede ser implementada en circuitos analógicos. En su forma original, esta operación está definida para aplicarse sobre coeficientes wavelet discretos obtenidos al aplicar la transformada wavelet discreta sobre una señal ruidosa.

La idea central en esta operación es mantener los coeficientes cuyo valor absoluto es mayor que un nivel de umbral dado, ya que se considera que éstos contienen información de la señal original, mientras que los coeficientes por debajo del nivel de umbral son removidos pues se considera que éstos están conformados por ruido en su mayor parte. Para poder ser aplicada a señales continuas, esta operación de filtrado se modifica de la siguiente manera.

Sea \( w(t) \) la entrada a la operación de filtrado wavelet y \( \tilde{w}(t) \) la salida. La operación hard thresholding está definida por

\[
\tilde{w}(t) = \begin{cases} 
  w(t) & \text{para } |w(t)| > \lambda \\
  0 & \text{para } |w(t)| \leq \lambda 
\end{cases}
\]  

(III.1)

donde el nivel de umbral \( \lambda > 0 \) puede ser una constante o una función de la escala.
Figura III.1: Procesamiento de señales en el dominio wavelet.

Esta operación depende sólo de los valores actuales de la componente wavelet \( w(t) \) según se ilustra en la Figura III.2(a).

**Hard thresholding con retardo**

Supóngase que aplicamos la operación hard thresholding a la señal \( w(t) \) compuesta por una componente sinusoidal más una componente de ruido. Cuando la sinusoidal cruza el eje horizontal, en las proximidades de los niveles \( \lambda \) o \(-\lambda\), las oscilaciones de alta frecuencia de la componente de ruido son capaces de introducir discontinuidades de amplitud \( \lambda \) sobre la señal de salida \( \tilde{w}(t) \). Estas discontinuidades
pueden ser más grandes que el ruido que se desea eliminar, como se ilustra en el ejemplo de la Figura III.2(a).

Para evitar estos inconvenientes proponemos el uso de la siguiente variante de la operación hard thresholding.

Sea \( w(t) \) la señal de entrada a la operación de filtrado no lineal y \( \tilde{w}(t) \) su salida. La operación hard thresholding con retardo se define mediante

\[
\tilde{w}(t) = \begin{cases} 
  w(t) & \text{si } |w(t)| \geq \lambda \text{ para algún } t \in [t - T, t] \\
  0 & \text{si } |w(t)| < \lambda \text{ para todo } t \in [t - T, t]
\end{cases} \tag{III.2}
\]

donde el nivel de umbral \( \lambda > 0 \) puede ser una constante o una función de la escala, y el tiempo de retardo \( T \) es una función de la escala definida por

\[
T_m = r^m T_0 \tag{III.3}
\]

donde \( T_0 \) y \( T_m \) son los retardos en las escalas 0 y \( m \), y \( r \) es la razón de escala del sistema wavelet.

El tiempo de retardo incluido en esta operación modificada previene que la señal sea cancelada cuando atraviesa el rango de valores dado por \([-\lambda, \lambda]\), como se ilustra en la Figura III.2(b).

**Suavidad de las componentes wavelet**

En este trabajo, la suavidad de una señal se cuantifica con el número de derivadas continuas que esta posee [10].

Por un lado, el efecto de conmutación de la operación hard thresholding introduce discontinuidades en la componente wavelet. Por otra parte cuando la componente wavelet pasa por el filtro de reconstrucción gana una derivada continua por cada \(-20 \text{ dB}_\text{dec}\) de pendiente en la asíntota derecha del diagrama de Bode del filtro.

**Comportamiento del ruido blanco**

Asumiendo que la señal de entrada es contaminada con ruido blanco aditivo, estamos interesados en el comportamiento del valor rms de ruido en cada escala de
la transformada wavelet. El conocimiento de esto, nos permitirá fijar adecuadamente los valores de umbral $\lambda_m$ para el filtrado de la componente de ruido.

Suponiendo que la señal de entrada $f_{in}(t)$ está dada por

$$f_{in}(t) = f_o(t) + n(t),$$  \hspace{1cm} (III.4)

donde $f_o(t)$ es la señal original, y $n(t)$ es una fuente de ruido blanco de banda limitada, con ancho de banda $f_{BW}$ que sobrepasa la banda de trabajo del sistema wavelet.

Si la amplitud de la señal original $f_{in}(t)$ es igual a cero, se puede demostrar que para las componentes de escala de la transformada wavelet semidiscreta el ruido se distribuye de manera inversamente proporcional a la raíz cuadrada de la escala, es decir

$$\bar{n} = \frac{k}{\sqrt{r^m}},$$  \hspace{1cm} (III.5)

La amplitud de la componente de ruido $n(t)$ está relacionada con su valor rms $\bar{n}$ mediante una distribución de probabilidad Gaussiana.

Esto significa que el valor pico-pico de la señal $n(t)$ estará por debajo de $\pm \bar{n}$ durante 68% del tiempo, debajo de $\pm 2\bar{n}$ durante 95.4% del tiempo, y así sucesivamente.

**Ejemplos de filtrado no lineal**

Para cuantificar la capacidad para reducir el ruido que manifiestan las técnicas aquí expuestas empleamos la razón de señal a ruido definida como

$$\text{SNR} = 20 \log \frac{\|f_{in}\|}{\|f_{out} - f_{in}\|} \text{ (dB)},$$  \hspace{1cm} (III.6)

donde $f_{in}(t)$ es la señal original y $f_{in}(t)$ es la señal filtrada.

En la Figura III.3(a) se muestra una señal de prueba libre de ruido *tone2*, mientras que en la Figura III.3(b) se muestra la misma señal pero contaminada con ruido blanco de banda limitada con un valor rms de $\bar{=} 0.8$. En la Figura III.3(c) se muestra la transformada wavelet de esta señal ruidosa graficada como una imagen de su valor absoluto. Para esta transformada se empleó $\text{biqu}(\sqrt{2}, 1)$ como filtro de descomposición y una razón de escala de 1 banda por octava, $D = 1$. En la Figura III.3(d) se
Figura III.3: Operación de filtrado no lineal con la transformada wavelet analógica.

muestra la transformada wavelet previa después de serle aplicada la operación *hard thresholding*.

En la Figura III.3(e) se muestran la señal reconstruida donde para la aplicación de la transformada wavelet inversa fue usado el filtro hecho a la medida \( H_\chi(s) = \frac{6.468s^3 + 4.565s^2 + 4.648s}{s^4 + 4.5s^3 + 7s^2 + 4.5s + 1} \).

En la Figura III.3(f) se muestra, como referencia, la señal filtrada con un filtro Butterworth \( H_{Butt}(s) = \frac{(\omega_2 - \omega_1)s}{s^2 + (\omega_2 - \omega_1)s + 1} \), cuyas frecuencias de esquina, \( \omega_1 \) y \( \omega_2 \), coinciden con el ancho de banda del sistema wavelet empleado.
El sistema wavelet de este ejemplo tiene la ventaja de requerir sólo una escala por octava, pero el filtro de reconstrucción es complicado y posee una gran sensibilidad comparado con filtros obtenidos con el método de incrementar la densidad de escalas.
IV Implementación de la transformada wavelet en un circuito integrado analógico

Sabemos que el error de reconstrucción que presentan los sistemas wavelet aquí desarrollados, proviene del rizo que se encuentra presente en la suma de las respuestas en frecuencia de los filtros escalados. De los dos métodos para diseñar sistemas wavelet que fueron arriba presentados, el que nos permite el empleo de filtros más simples es el método en el cual se incrementa la densidad de escalas, lo cual a su vez produce un mayor traslape entre los espectros de filtros adyacentes, ayudando a aliviar los efectos negativos de las variaciones de proceso aleatorias.

Descripción del sistema implementado

El sistema implementado consta de un total de 16 escalas con una densidad de 2 escalas por octava, D=2, lo cual implica una razón de escala $r = 2^{1/D} = \sqrt{2}$. Los filtros de descomposición están dados por

$$H_{\psi,m}(s) = \frac{\omega_m}{q} \frac{s}{s^2 + \frac{\omega_m}{q} s + \omega_m^2},$$

con $q = \sqrt{2}$, $\omega_m = 2\pi f_m$, para $m = 1, 2, 3, \ldots, 16$. El filtro de reconstrucción es simplemente una constante de normalización $H_{\chi,m}(s) = 0.3795$. La frecuencia central $f_m$ de los 16 filtros bicuadráticos se muestra en la Tabla 4.1.

El error máximo de reconstrucción de este sistema debido al rizo en una sumatoria infinita sería de $\varepsilon_{max} = 0.38\%$. No tendría caso aumentar la densidad de escalas para lograr un error de reconstrucción menor debido a que se espera un error en el rizo de entre 2% y 3% debido al mismatch producido por las variaciones de proceso durante la fabricación del chip.

El sistema completo está dado por la sumatoria parcial

$$\sum_{m=1}^{16} H_{\psi,m}(s) H_{\chi,m}(s).$$
Tabla IV.1: Frecuencia central de los 16 filtros bicuadráticos pasabanda.

<table>
<thead>
<tr>
<th>escala</th>
<th>frecuencia</th>
<th>escala</th>
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<th>frecuencia</th>
<th>escala</th>
<th>frecuencia</th>
</tr>
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<tbody>
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<td>m</td>
<td>( f_m )</td>
<td>m</td>
<td>( f_m )</td>
<td>m</td>
<td>( f_m )</td>
</tr>
<tr>
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<td>74.33 Hz</td>
<td>5</td>
<td>297.3 Hz</td>
<td>9</td>
<td>1.189 kHz</td>
<td>12</td>
<td>4.757 kHz</td>
</tr>
<tr>
<td>2</td>
<td>105.2 Hz</td>
<td>6</td>
<td>420.4 Hz</td>
<td>10</td>
<td>1.682 kHz</td>
<td>14</td>
<td>6.727 kHz</td>
</tr>
<tr>
<td>3</td>
<td>148.7 Hz</td>
<td>7</td>
<td>594.6 Hz</td>
<td>11</td>
<td>2.378 kHz</td>
<td>15</td>
<td>9.514 kHz</td>
</tr>
<tr>
<td>4</td>
<td>210.2 Hz</td>
<td>8</td>
<td>840.9 Hz</td>
<td>12</td>
<td>3.363 kHz</td>
<td>16</td>
<td>13.454 kHz</td>
</tr>
</tbody>
</table>

La respuesta en frecuencia de este sistema se muestra en la Figura II.4(a) y II.4(b) para las partes real e imaginaria de la suma total y para cada producto individual \( H_{\psi,m}(s)H_{x,m}(s) \). Mientras que en la Figura II.4(c) se muestra la magnitud y fase de la sumatoria, y en la Figura II.4(d) se muestra el error de reconstrucción debido al comportamiento pasabanda del sistema completo.

**Proceso AMIS-C5**

El proceso seleccionado para la fabricación del sistema fue el AMI Semiconductors C5. Esta es una tecnología CMOS de 0.5 \( \mu m \) destinada para aplicaciones de señal mixta. El kit de fabricación proporcionado por el vendedor incluye modelos típicos y de peores casos para los dispositivos N-MOS y P-MOS (parámetros Spice BSIM3 Version 3.1), así como parámetros para un modelo Pelgrom generalizado usado para estimar el efecto que las variaciones de proceso tienen sobre los transistores, resistores y capacitores mediante simulaciones Monte Carlo.

Considerando la capacitancia en el Pad de salida, en el PCB, en la entrada a un OpAmp y la capacitancia de la punta del osciloscopio, se estimó que cada salida del chip debía manejar una carga de alrededor de 20 pF. Pero el sistema también se simuló manejando cargas de hasta 100 pF.
Diseñando en subumbral

El sistema emplea transistores CMOS trabajando en subumbral debido a que en esta región la corriente de drenaje del transistor tiene un comportamiento exponencial respecto a cambios lineales en el voltage de compuerta. Con este comportamiento se simplifica el escalamiento exponencial en frecuencia de los filtros tipo $g_m C$.

Modelo del CMOS en subumbral

Un modelo para la corriente de drenaje del transistor PMOS trabajando en subumbral está dado por \[ I_D = I_0 \frac{W}{L} e^{\frac{V_{BG}}{U_t}} \left( e^{-\frac{V_{BS}}{U_t}} - e^{-\frac{V_{BD}}{U_t}} \right), \] (IV.3)
donde $V_{BG}$, $V_{BS}$ y $V_{BD}$ son los voltajes de substrato-compuerta, substrato-fuente y substrato-drenaje, $W$ y $L$ son el ancho y largo de canal del transistor, $I_0$ es la corriente en ausencia de polarización, $\rho$ es el coeficiente de acoplamiento electrostático entre canal y compuerta, y $U_t = kT/q$ es el voltaje térmico. Cuando $V_{BD} > V_{BS} + 5 U_t$ decimos que el transistor está en saturación [34].

En el sistema diseñado se emplean transistores PMOS con una razón de aspecto $W/L = 60\lambda/15\lambda$ como fuentes de corriente.

Pares diferenciales en subumbral

En la Figura IV.1(a) se muestra el par diferencial simple compuesto por dos transistores idénticos y una fuente de corriente. Su transconductancia se define como $G = \frac{\partial I_{DM}}{\partial V_{DM}}$. Mientras que la transconductancia normalizada se define como $G_{norm} = G/G_{max}$. En [34] la distorsión se define como la mayor desviación de la $G_{norm}$ de la unidad.

Partiendo de (IV.3) se puede demostrar que el par diferencial básico alcanza el 1% de distorsión en un intervalo de $\pm 7.7$ mV para $V_{DM}$, mientras que el par diferencial con degeneración de fuente mostrado en la Figura IV.1(b) lo alcanza en el un intervalo
Figura IV.1: (a) Par diferencial simple. (b) Par diferencial con rango lineal mejorado mediante degeneración de fuente empleando difusores simétricos.

±30.7 mV para $V_{DM}$, el cual es 4 veces el rango original. Este par diferencial con rango lineal mejorado es el que se emplea en el sistema diseñado.

**Escalamiento de filtros $g_mC$ en subumbral**

Se puede demostrar que la frecuencia central de un filtro $g_mC$ cuya fuente de polarización es un transistor P-MOS trabajando en subumbral, Figura IV.2(a), varía exponencialmente en función de su voltaje substrato-compuerta

$$\omega_c \propto e^{V_{BG}}. \quad (IV.4)$$

Esta dependencia exponencial de la frecuencia central $\omega_c$ ante cambios lineales en el voltaje $V_{BG}$ del transistor de polarización, se aprovecha en el sistema diseñado para polarizar los 16 filtros escalados en frecuencia de manera exponencial. Tal como se sugiere en [36], se emplea una escalera de resistores fabricada con una línea de polisilicio, Figura IV.2(b).

**Fuentes de alimentación**

El sistema está proyectado para trabajar con una alimentación de 2.5V. En el sistema se incluyeron 2 pares de líneas de alimentación, esto para separar la alimentación de los circuitos analógicos de la alimentación de circuitos lógicos, los cuales generan
Figura IV.2: (a) Transistor PMOS en subumbral empleado para obtener la corriente de polarización $I_b$. (b) Polarización de los 16 filtros $g_m-C$ de estructura idéntica pero escalados en frecuencia. La corriente de polarización de cada filtro es producida mediante un transistor PMOS como el mostrado en (a).

ruido durante la conmutación. Se incluyó además una tercera línea para alimentar a los circuitos analógicos más sensibles.

**Amplificador operacional de transconductancia $TL$**

Los 16 filtros bicuadráticos de que consta el sistema son de tipo $g_m-C$ y emplean el OTA de la Figura IV.3, el cual emplea el par diferencial con rango lineal ampliado que ya ha sido explicado, junto con un transistor en subumbral actuando como fuente de corriente.

El objetivo principal durante el diseño eléctrico de los filtros de descomposición no fue tanto obtener una implementación óptima sino solamente una implementación simple y fabricable. Para seleccionar el tipo de filtros a usar se revisaron varias implementaciones de filtros y de OTAs. Los criterios que se tomaron en cuenta fueron la cantidad de transistores requeridos, el tamaño de la implementación, el rango lineal, el consumo de potencia y la sensibilidad.
Filtro bicuadrático pasabanda

La implementación del filtro bicuadrático (IV.1) se muestra en la Figura IV.4. El circuito está compuesto por dos OTAs y tres capacitores, donde los voltajes \( v_x \) y \( v_y \) son las señales de entrada y salida del filtro.

Este filtro, como otros circuitos analógicos dentro del sistema diseñado, requiere del voltage de referencia externo \( A_{Gnd} = 0.7 \) V.

La función de transferencia del filtro está dada por

\[
\frac{V_w(s)}{V_x(s)} = \frac{q \omega_c s}{s^2 + \frac{q}{\omega_c} s + \omega_c^2},
\]

donde \( q = \sqrt{2} \) y \( \omega_c = \sqrt{2} \times 10^{11} \) G, tal que \( G_1 = G_2 = G \). Lo cual significa que la frecuencia de los filtros \( \omega_c \), es proporcional a la transconductancia \( G \), la cual a su vez depende exponencialmente del voltaje de polarización \( V_b \), como ya ha sido explicado.

La respuesta de los 16 filtros escalados se muestra en la Figura IV.5.

El nodo \( V_{Gnd} \approx 0.7 \) V es una tierra virtual a la entrada de un amplificador operacional conectada en configuración de sumador inversor como se muestra más adelante.
Monitoreo de señales

El amplificador TXBfer, mostrado en la Figura IV.4, fue diseñado para manejar cargas hasta de 100 pF y se usa para mandar la componente wavelet \( v_w \) al Pad de salida del chip. El voltaje \( v_b \), empleado para polarizar TL1 y TL2, es también usado para polarizar el amplificador TXBfer después de pasar a través de los amplificadores de corriente BAMP y TXAMP.

Circuito de Suma

En la Figura IV.4, el voltaje \( v_w \) en el capacitor \( C_{2a} \) produce la corriente \( i_w \). Las 16 corrientes \( i_w \) de los 16 filtros se suman en el nodo de tierra virtual \( V_{Gnd} \). En esta forma se implementa la suma involucrada en la transformada wavelet semidiscreta.
inversa (I.5). Dicha suma se efectúa empleando el circuito de la Figura IV.6, que consiste en un amplificador en configuración de sumador inversor empleando capacitores como impedancias en lugar de resistores [39]. Lo cual es conveniente debido al menor tamaño de los capacitores implicados, y a la simplicidad del circuito resultante.

Para este circuito, sin embargo, se requiere de un resistor de gran valor $R_f$ el cual es implementado en [39] mediante un transistor PMOS de dimensiones mínimas trabajando en subumbral dentro de un pozo-N dedicado. El valor de resistencia de este transistor es controlado a través su voltaje de compuerta.

**Respuesta del sistema completo**

En la Figura IV.7 se muestra la repuesta en frecuencia del sistema completo obtenido mediante simulaciones en Hspice a nivel transistor.
Simulaciones de descomposición-reconstrucción a nivel transistor

La siguiente simulación fue hecha a nivel transistor usando Hspice. Se empleó la señal de prueba *tone2* pero escalada en frecuencia por 3 décadas para asegurar que su espectro en frecuencia cayera dentro de la banda de trabajo del sistema diseñado. La máxima amplitud de la señal de prueba es de ±30 mV, y para las simulaciones se empleó una carga de 20 pF.

En la Figura IV.8(a) se muestra la señal de prueba original *tone2*.

En la Figura IV.8(b) se muestra la transformada wavelet de la señal original, donde el valor absoluto de las 16 componentes wavelet fue graficado como una imagen. A cada componente wavelet se le substraigo su nivel de DC ya que están componentes están acopladas en AC a través del capacitor $C_2$. Para una mejor visualización de la transformada wavelet cada componente se multiplicó por el factor $\sqrt{r^m}$.

En la Figura IV.8(d) se muestran la señal de prueba original junto con la señal reconstruida multiplicada por un factor de normalización $k = -0.897$. La razón de señal a error resultante fue SER=17.11 dB.
Figura IV.8: Descomposición y reconstrucción de la señal de prueba \textit{tone2} simulada con Hspice a nivel transistor.
V Implementación de un sistema de filtrado empleando la transformada wavelet analógica

En la presente sección se incorpora la operación de filtrado al sistema wavelet que fue diseñado en la sección anterior.

Función de filtrado simplificada

La operación *hard thresholding* descrita con anterioridad puede ser representada mediante el bloque de la Figura V.1(a). Mas para los propósitos de la implementación analógica es mejor representar esta operación usando operaciones más simples como se indica en la Figura V.1(b). El funcionamiento de los bloques *Level* y *Delay* se describe en seguida.

Sean \( w(t) \) y \( L(t) \) la entrada y la salida del bloque *Level*. La operación de este bloque se define como sigue

\[
L(t) = \begin{cases} 
1 & \text{si } |w(t)| \geq \lambda \\
0 & \text{si } |w(t)| < \lambda 
\end{cases}. \quad (V.1)
\]

Sean \( L(t) \) y \( \tilde{L}(t) \) la entrada y la salida del bloque *Delay*. La operación de este bloque se define como sigue

\[
\tilde{L}(t) = \begin{cases} 
1 & \text{si } L(t) = 1 \text{ para algun } t \in [t - T, t] \\
0 & \text{si } L(t) = 0 \text{ para todo } t \in [t - T, t] 
\end{cases}. \quad (V.2)
\]

En la Figura V.1(c) se muestran las configuraciones convencionales de la función de filtrado en un sistema cuyo filtro de reconstrucción es simplemente una constante de normalización. Obsérvese que esta configuración no contiene un filtro después del switch que suavice las discontinuidades introducidas por éste.

Configuración de filtrado alternativa

En la Figura V.2 se muestra una configuración alternativa para suavizar la señal de salida pero que no requiere de un filtro pasabajos o pasabanda de reconstrucción.
Figura V.1: (a) Bloque de la función thresholding. (b) Bloque equivalente de la función delayed hard thresholding. (c) Configuración de filtrado convencional.

Lo que se hace es simplemente poner un filtro racional pasabanda repetido $H_φ(s)$, el cual es factor del filtro de descomposición original $H_ψ(s)$.

Con el empleo de este filtro extra, se obtiene una señal reconstruida continua sin necesidad de diseñar un filtro de reconstrucción a la medida. Cuando la descomposición del filtro original $H_ψ(s)$ es igual a su factor $H_φ(s)$ se obtiene la configuración de la Figura V.2(b). Esta última configuración es la que elegimos para implementar en el circuito integrado diseñado.

En la Sección 5.1.4 se compara el desempeño de las estructuras de filtrado convencional y alternativa.

Configuración de filtrado implementada

La configuración de filtrado alternativa mostrada en la Figura V.2(b) se incorporó en el sistema wavelet descrito en la Sección IV. Este sistema consta de 16 escalas, con una densidad de 2 escalas por octava. El filtro de descomposición es una función
bicuadrática pasabanda con \( q = \sqrt{2} \), mientras que el filtro de reconstrucción es simplemente una constante de normalización.

Los filtros \( H_\phi(s) \) y \( H'_\phi(s) \) a que se refiere la Figura V.2(b) son idénticos e iguales a biqu(\( \sqrt{2}, 1 \)), en tanto que el filtro \( H_\psi(s)/H_\phi(s) \) es igual a 1.

**Circuito de detección de nivel**

Como puede observarse en la Figura V.2, la operación de filtrado requiere la operación \( \text{Level} \) seguida de la operación \( \text{Delay} \). En tanto que para la operación \( \text{Level} \) se requiere de la comparación de la componente wavelet que se encuentra a la salida del biquad contra el voltaje de umbral.

Después de realizar simulaciones Monte Carlo se encontró una incertidumbre en el nivel de DC a la salida de los biquads de ±7 mV. Dado que el filtro puede manejar señales con una amplitud máxima de ±30 mV, se hace necesario amplificar la componente wavelet \( v_{w_m}(t) \) para reducir el efecto de este indeseado nivel de offset.

Dicha amplificación es realizada por el amplificador \( TAmp1 \) que se muestra en la Figura V.3, el cual está en configuración de amplificador no inversor empleando
capacitores como impedancias en vez de resistores. Mediante el transistor \( m_{f1} \), PMOS de dimensiones mínimas en un pozo-N dedicado, se implementa un resistor de muy alto valor \([39]\) para mantener el nivel de DC a la salida del amplificador cercano a \( A_{Gnd} \).

La amplificación se fijó para lograr que el rango de operación de ±30mV aumente hasta ±200mV.

El amplificador \( TAmp2 \) está en configuración de amplificador inversor con ganancia de –1. Ambas señales \( v_{wa} \) y su inversa \( v_{wb} \) se comparan contra un mismo nivel de umbral \( V_{th} \). Las salidas de los comparadores entran a una compuerta Nand cuya salida se activa cuando alguna de estas dos señales se encuentra por encima del nivel de umbral.

**Circuito de retardo**

En la operación de retardo, realizada por el circuito de la Figura V.4, la señal \( Sw \) se habilita cuando la señal \( DlyE \) se activa. Pero \( Sw \) se deshabilita con un tiempo de retardo respecto a la desactivación de \( DlyE \), tal como se requiere. Mediante la señal
Figura V.4: Circuito de retardo.

Figura V.5: Implementación del switch analógico requerido para la operación de filtrado.

*DlyS* se seleccionan dos niveles de retardo.

**Analog switch**

En la Figura V.5 se muestra el circuito para implementar el switch analógico requerido para la función de filtrado wavelet de la Figura V.2(b). Los transistores *m*x*₁* y *m*x*₂* controlan el paso de la señal *v*x al punto *v*x′. Los transistores *m*g*₁* y *m*g*₂* controlan la conexión entre *V*AGnd y *v*x′.

**Simulaciones del filtrado wavelet a nivel transistor**

En esta sección presentamos la simulación del filtrado wavelet a nivel transistor empleando la señal de prueba *tone2* que se muestra en la Figura V.6(a). La amplitud
Figura V.6: Simulaciones en Hspice a nivel transistor de la función de filtrado wavelet. 

máxima de esta señal ha sido fijada a ±10 mV. En la Figura V.6(b) se muestra la señal de prueba contaminada con ruido blanco de banda limitada de rms = 4.0 mV.

En la Figura V.6(c) están graficados como una imagen las 16 componentes $v_{wa}$ a la salida del amplificador $Amp1$. De cada componente ha sido sustraído el nivel de $AGnd = 700$ mV. Estas señales están acopladas en DC con los comparadores $TComp$. Esta imagen es equivalente a la transformada wavelet de la señal ruidosa pero con cada componente multiplicada por un factor de $\sqrt{r^{m/2}}$.

En la Figura V.6(d) se muestran las 16 salidas de la función de retardo $L_m$ graficadas como una imagen, donde negro=1 y blanco=0.
En la Figura V.6(e) se muestran la señal de prueba original y la señal reconstruida cuyo nivel de DC ha sido substraído y multiplicada por un factor de $k = -0.834$. La razón de señal a ruido resultante es de SNR=6.85 dB.
VI Diseño del layout

Una vez diseñado el chip, se mandó fabricar a través de la organización MOSIS en el proceso C5 de AMIS Semiconductor, el cual es una tecnología CMOS de 0.5 \( \mu m \) para aplicaciones de señal mixta.

El layout del sistema se diseñó empleando las herramientas de Tanner: editor de esquemáticos, editor de layouts, herramienta de extracción y la herramienta para la verificación de esquemático versus layout (LVS).

En esta sección se presentan las particularidades del diseño del layout.

Planeación del chip

El sistema consta de 16 bloques idénticos. El *floor planning* del chip se muestra en la Figura VI.1(a). Los 8 filtros a la derecha del *core* son imágenes especulares de los 8 bloques de la izquierda. En la Figura VI.1(b) se muestra una vista interna del bloque.

Reglas de diseño

El layout fue elaborado con la ayuda del editor *Tanner L-Edit*. El diseño fue completamente *full-custom*, pero se simplificó efectuando un diseño modular y jerárquico.

Para la mayor parte del diseño se utilizaron las reglas de diseño independientes del fabricante proporcionadas por MOSIS, *scalable design rules*, específicamente se usó el conjunto de reglas *SCN3ME_SUBM*. Solamente en algunos casos las reglas de diseño nativas se utilizaron para realizar un ajuste fino.

Reglas de electromigración

MOSIS recomienda que en el diseño de prototipos no se preste mucha atención al cumplimiento de las reglas de electromigración. A pesar de esto se verificó que para el consumo de potencia del sistema diseñado, menos de 1 mA, estas reglas son siempre respetadas.
Figura VI.1: Floor Planning.

Reglas de densidad mínima

Se verificó que el sistema diseñado cumple con las siguientes reglas de densidad mínima requeridas para el proceso seleccionado:

- Densidad mínima para la capa de polisilicio: 15%
- Densidad mínima para la capa de metal-1: 30%
- Densidad mínima para la capa de metal-2: 30%

Reglas de antena

Se revisó el cumplimiento de las reglas de antena, y se detectaron y corrigieron los pocos casos en que esta regla no se cumplía, que es en las entradas de control a las compuertas de los transistores de dimensiones mínimas usadas para emular los resistores de alto valor.
Buenas prácticas de diseño

Se cuenta con ocho bien conocidas reglas para reducir el efecto del mismatch durante el diseño del layout:

1. No emplear dimensiones mínimas
2. Emplear dimensiones idénticas
3. Emplear distancias mínimas
4. Emplear estructuras de centroide común
5. Emplear la misma orientación
6. Emplear la misma estructura
7. Emplear la misma temperatura
8. Emplear el mismo entorno

El propósito de la primer regla es el de reducir los efectos de las fluctuaciones aleatorias locales. Esta regla está implícita en el modelo Pelgrom generalizado usado para las simulaciones Monte Carlo durante el diseño eléctrico del sistema.

La mayoría de las otras reglas tienen como propósito el reducir el efecto de los gradientes sistemáticos en los parámetros de proceso. Efectos que no están consideradas en el modelo Pelgrom empleado. Por lo tanto, el efecto de los resultados de mismatch obtenido mediante las simulaciones Monte Carlo se incrementaría en el chip fabricado si estas reglas no se siguen.

Para el diseño del layout, se tuvo que hacer un compromiso entre el cumplimiento de estas reglas y el consumo de área. Sin embargo, las reglas referentes a distancias mínimas, centroide común, misma orientación y misma estructura siempre se cumplieron durante el diseño de los pares de transistores apareados.
Ruido en el substrato

Para reducir el ruido en el substrato, se colocaron anillos de guarda alrededor de cada par de transistores apareados en los diferentes amplificadores, así como alrededor de otras estructuras analógicas sensibles.

Algunos layouts

En el presente resumen solo presentamos algunos layouts de ejemplo.

En la Figura VI.2 se muestra un par de OTAs TL. Puede observarse que los pares de transistores apareados fueron diseñados siguiendo estructuras de centroide común y se colocó un anillo de guarda alrededor de cada uno de estos pares.

La mayoría de los capacitores necesarios son del orden de los pF, los cuales requieren una gran cantidad de área para su implementación. Estos capacitores se obtuvieron al rellenar los huecos resultantes después de que los demás bloques del sistema estaban colocados.

En la Figura VI.3 se muestra el layout de los capacitores $C_{f1}$, $C_{in2}$ y $C_{f2}$ a los...
que se refiere el esquemático de la Figura V.3. Los efectos de la capacitancia de perímetro, los efectos de las capacitancias parásitas en las interconexiones, y las reglas de la misma estructura y del mismo entorno fueron cuidadosamente consideradas en el diseño de este layout con el propósito de garantizar las razones de capacitancia deseadas en estos capacitores de valor pequeño.

En la Figura VI.4 se muestra el layout del chip completo. Su tamaño fue de 3000$\mu$m \times 1500$\mu$m. El área resultante fue de 4.5mm$^2$. De esta área el 48% corresponde al core y el 52% corresponde al Pad frame.
Figura VI.4: Layout del chip completo.
VII  Mediciones en el chip

Corrida de fabricación AMIS C5 - 4 de febrero de 2008

El chip diseñado se fabricó en la corrida AMIS C5 del día Feb/4/08. La primera tarea fue el verificar mediante simulaciones en Hspice, qué tan bien se ajusta esta corrida a los modelos de peores casos proporcionados por el fabricante, con ayuda de los cuales el sistema fue diseñado.

Empleando los modelos de los peores casos y los parámetros de la corrida del día Feb/4/08 (publicados por MOSIS en su página web) se obtuvo la respuesta en frecuencia del sistema completo. Los resultados se muestran en la Figura VII.1. Cómo puede observarse, la respuesta del sistema obtenida con el modelo del día Feb/4/08 está inclinada. Esto quizás se debe a que algunos de los parámetros de esta corrida caen fuera de las esquinas de los peores casos.

Preparación de las mediciones

Debido a que el sistema está planeado para trabajar con señales menores a ±30 mV, la principal consideración durante las mediciones del chip es el controlar el ruido del medio ambiente, es decir, mantener bajas las EMIs radiadas y conducidas.

Entre las precauciones tomadas para este efecto están la fabricación de un PCB especial mostrado en la Figura VII.2, fuentes de alimentación en base a baterías, el empleo de un filtro de línea para la toma de corriente del generador de ondas y del osciloscopio, el apantallado introduciendo la tarjeta de prueba dentro de una caja de material ferromagnético aterrizada y la desconexión de los aparatos eléctricos y electrónicos en las cercanías del laboratorio. De esta manera se logró reducir el ruido por debajo de 2 mV pico-pico.
Figura VII.1: Respuesta en frecuencia del sistema wavelet diseñado, obtenido mediante simulaciones Hspice a nivel transistor, empleando el modelo típico, los 4 modelos de los peores casos y el modelo correspondiente a la corrida día Feb/4/08.

**Prueba de descomposición-reconstrucción**

Para la prueba de descomposición-reconstrucción se preparó en Matlab la señal de prueba *tone3* y se cargó en la memoria de un Generador de Señales Arbitrarias, la captura de esta señal mediante un Osciloscopio Digital se muestra en la Figura VII.3(a). Esta señal de prueba consta de 3 pulsos de 500 Hz, 1 kHz y 2 kHz. Para reducir el efecto de la componente de ruido proveniente del medio ambiente se hizo un promedio de 128 muestras con el Osciloscopio Digital.

En la Figura VII.3(b) se grafican como una imagen los valores absolutos de las componentes wavelet $v_{wa}$ que se emplean en la etapa de comparación. Esta imagen corresponde con la transformada de la señal de entrada excepto que cada componente está multiplicada por el factor $\sqrt{r^m/r^1}$. Se muestran dos imágenes, a una se le ha restado el nivel promedio de la imagen total, es decir, esta acoplada en DC. Mientras que a la segunda imagen se le ha restado el nivel de DC a cada componente wavelet de manera individual, por lo que podemos decir que esta transformada se encuentra acoplada en AC.

En la Figura VII.3(c) se muestran juntas la señal de prueba de entrada y la señal
reconstruida a la cual se le ha restado su componente de DC y se ha multiplicado por un factor $k = -0.878$.

En todas estas mediciones se tomó un promedio de 128 muestras con el Osciloscopio Digital para reducir el efecto del ruido blanco del medio ambiente. Sobre los datos digitales capturados se realizó un postprocesamiento en Matlab para obtener estas gráficas.

**Prueba de filtrado wavelet**

Para la prueba de filtrado se creo en Matlab la señal de prueba $tone3$. Se le añadió ruido blanco de banda limitada y se filtro con un filtro Butterworth cuyo ancho de banda cae en la banda de trabajo del sistema wavelet fabricado.

La señal ruidosa de prueba se muestra en la Figura VII.4(a). En la Figura VII.4(b) se muestra la transformada wavelet de la señal ruidosa acoplada tanto en DC como en AC. Observe como el ruido se distribuye en las diferentes componentes wavelet.

En la Figura VII.4(c) se muestra la señal reconstruida sin aplicar la función de filtrado del sistema wavelet. Por último, en la Figura VII.4(d) se muestran la señal de salida filtrada mediante el sistema wavelet la cual se ha multiplicado por el factor $k = -0.573$ junto con la señal original libre de ruido.
Figura VII.3: Prueba de reconstrucción (mediciones con osciloscopio). (a) Señal de prueba *tone3*, compuesta por pulsos de 500Hz, 1kHz y 2kHz. (b) Imágenes de la transformada wavelet de la señal de prueba obtenidas a partir de las componentes $v_{wu}$. (c) Señal de prueba original graficada junto con la señal reconstruida tras ser multiplicada por un factor $k = -0.878$. 
Figura VII.4: Prueba de filtrado (mediciones con osciloscopio). (a) Señal de prueba tone3 con ruido Gausiano aditivo de banda limitada. (b) Imágenes de la transformada wavelet obtenidas con las componentes $v_{wa}$. (c) Señal reconstruida sin activar la función de filtrado. (d) Señal original libre de ruido graficada junto con la señal ruidosa después de haber sido filtrada y multiplicada por un factor $k = -0.573$. 

$SNR = -1.52 \, dB$

$SNR = 4.79 \, dB$
En todas estas mediciones se tomó un promedio de 128 muestras con el Osciloscopio Digital para reducir el efecto del ruido blanco del medio ambiente. Los datos digitales fueron procesados en Matlab para obtener estas gráficas.
VIII Conclusiones

En esta sección se enumeran los resultados teóricos y aplicados obtenidos durante el desarrollo de este trabajo.

Resultados teóricos

Entre los principales resultados teóricos se cuentan los siguientes.

1. Cualquier filtro racional pasabanda realiza la transformada wavelet semidiscreta.

2. La wavelet es el inverso en el tiempo de la respuesta al impulso del filtro pasabanda.

3. El tipo de filtro de reconstrucción puede ser cualquiera físicamente realizable donde el error de reconstrucción proviene del rizo presente en la sumatoria de los espectros en frecuencia de los filtros.

Se propusieron 2 métodos para diseñar sistemas analógicos de descomposición-reconstrucción mediante la transformada wavelet.

5. El primer método consiste simplemente en incrementar la densidad de escalas. Con este método se requieren filtros más sencillos y se reducen los efectos de las variaciones de proceso debido a que aumenta el traslape entre los espectros de filtros adyacentes.

6. En el segundo método se encuentran filtros de reconstrucción hechos a la medida. Se mantiene una baja densidad de escalas a cambio de un aumento en la complejidad y sensibilidad de los filtros.

7. Las wavelets semidiscretas analógicas aquí descritas satisfacen la propiedad de retardo cero debido a que son wavelets totalmente causales. Esta propiedad no la poseen las wavelets clásicas continuas o discretas.
8. Como ejemplo de aplicación, proponemos una modificación de la función de filtrado *hard thresholding* para ser usada con la transformada wavelet analógica aquí descrita.

**Resultados aplicados**

Tomando en cuenta que el principal propósito del circuito integrado diseñado fue el mostrar la aplicabilidad de los resultados teóricamente obtenidos, a continuación se presentan algunos puntos destacables que se obtuvieron durante las etapas de diseño y caracterización del sistema implementado.

1. El sistema diseñado realiza procesamiento altamente paralelo, para lo cual se requiere de una gran cantidad de estructuras idénticas. Esta redundancia ayuda a aliviar el efecto de las variaciones de proceso. Por esta razón, la filosofía de diseño de los amplificadores y OTAs fue la simplicidad. De hecho, no eran necesarias altas ganancias, mientras que una baja cantidad de nodos internos ayudó en la reducción del ruido producido internamente.

2. El sistema wavelet se diseñó para operar en el rango de audiofrecuencias. Para los filtros se emplearon filtros construidos con CMOS trabajando en subumbral, debido a la facilidad que tienen estos filtros para ser escalados en frecuencia de manera exponencial.

3. Para el diseño del chip, incluyendo la etapa de diseño del layout, se siguieron las precauciones comúnmente recomendadas: no dimensiones mínimas, mismas dimensiones, mínimas distancias, estructuras de centroide común, misma orientación, misma estructura, misma temperatura y mismo entorno.

4. Otras precauciones tomadas fueron: líneas separadas para circuitos analógicos y digitales, anillos de guarda alrededor de estructuras analógicas sensibles.

5. Durante la etapa de diseño se uso ampliamente el análisis Monte Carlo con un modelo Pelgrom generalizado.
6. Se obtuvieron buenos resultados en las mediciones de descomposición y reconstrucción con el chip, a pesar de que la corrida con que se fabricó no cae dentro de los modelos de peor caso con los cuales el sistema fue diseñado.

7. Se obtuvieron sólo modestos resultados en las mediciones de la función de filtrado con el chip. Hemos identificado dos causas probables para este pobre desempeño: a) Ruido introducido por las conmutaciones digitales. b) La no-linealidad del transistor de dimensiones mínimas usado como resistencia de alto valor. Para trabajo futuro se deberá poner más atención en estas cuestiones.

La filosofía del presente trabajo fue la de trabajar a profundidad con la teoría de la transformada wavelet con el propósito de obtener implementaciones analógicas con estructuras simples.