# Nonlinear coherent states for the Susskind-Glogower operators

by

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# Abstract

In this thesis, we construct nonlinear coherent states for the Susskind-Glogower operators by the application of the displacement operator on the vacuum state. We also construct nonlinear coherent states as eigenfunctions of a Hamiltonian constructed with the Susskind-Glogower operators. We generalize the solution to the eigenfunction problem to an arbitrary $|m\rangle$ initial condition. To analize the obtained results, we plot the Husimi $\mathcal{Q}$ function, the photon number probability distribution and the Mandel Q-parameter. For both cases, we find that the constructed states exhibit interesting nonclassical features such as amplitude squeezing and quantum inteferences due to a self-splitting into two coherent-like states. Additionally, we show that nonlinear coherent states may be modelled by propagating light in semi-infinite arrays of optical fibers.

# Resumen

En la presente tesis, construímos estados coherentes no lineales para los operadores de Susskind-Glogower mediante la aplicación del operador de desplazamiento al estado de vacío. Construímos estados no lineales como eigenfunciones de un Hamiltoniano construído con los operadores de Susskind-Glogower. Generalizamos la solución al problema de eigenfunciones para una condición inicial arbitraria $|m\rangle$. Para analizar los resultados obtenidos, graficamos la función $\mathcal{Q}$ de Husimi, la distribución de fotones y el parámetro Q de Mandel. En ambos casos, encontramos que los estados construídos presentan características no lineales tales como compresión en amplitud e interference cuántica debida a la separación del estado en dos estados semejantes a los estados coherentes. Adicionalmente, mostramos que los estados coherentes no lineales pueden ser modelados mediante luz propagada en arreglos semi-infinitos de fibras ópticas.

# Preface

I was told once that a master thesis gives us the preparation we need to face the challenge of a PhD and, I got to say, that is completly true. I remembered that I wanted to start my PhD immediately after obtaining my Bachelor degree but, after listening to a friend saying that phrase about the master, I definitely changed my mind. A bachelor degree helps us to develope some skills however, an MSc helps us to be more experienced and, according to my own experience, it helps us to develope the spirit of research that a researcher needs to successfully perform his job. That research spirit led me to work on a thesis within the framework of Quantum Optics, particularly in the topic of nonlinear coherent states.

Coherent states play an important role in quantum optics. The development of the laser made it possible to prepare fields that are very close to such states. Their behavior shows a close correspondence to that of a classical wave which is why coherent states define the limit between the classical and nonclassical behavior of a certain state that we want to analyze.

Coherent states are mathematically obtained using the harmonic oscillator algebra; however, new methods have been proposed to generalize the idea of coherent states for systems with different dynamical features. This generalization led people to obtain states that not only maintain typical features of the coherent states but also present nonclassical properties, such as amplitude squeezing and quantum interferences. Those states are called *nonlinear coherent states.*

In this thesis, We present the construction of nonlinear coherent states for the Susskind-Glogower operators, first as those obtained by the application of the displacement operator on the vacuum state and second, as the eigenfunctions of a particular Hamiltonian to finally generalize the solution to an arbitrary initial condition. Additionally, we cite a paper that allows us to propose that nonlinear coherent states may be modeled by propagating light in semi-infinite arrays of optical fibers.

I want to express my sincere thanks to my teachers. My thesis director Dr. Héctor Moya Cessa for helping me to develope the thesis and whose lectures at the Institute made me realize that quantum optics is the area of physics where I want to work in the future and Dr. Víctor Arrizón who gave us a very useful idea to calculate Bessel series that we obtain during the development of the thesis.

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# Chapter 1

# Introduction

Coherent states for the electromagnetic field, introduced by Glauber [1], [2], [3], have been an important part in the development of quantum optics. The importance of these states resides in the relatively easy way to produce them in the lab and in their classical wave behavior. These states can be obtained by different mathematical definitions:

a) As the right-hand eigenstates of the annihilation operator [2]

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle ,$$

with $\alpha$ a complex number.

b) As the those states obtained by the application of the displacement operator $D\left(\alpha\right)$ on the vacuum state of the harmonic oscillator [2]

$$\left|\alpha\right\rangle = \hat{D}\left(\alpha\right)\left|0\right\rangle ,$$

with $\hat{D}\left(\alpha\right) = e^{\alpha\hat{a}^{\dagger}-\alpha^{*}\hat{a}}$.

c) As those states which their time-evolving wave function shape does not change with time and whose centroid follows the motion of a classical point particle in a harmonic oscillator potential [13].

Using the harmonic oscillator algebra one can obtain the same coherent states from the three different definitions; however, for systems with complex dynamical properties, the harmonic oscillator model is not adequate, therefore, new methods to generalize the idea of coherent states for systems like those have been proposed.

Nieto and Simmons [4], [5], [6], constructed coherent states for potentials other than the harmonic oscillator whose energy spectra have unequally spaced energy levels such as the Poschl-Teller potential, the harmonic oscillator with centripetal barrier and the Morse potential.

Gazeau and Klauder [7] proposed a generalization for systems with one degree of freedom possessing discrete and continuous spectra. These states were constructed performing a parametrization of the coherent states by two real values: an amplitude $J$, and a phase $\gamma$, instead of a complex value $\alpha$ .

Man'ko *et al.* [8] introduced coherent states of an f-deformed algebra as eigenstates of the annihilation operator $A = \hat{a}f(\hat{n})$ where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the number operator and $\hat{a}$, $\hat{a}^\dagger$ are the annihilation and creation boson operators of the harmonic oscillator algebra, respectively. A remarkable result is that these states present nonclassical properties such as squeezing and antibunching [9].

Récamier *et al.* [10] proposed coherent states using a *deformed* version of the displacement operator method generalized to the case of f-deformed oscillators, assuming that the number operator function appearing in the commutator between the deformed operators can be replaced by a number. The method yields a displacement operator which is *approximately* unitary and displaces the deformed annihilation $\hat{A}$ and creation $\hat{A}^\dagger$ operators in the usual way.

In this thesis, we construct nonlinear coherent states by the application of the displacement operator (for the Susskind-Glogower operators [11]) on the vacuum state and; time-dependent, $|m\rangle$ displaced number states as eigenfunctions of a Hamiltonian representing the fundamental physical coupling to the radiation field via the Susskind-Glogower operators. We show the $\mathcal{Q}$ (Husimi) function [26] for the resulting states as well as their photon distribution and Mandel Q-parameter [28] in order to determine the nonclassical features of the constructed states. Additionally, we show that nonlinear coherent states may be modelled by propagating light in semi-infinite arrays of optical fibers.

The content is organized as follows. In Chapter 2 we present a brief description of the field quantization using, for simplicity, a single-mode field. In Chapter 3 we present the linear and nonlinear coherent states and the mathematical formulation for constructing them. In Chapter 4 we introduce the phase operators and we give a complete description of the Susskind-Glogower operators and some of their properties. Chapter 5 and Chapter 6 correspond to the work done during the thesis project. In Chapter 5 we construct coherent states by the application of the displacement operator for the Susskind-Glogower operators on the vacuum state, we show their corresponding $\mathcal{Q}$ function, photon distribution and Mandel's Q-parameter. In Chapter 6 we construct time-dependent coherent states as eigenfunctions of the Susskind-Glogower Hamiltonian, we show their time-dependent $\mathcal{Q}$ function, photon distribution and Mandel's Q-parameter and, also, we show how nonlinear coherent states may be modelled by propagating light in semi-infinite arrays of optical fibers and in Chapter 7 we present remarks and conclusions.

# Chapter 2

# Field Quantization

In this chapter we present the result on which quantum optics is based, this is, the quantization of the electromagnetic field. For simplicity we present the case of a single-mode field confined by conducting walls in a one-dimensional cavity. We introduce the photon states and present the fluctuations of the field with respect to these states.

## 2.1 Quantization of a single-mode field

For this section we consider a radiation field in a one-dimensional cavity along the $z$-axis with perfectly conducting walls at $z = 0$ and $z = L$, where $L$ corresponds to the cavity length as shown in Fig. 2.1.

The boundary conditions consist of the vanishing of the electric field on the conducting walls. We assume there are no sources of radiation, this is, no currents nor charges.

The field is assumed to be polarized along the $x$-direction, $\mathbf{E}(\mathbf{r}, t) =$

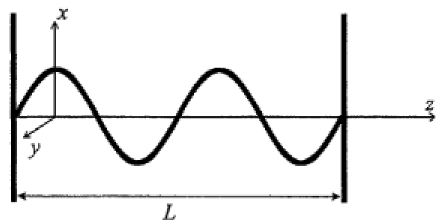


Figure 2.1: Cavity with conducting walls located at $z = 0$ and $z = L$. The electric field is polarized along the $x$-direction.

$E_x(z,t)\,\hat{e}_x$, where $\hat{e}_x$ is a unit polarization vector. Maxwell's equations without sources are, in SI units,

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \tag{2.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.3}$$

$$\nabla \cdot \mathbf{E} = 0. \tag{2.4}$$

A single-mode field satisfying Maxwell's equations and the boundary conditions is given by

$$E_x(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{\frac{1}{2}} q(t) \sin(kz), \tag{2.5}$$

where $\omega$ is the frequency of the mode and $k = \omega/c$ is the wave number, $c$ is the speed of light, $V$ is the volume of the cavity and $q(t)$ is a time-dependent factor having dimensions of length. We will see that $q(t)$ is, in fact, the canonical position. The boundary condition at $z = L$ yields the allowed frequencies $\omega_m = c(m\pi/L)$ , $m = 1, 2, ....$ Using Eq. (2.2) and Eq. (2.5) we obtain the magnetic field in the cavity $\mathbf{B}(\mathbf{r},t) = B_y(z,t)\,\hat{e}_y$, where

$$B_y(z,t) = \left(\frac{\mu_0\epsilon_0}{k}\right)\left(\frac{2\omega^2}{V\epsilon_0}\right)^{\frac{1}{2}} \dot{q}(t) \cos(kz). \tag{2.6}$$

The function $\dot{q}(t)$ will play the role of a canonical momentum, this is, $p(t) = \dot{q}(t)$.

The Hamiltonian (classical) $H$, is given by

$$\begin{aligned} H &= \frac{1}{2}\int dV\left[\epsilon_0\mathbf{E}^2\left(\mathbf{r},t\right)+\frac{1}{\mu_0}\mathbf{B}^2\left(\mathbf{r},t\right)\right], \\ &= \frac{1}{2}\int dV\left[\epsilon_0 E_x^2\left(z,t\right)+\frac{1}{\mu_0}B_y^2\left(z,t\right)\right]. \end{aligned} \tag{2.7}$$

Substituting Eq. (2.5), Eq. (2.6) and integrating we obtain

$$H=\frac{1}{2}\left(p^2+\omega^2q^2\right). \tag{2.8}$$

It is important to notice that the resulting Hamiltonian is equivalent to the harmonic oscillator of unit mass, where the electric and magnetic fields, apart from some scale factors, play the roles of canonical position $q$ and canonical momentum $p$.

Having identified the canonical variables $q$ and $p$ for the classical system, we use the correspondence principle to replace them by their operator equivalents $\hat{q}$ and $\hat{p}$ that satisfy the commutation relation[1]

$$\left[\hat{q},\hat{p}\right]=i\hbar\hat{I}, \tag{2.9}$$

where $\hat{I}$ corresponds to the identity operator. The electric and magnetic fields of the single mode become the operators

$$\hat{E}_x\left(z\right)=\left(\frac{2\omega^2}{V\epsilon_0}\right)^{\frac{1}{2}}\hat{q}\sin\left(kz\right), \tag{2.10}$$

and

$$\hat{B}_y\left(z\right)=\left(\frac{\mu_0\epsilon_0}{k}\right)\left(\frac{2\omega^2}{V\epsilon_0}\right)^{\frac{1}{2}}\hat{p}\cos\left(kz\right), \tag{2.11}$$

where we have not considered the time dependance of the operators, since it is not necessary for what we want to show next.

[1] A commutator tells us if we can measure the expectation value of two observables at the same time. If the commutator of two observables is zero, then they can be measured at the same time, otherwise there exists an uncertainty relation between them.

Performing the integration in the same way as in Eq. (2.7), the Hamiltonian (quantized) becomes

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{q}^2\right). \tag{2.12}$$

We now use the non-Hermitian boson annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators to construct the combinations

$$\hat{a} = (2\hbar\omega)^{-\frac{1}{2}} \left(\omega\hat{q} + i\hat{p}\right), \tag{2.13}$$

$$\hat{a}^\dagger = (2\hbar\omega)^{-\frac{1}{2}} \left(\omega\hat{q} - i\hat{p}\right). \tag{2.14}$$

Using these operators, the electric and magnetic field may be written as

$$\hat{E}_x(z) = E_0 \left(\hat{a} + \hat{a}^\dagger\right) \sin(kz), \tag{2.15}$$

$$\hat{B}_y(z) = B_0 \frac{1}{i} \left(\hat{a} - \hat{a}^\dagger\right) \cos(kz), \tag{2.16}$$

where

$$E_0 = \left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{\frac{1}{2}}, \tag{2.17}$$

and

$$B_0 = \left(\frac{\mu_0}{k}\right)\left(\frac{\epsilon_0 \hbar\omega^3}{V}\right)^{\frac{1}{2}}. \tag{2.18}$$

Using Eq. (2.13) and Eq. (2.14) we find the following relations

$$\hat{q} = \left(\frac{\hbar}{2\omega}\right)^{\frac{1}{2}} \left(\hat{a}^\dagger + \hat{a}\right), \tag{2.19}$$

$$\hat{p} = i\left(\frac{\omega\hbar}{2}\right)^{\frac{1}{2}} \left(\hat{a}^\dagger - \hat{a}\right), \tag{2.20}$$

and considering that the operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the commutation relation

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1, \tag{2.21}$$

the Hamiltonian operator takes the form

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{2.22}$$

The operator product $\hat{a}^\dagger\hat{a}$ has an important significance and is called the number operator and denoted as $\hat{n}$. Let $|n\rangle$ be an energy eigenstate of the single mode field with the energy eigenvalue $E_n$ such that

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle = E_n|n\rangle . \tag{2.23}$$

If we multiply on the left of the Eq. (2.23) by $\hat{a}^\dagger$ and we use the relation (2.21) we generate the eigenvalue equation

$$\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\left(\hat{a}^\dagger|n\rangle\right) = (E_n + \hbar\omega)\left(\hat{a}^\dagger|n\rangle\right), \tag{2.24}$$

which is the eigenvalue problem for the eigenstate $\left(\hat{a}^\dagger|n\rangle\right)$ with the energy eigenvalue $E_n + \hbar\omega$. Similarly, if we perform the same procedure using the operator $\hat{a}$ we obtain

$$\hat{H}\left(\hat{a}|n\rangle\right) = (E_n - \hbar\omega)\left(\hat{a}|n\rangle\right). \tag{2.25}$$

From Eq. (2.25) we see that as we keep applying the annihilation operator we low the energy eigenvalue by integer multiples of $\hbar\omega$. We have to consider that the energy is always positive and there must exist a condition in such a way that the annihilation operator stops acting at the ground state $|0\rangle$, this is,

$$\hat{a}|0\rangle = 0. \tag{2.26}$$

Thus, the eigenvalue problem for the ground state is

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle , \tag{2.27}$$

so the lowest-energy eigenvalue is $\hbar\omega/2$ and since $E_{n+1} = E_n + \hbar\omega$, the energy eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), n = 0, 1, 2, ... \tag{2.28}$$

Using previous results we are now able to construct an elegant and very useful representation for the number states. If we substitute Eq. (2.28) in Eq.

(2.23) we obtain

$$\hat{n}\left|n\right\rangle = n\left|n\right\rangle . \tag{2.29}$$

These number states must be normalized according to $\langle n|n\rangle = 1$. For the state $\hat{a}\left|n\right\rangle$ we have

$$\hat{a}\left|n\right\rangle = c_n\left|n-1\right\rangle , \tag{2.30}$$

where $c_n$ is the constant to be determined. Then the inner product of $\hat{a}\left|n\right\rangle$ with itself is

$$\left\langle n\left|\hat{a}^\dagger\hat{a}\right|n\right\rangle = \left\langle n-1\left|c_n^* c_n\right|n-1\right\rangle = \left|c_n^2\right| , \tag{2.31}$$

so we write $c_n = \sqrt{n}$, to obtain

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle . \tag{2.32}$$

Similarly we have that

$$\hat{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle . \tag{2.33}$$

From Eq. (2.33) we find that the number states $|n\rangle$ may be generated from the ground state $|0\rangle$ by the repeated action of the creation operator

$$\left|n\right\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}\left|0\right\rangle . \tag{2.34}$$

Number states are taken to represent a state of the field containing $n$ photons. Furthermore, from Eq. (2.9), we see that number states do not behave like classical oscillators, this is, we cannot measure simultaneously the expectation value for the position $q$ and the momentum $p$.

Finally, we can see that because $\hat{H}$ and $\hat{n}$ are Hermitian (observables), states of different number are orthogonal, this is $\langle n|m\rangle = \delta_{nm}$ and, additionally, the number states form a complete set, this is

$$\sum_{n=0}^{\infty}\left|n\right\rangle\left\langle n\right| = 1. \tag{2.35}$$

## 2.2 Quantum fluctuations of a single-mode field

In this section we show the need of defining a new state to describe correctly the electric field and we introduce the notion of quantum phase.

The number state $|n\rangle$ is a well-defined energy state, however the electric field is not since

$$\left\langle \hat{E}_x(z) \right\rangle = \langle n| \hat{E}_x(z) |n\rangle = E_0 \sin(kz) \left[\langle n| \hat{a} |n\rangle + \langle n| \hat{a}^\dagger |n\rangle\right] = 0, \tag{2.36}$$

this is, the mean field is zero. However, the mean of the square of this field, that contributes to the energy density, is not zero

$$\begin{aligned} \left\langle \hat{E}_x^2(z) \right\rangle &= \langle n| \hat{E}_x^2(z) |n\rangle , \\ &= E_0^2 \sin^2(kz) \langle n| \hat{a}^{\dagger 2} + \hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger |n\rangle , \\ &= E_0^2 \sin^2(kz) \langle n| \hat{a}^{\dagger 2} + \hat{a}^2 + 2\hat{a}^\dagger \hat{a} + 1 |n\rangle , \\ &= 2E_0^2 \sin^2(kz) \left(n + \frac{1}{2}\right). \end{aligned} \tag{2.37}$$

The fluctuations of the electric field (and any statistical quantity) may be characterized by the variance but, in order to determine uncertainty relations, we should use the standard deviation that corresponds to the square root of the variance

$$\begin{aligned} \Delta E_x &= \left(\left\langle \hat{E}_x^2(z) \right\rangle - \left\langle \hat{E}_x(z) \right\rangle^2\right)^{\frac{1}{2}} , \\ &= \sqrt{2} E_0 \sin(kz) \left(n + \frac{1}{2}\right)^{\frac{1}{2}} . \end{aligned} \tag{2.38}$$

We see that even for the case when there are no photons $n = 0$, the field has fluctuations. This interesting result arises as a consequence of the uncertainty principle because the number operator $\hat{n}$ does not commute with the electric field

$$\left[\hat{n}, \hat{E}_x\right] = E_0 \sin(kz)\left(\hat{a}^\dagger - \hat{a}\right). \tag{2.39}$$

Using the result [13]

$$If \quad \left[\hat{A}, \hat{B}\right] = \hat{C} \qquad \Rightarrow \qquad \Delta A \Delta B \geq \frac{1}{2}\left|\left\langle \hat{C}\right\rangle\right|, \tag{2.40}$$

we obtain the uncertainty relation

$$\Delta n \Delta E_x \geq \frac{1}{2} E_0 \left|\sin(kz)\right| \left|\left\langle \hat{a}^\dagger - \hat{a}\right\rangle\right|. \tag{2.41}$$

The previous result tells us something important about the relation between the number of photons and the electric field. If the field were accurately known, then the number of photons would be uncertain and viceversa. Furthermore, as we can see from Eq. (2.10), the electric field carries information about the phase of the field, this suggests that there must exist a connection between the photon number and the phase of the field.

Considering Eq. (2.41) we can conclude that the photon number and phase of the field cannot be simultaneously well defined, this, we can consider that the phase is in some sense complementary to the photon number much in the way that time is complementary to energy (frequency).

In the previous description, we have used the term phase in a classical way; however, there have been many attempts to create a formalism for the description of the quantum phase. In chapter 4, we will give a brief description of how the quantum phase issue has been treated and we present the Susskind-Glogower operators, not as a concluding solution to the phase problem but as a useful tool to construct nonlinear coherent states.

# Chapter 3

# Coherent states

## 3.1 Coherent states

Coherent states represent an important part in the development of quantum optics, first derived in 1926 by Erwin Schrödinger while searching for solutions to the Schrödinger equation that satisfy the correspondence principle and later, within the framework of quantum optics, introduced by Roy J. Glauber in 1963. The importance of these states resides in the relatively easy way to produce them in the lab and in their classical behavior.

To introduce coherent states we will use Sakurai's approach [12]. We have seen that an energy eigenstate $|n\rangle$ does not behave like the classical oscillator, this is, it results impossible to measure simultaneously the expectation value for the position $q$ and the momentum $p$, furthermore, the expectation value of the electric field vanishes Eq. (2.36). A question arises from these results. How can we construct states that most closely imitates the classical oscillator and where the expectation value of the electric field is not zero. The answer to this question lies in the superposition of all the number states.

### 3.1.1 Eigenstates of the annihilation operator

The replacement of the creation and annihilation operators by continuous variables produces a classical field and, the way to make this replacement, is by seeking eigenstates of the annihilation operator [13]. Coherent states denoted as $|\alpha\rangle$ may be constructed as eigenstates of the annihilation operator as follows

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{3.1}$$

where $\alpha = q + ip$ is an arbitrary complex number. The operator $\hat{a}$ acts as Eq. (3.1) while $\hat{a}^\dagger$ acts as

$$\langle\alpha|\,\hat{a}^\dagger = \alpha^*\,\langle\alpha|\,. \tag{3.2}$$

Following the idea of the superposition of number states and considering that they form a complete set, we write

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle\,. \tag{3.3}$$

Substituting in Eq. (3.1) we have

$$\hat{a}\,|\alpha\rangle = \sum_{n=1}^{\infty} c_n\sqrt{n}\,|n-1\rangle = \alpha\sum_{n=0}^{\infty} c_n\,|n\rangle = \sum_{n=1}^{\infty} \alpha c_{n-1}\,|n-1\rangle\,. \tag{3.4}$$

Equating the coefficients of $|n-1\rangle$ we find the recurrence relation

$$c_n\sqrt{n} = \alpha c_{n-1}, \tag{3.5}$$

from where we have

$$c_n = \frac{\alpha^n}{\sqrt{n!}}c_0. \tag{3.6}$$

Substituting in Eq. (3.3) we obtain

$$|\alpha\rangle = c_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{3.7}$$

From the normalization requirement we determine $c_0$

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= |c_0|^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*^n}\alpha^m}{\sqrt{n!}\sqrt{m!}}\langle n|m\rangle\,, \\
&= |c_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}, \\
&= |c_0|^2 e^{|\alpha|^2}, \\
&\equiv 1,
\end{aligned} \tag{3.8}$$

that implies that

$$c_0 = e^{-\frac{1}{2}|\alpha|^2}. \tag{3.9}$$

Substituting this expression into Eq. (3.7) we finally obtain

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,. \tag{3.10}$$

### 3.1.2 Displaced vacuum states

Now we derive an elegant result that is widely used in quantum optics, within the framework of group theory. From previous section, we found that an expression for $|\alpha\rangle$ in terms of the number states is given by the Eq. (3.10) and, because we know that $|n\rangle$ can be expressed by Eq. (2.34), we have

$$\begin{aligned}
|\alpha\rangle &= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\left(\hat{a}^{\dagger^n}\right)}{\sqrt{n!}}|0\rangle\,, \\
&= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\left(\alpha\hat{a}^\dagger\right)^n}{n!}|0\rangle\,, \\
&= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}|0\rangle\,.
\end{aligned} \tag{3.11}$$

Considering Eq. (2.26) we can write

$$|0\rangle = e^{-\alpha^*\hat{a}}|0\rangle\,. \tag{3.12}$$

Substituting in Eq. (3.11) we finally obtain

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle, \tag{3.13}$$

where $\hat{D}(\alpha) = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}$ is called the **Displacement Operator**.

Now, in view of the Baker-Hausdorff theorem of group theory [13], [14], if $A$ and $B$ are any two operators such that

$$[[A,B],A] = [[A,B],B] = 0, \tag{3.14}$$

then

$$e^{A+B} = e^{-\frac{[A,B]}{2}} e^A e^B. \tag{3.15}$$

Taking $A = \alpha\hat{a}^\dagger$ and $B = -\alpha^*\hat{a}$, it follows that

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}. \tag{3.16}$$

### 3.1.3 Time evolution

In order to obtain the time-evolved wave function for coherent states, let us consider the wave function for the number states [15]

$$\psi_n(q) = \langle q|n\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\frac{\xi^2}{2}}, \tag{3.17}$$

where $\xi = q\sqrt{\omega/\hbar}$ and where $H_n(\xi)$ are the Hermite polynomials. The corresponding wave function for the coherent states is then

$$\psi_\alpha(q) = \langle q|\alpha\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\left(\alpha/\sqrt{2}\right)^n}{n!} H_n(\xi)\, e^{-\frac{\xi^2}{2}}. \tag{3.18}$$

Using the generating function for Hermite polynomials, we can simplify the expression for the coherent state wave function

$$\psi_\alpha(\xi) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{|\alpha|^2}{2}} e^{\frac{\xi^2}{2}} e^{-\left(\xi - \frac{\alpha}{\sqrt{2}}\right)^2}. \tag{3.19}$$

We now consider the time evolution of a coherent state for a single-mode free field where the Hamiltonian is given by Eq. (2.22). The time-evolved coherent state is given by [13]

$$\begin{aligned} |\alpha, t\rangle &= e^{-i\frac{\hat{H}t}{\hbar}} |\alpha\rangle , \\ &= e^{-i\frac{\omega t}{2}} e^{-i\omega t\hat{n}} |\alpha\rangle , \\ &= e^{-i\frac{\omega t}{2}} \left|\alpha e^{-i\omega t}\right\rangle , \end{aligned} \tag{3.20}$$

and so the coherent state remains a coherent state under free field evolution. The corresponding time-evolved wave function is, from Eq. (3.19),

$$\psi_\alpha(\xi, t) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{|\alpha|^2}{2}} e^{\frac{\xi^2}{2}} e^{-\left(\xi - \frac{\alpha e^{-i\omega t}}{\sqrt{2}}\right)^2}. \tag{3.21}$$

Eq. (3.21) corresponds to a Gaussian whose shape does not change with time and whose centroid follows the motion of a classical point particle in a harmonic oscillator potential (See Figure 3.1). The behavior of coherent states wave function allows us to establish a connection between quantum and classical oscillators via the coherent states.

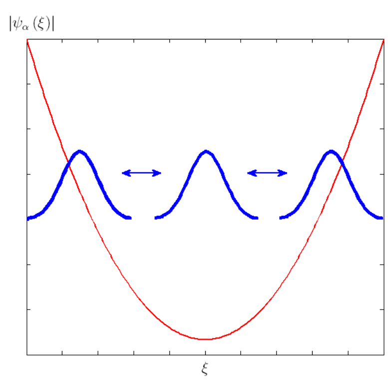


Figure 3.1: A coherent state wave function moves through the harmonic oscillator potential, between the classical turning points, without dispersion.

### 3.1.4 Minimum uncertainty states (MUS)

Coherent states minimize the uncertainty relationship given by Eq. (2.40). To prove this, we consider the definition used to obtain Eq. (2.38), this is, we use the standard deviation of the position and momentum for coherent states.

$$\Delta q = \left(\left\langle \hat{q}^2 \right\rangle - \left\langle \hat{q} \right\rangle^2\right)^{\frac{1}{2}}, \tag{3.22}$$

$$\Delta p = \left(\left\langle \hat{p}^2 \right\rangle - \left\langle \hat{p} \right\rangle^2\right)^{\frac{1}{2}}, \tag{3.23}$$

where $\hat{q}$ and $\hat{p}$ are defined by Eq. (2.19) and Eq. (2.20), respectively. By direct substitution and using Eq. (2.21), Eq. (3.1) and Eq. (3.2) we obtain the following results

$$\langle \hat{q} \rangle = \langle \alpha | \, \hat{q} \, | \alpha \rangle = \left(\frac{\hbar}{2\omega}\right)^{\frac{1}{2}} \left(\alpha^* + \alpha\right), \tag{3.24}$$

$$\left\langle \hat{q}^2 \right\rangle = \langle \alpha | \, \hat{q}^2 \, | \alpha \rangle = \frac{\hbar}{2\omega} \left(1 + 2\left|\alpha\right|^2 + \alpha^2 + \alpha^{*^2}\right), \tag{3.25}$$

$$\langle \hat{p} \rangle = \langle \alpha | \, \hat{p} \, | \alpha \rangle = i \left(\frac{\omega\hbar}{2}\right)^{\frac{1}{2}} \left(\alpha^* - \alpha\right), \tag{3.26}$$

$$\left\langle \hat{p}^2 \right\rangle = \langle \alpha | \, \hat{p}^2 \, | \alpha \rangle = \frac{\omega\hbar}{2} \left(1 + 2\left|\alpha\right|^2 - \alpha^2 - \alpha^{*^2}\right), \tag{3.27}$$

Substituting in Eq. (3.22) and Eq. (3.23) we finally write

$$\Delta q \Delta p = \frac{\hbar}{2}. \tag{3.28}$$

We see that coherent states minimize the value of the uncertainty relation given by Eq. (2.40); however, there exist some other states known as squeezed states that also minimize the uncertainty relation, all of them form a set called MUS.

Fluctuations associated to coherent states are referred to as the standard quantum limit (SQL) because, when we analyze the fluctuations of a new state, any deviation from these fluctuations gives us information about these new states, for example, amplitude squeezing.

### 3.1.5 Photon number probability distribution for coherent states

In this section we analyze the photon number distribution for coherent states, this result will be very useful in the following chapters because it will help us to find out how far from a classical-like state are those we will obtain.
The mean number of photons in the coherent state $|\alpha\rangle$ is given by

$$\langle\alpha|\,\hat{n}\,|\alpha\rangle = \langle\alpha|\,\hat{a}^{\dagger}\hat{a}\,|\alpha\rangle = |\alpha|^{2}\,. \tag{3.29}$$

The probability of finding $n$ photons in $|\alpha\rangle$ is given by

$$P\left(n\right) = \langle n|\,\hat{\rho}\,|n\rangle\,, \tag{3.30}$$

where $\hat{\rho}$ is called the density matrix and for a pure state $|\psi\rangle$ it could be defined as [13]

$$\hat{\rho} = |\psi\rangle\,\langle\psi|\,. \tag{3.31}$$

Using Eq. (3.31) we rewrite Eq. (3.30) as

$$P\left(n\right) = \langle n|\alpha\rangle\,\langle\alpha|n\rangle = |\langle n|\alpha\rangle|^{2}\,, \tag{3.32}$$

to obtain

$$P\left(n\right) = |\alpha|^{2n}\,\frac{e^{-|\alpha|^{2}}}{n!} = \langle n\rangle^{n}\,\frac{e^{-\langle n\rangle}}{n!}. \tag{3.33}$$

Eq. (3.33) corresponds to a Poisson distribution with a mean of $\langle n\rangle$, where $\langle n\rangle = |\alpha|^{2}$ is the average photon number of the field. Figure 3.2 shows the photon number probability distribution for different values of $\langle n\rangle$.

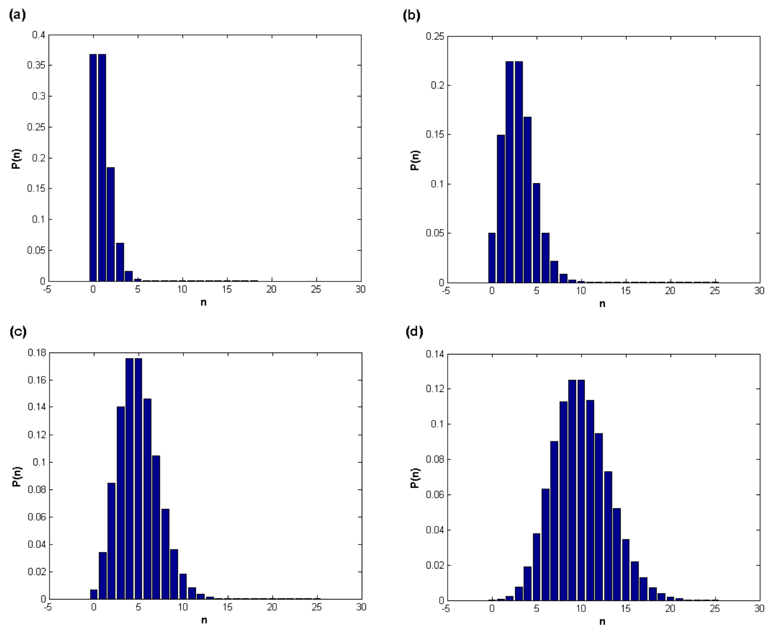


Figure 3.2: Coherent state photon number probability distributions for (a) $\langle n \rangle = 1$; (b) $\langle n \rangle = 3$; (c) $\langle n \rangle = 5$ and (d) $\langle n \rangle = 10$.

## 3.2 Nonlinear coherent states

We have seen that coherent states, defined through creation and annihilation operators, provide us with a beautiful connection between quantum and classical oscillators. The notion of coherent states permitted the use of the language and intuition developed from the study of classical harmonic oscillators in order to treat their quantum counterpart. Coherent states turned out to be appropriate also to describe quantum systems like spin [16] and cyclotron motion of a charge in a magnetic field [17], but for systems with different dynamical properties the harmonic oscillator model is not adequate, therefore, there is an interest to generalize these states to other systems.

A method to generalize coherent states to other systems is based on $q$-deformed algebras, these are deformed versions of the standard Lie algebras, which are recovered as the deformation parameter $q$ goes to unity. The basic interest in

$q$-deformed algebras resides in the fact that they encompass a set of symmetries that is richer than that of the standard Lie algebras.

A $q$-deformed algebra was used to introduce the idea of quantum $q$-oscillators and, since the complete study of $q$-algebras is not necessary to accomplish the objectives of the thesis, we only mention their physical interpretation.

A $q$-oscillator was interpreted as a nonlinear oscillator with a very specific type of nonlinearity in which the frequency of vibration depends on the energy of these vibrations through the hyperbolic cosine function containing a nonlinear parameter [18],[19]. This observation suggested that there would exist other types of nonlinearities for which the frequency of oscillation varies with the amplitude in a different manner from the one obtained with the $q$-deformed algebra. Such oscillators are called $f$-oscillators [18]. One can extend the notion of coherent states by using $f$-oscillators to construct $f$-coherent states (called also as nonlinear coherent states) by means of "deformed" creation and annihilation operators representing the dynamical variables to be associated with the quantum $f$-oscillators [8]. These operators are defined through

$$A = \hat{a} f(\hat{n}) = f(\hat{n}+1)\,\hat{a}, \tag{3.34}$$

$$A^{\dagger} = f(\hat{n})\,\hat{a}^{\dagger} = \hat{a}^{\dagger} f(\hat{n}+1)\,. \tag{3.35}$$

The importance of studying nonlinear coherent states resides in their physical consequences such as amplitude squeezing, quantum interferences and the possibility of having super- or sub-Poissonian statistics. Furthermore, nonlinear coherent states can be realized in the motion of a trapped ion [9] and, as we will see at the end of this thesis, we may model them by propagating light in a semi-infinite array of optical fibers.

From Eq. (3.34) and Eq. (3.35) one can see that phase operators could be of the kind we need to construct nonlinear coherent states. We will see the phase operators and some of their properties in the following chapter.

# Chapter 4

# Phase operators

Over the years, there have been many attempts to create a formalism for the description of the quantum phase, which is a subject of great interest and, nowadays, an open topic in the area of quantum optics. In 1927, Dirac [20] postulated the existence of Hermitian, canonical number and phase variables in his description of the quantized electromagnetic field.

$$\hat{a} = e^{i\hat{\phi}}\sqrt{\hat{n}}, \tag{4.1}$$

$$\hat{a}^{\dagger} = \sqrt{\hat{n}}e^{-i\hat{\phi}}. \tag{4.2}$$

From the commutation relation (2.21) it follows that the number and phase operators obey the commutation relation

$$\left[\hat{n}, \hat{\phi}\right] = i. \tag{4.3}$$

According to Eq. (4.3) the number and phase operators are complementary observables and therefore the fluctuation in these quantities should satisfy the uncertainty relation defined by Eq. (2.40). Unfortunately, the definition given by Dirac has inconsistencies such as the following. Consider the matrix element of the commutator for the arbitrary number states $|n\rangle$ and $|m\rangle$

$$\left\langle n \left| \left[\hat{n}, \hat{\phi}\right] \right| m \right\rangle = i\delta_{nm}. \tag{4.4}$$

Expanding the left side results in

$$(n-m)\left\langle n\right| \hat{\phi} \left|m\right\rangle = i\delta_{nm}, \tag{4.5}$$

which contains a contradiction in the case when $n = m$. Dirac's approach fails because the associated phase operator is not Hermitian since $\hat{U}^{\dagger} = e^{i\hat{\phi}} = \hat{a}\frac{1}{\sqrt{\hat{n}}}$ is not unitary, this is, $\hat{U}\hat{n}\hat{U}^{\dagger} = \hat{n} + 1$.

In 1963, Louisell [21] suggested that the problem presented in Eq. (4.5) could be overcome by considering periodic functions of the Dirac phase operator, he introduced the periodic operator functions $\cos\left(\hat{\phi}\right)$ and $\sin\left(\hat{\phi}\right)$ that satisfy the commutation relations

$$\left[\cos\left(\hat{\phi}\right), \hat{n}\right] = i\sin\left(\hat{\phi}\right), \tag{4.6}$$

$$\left[\sin\left(\hat{\phi}\right), \hat{n}\right] = -i\cos\left(\hat{\phi}\right). \tag{4.7}$$

In 1964, Susskind and Glogower [11] presented a description of the phase using exponential phase operators in a polar decomposition of the creation and annihilation operators. The work presented by Susskind and Glogower is a very useful tool for discussing properties of coherent states [22], squeezed states [23] and optical amplification processes [24] despite their non-commuting and non-unitary nature. In this section we present the formalism behind the Susskind-Glogower operators as well as their non-commuting problem, that is important to overcome when we construct nonlinear coherent states.

## 4.1 Susskind-Glogower (SG) operators

As we have seen, Susskind and Glogower proposed operators that were as close as possible to Dirac's proposal. The resulting exponential phase operators are

$$\hat{V} = e^{\hat{i\phi}} = \sum_{n=0}^{\infty} |n\rangle \langle n+1| = \frac{1}{\sqrt{\hat{n}+1}} \hat{a}, \tag{4.8}$$

$$\hat{V}^{\dagger} = e^{\hat{-i\phi}} = \sum_{n=0}^{\infty} |n+1\rangle \langle n| = \hat{a}^{\dagger} \frac{1}{\sqrt{\hat{n}+1}}, \tag{4.9}$$

satisfying the conditions

$$\hat{V} |n\rangle = |n-1\rangle \,, \tag{4.10}$$

$$\hat{V}^{\dagger} |n\rangle = |n+1\rangle \,. \tag{4.11}$$

Due to Eq. (4.10) and Eq. (4.11) one can call **SG** operators, the "true" boson annihilation and creation operators, respectively.
Additionally, we would like to make explicit the result

$$\hat{V} |0\rangle = 0, \tag{4.12}$$

that comes naturally from Eq. (2.26).
We mentioned above that **SG** operators possess a non-commuting and non-unitary nature, this resides in the expressions

$$\hat{V}\hat{V}^{\dagger} = 1, \tag{4.13}$$

$$\hat{V}^{\dagger}\hat{V} = 1 - |0\rangle \langle 0| \,. \tag{4.14}$$

From Eq. (4.13) and Eq. (4.14) we can see that the non-commuting and non-unitary nature of **SG** operators is only apparent for states of the radiation field that have a significant overlap with the vacuum

$$\left\langle \psi \left| \left[ \hat{V}, \hat{V}^{\dagger} \right] \right| \psi \right\rangle = \langle \psi | 0 \rangle \langle 0 | \psi \rangle \,. \tag{4.15}$$

Therefore, for states where the vacuum contribution is negligible, we can consider them as unitary and commutative and we can perform the following approximation

$$\hat{V}^{-1} \simeq \hat{V}^{\dagger}. \tag{4.16}$$

The properties of the **SG** operators play an important role in the development of the present thesis project. For instance, if we analyze Eq. (4.8) and Eq. (4.9) we find that **SG** operators have the same form of the ones we need to construct nonlinear coherent states, i.e., $\hat{V} = f\left(\hat{n}+1\right)\hat{a}$ and $\hat{V}^{\dagger} = \hat{a}^{\dagger} f\left(\hat{n}+1\right)$ .

# Chapter 5

# Displacement operator and coherent states for the SG operators

In previous chapters we defined important properties of coherent states, their nonlinear counterpart and the phase operators. The following chapters correspond to the work done during the thesis, that involves most of the properties we have seen and brings out very interesting results.

We observe, from Eqs. (3.29 - 3.30) and Eqs. (4.8 - 4.9), that nonlinear coherent states may be constructed using the **SG** operators, for instance, defining a displacement operator for the **SG** operators acting on the vacuum state. Following Récamier *et al.* [10] we define **SG** coherent states[1] by

$$|\alpha\rangle_{SG} = e^{x\left(\hat{V}^{\dagger}-\hat{V}\right)}|0\rangle\,, x\in\mathbb{R} \tag{5.1}$$

[1]From this point we will refer to these states as **SG** coherent states keeping in mind that, in fact, they present a nonlinear behavior.

As we have seen, commutation relations for the **SG** operators are not simple so we cannot factorize the displacement operator in a simple way (3.15) which is why, in this chapter, we propose two methods to solve the displacement operator.

In section 5.1, we use the approximation (4.16) to solve, approximately, the displacement operator. This solution helps us to understand how the exact solution for the displacement operator should be. In section 5.2, we solve the displacement operator in an exact way by developing it in a Taylor series, that allows us to introduce the exact solution for nonlinear coherent states, constructed with the **SG** operators. In section 5.3, we analyze the constructed states via the $\mathcal{Q}$ function [26], the Photon number distribution (3.32) and the Mandel Q-parameter [28] in order to show their nonclassical features such as amplitude squeezing and quantum interferences.

## 5.1 Approximated displacement operator

A first approach to factorize the displacement operator in a product of exponentials, is to consider the approximation (4.16).

Let us write the displacement operator as

$$D_{SG} \simeq e^{x\left(\hat{V}^\dagger - \left[\hat{V}^\dagger\right]^{-1}\right)}. \tag{5.2}$$

We find that the right-hand side of this equation corresponds to the generating function of Bessel functions, that implies that

$$D_{SG} \simeq \sum_{n=-\infty}^{\infty} \hat{V}^{\dagger n} J_n\left(2x\right), \tag{5.3}$$

where $J_n$ is the Bessel function of the first kind and order $n$.

Applying the displacement operator on the vacuum state, we have

$$D_{SG}\left|0\right\rangle \simeq c_0 \sum_{n=-\infty}^{\infty} \hat{V}^{\dagger n} J_n\left(2x\right)\left|0\right\rangle , \tag{5.4}$$

and, using Eq. (4.11) and Eq. (4.16) we obtain

$$\left|\alpha\right\rangle_{SG} = D_{SG}\left|0\right\rangle \simeq c_0 \sum_{n=0}^{\infty} J_n\left(2x\right)\left|n\right\rangle . \tag{5.5}$$

From the normalization requirement we determine $c_0$

$$\begin{aligned} {}_{SG}\left\langle\alpha|\alpha\right\rangle_{SG} &= c_0^2 \sum_{n=0}^{\infty} J_n\left(2x\right)\left\langle n\right| \sum_{m=0}^{\infty} J_m\left(2x\right)\left|m\right\rangle , \\ &= c_0^2 \sum_{n=0}^{\infty} J_n^2\left(2x\right) , \\ &= 1. \end{aligned} \tag{5.6}$$

Using the result [25]

$$1 = J_0^2\left(2x\right) + 2\sum_{n=1}^{\infty} J_n^2\left(2x\right) , \tag{5.7}$$

we obtain

$$c_0 = \sqrt{\frac{2}{1+J_0^2\left(2x\right)}}. \tag{5.8}$$

Substituting the above expression in Eq. (5.5) we finally obtain

$$\left|\alpha\right\rangle_{SG} \simeq \sqrt{\frac{2}{1+J_0^2\left(2x\right)}} \sum_{n=0}^{\infty} J_n\left(2x\right)\left|n\right\rangle . \tag{5.9}$$

As we mentioned before, solution (5.9) helps us to foresee the exact solution for these states. We can expect that the exact solution for **SG** coherent states corresponds to a linear combination of number states where the coefficients are, except for some terms, Bessel functions of the first kind and order $n$.

## 5.2 Exact solution for the displacement operator

In order to "disentangle" the displacement operator in an exact way we can develop the exponential (5.1) in a Taylor series, and then to evaluate the terms $\left(\hat{V}^\dagger - \hat{V}\right)^k$. For instance, for $k=7$ we have

$$\begin{aligned}\left(\hat{V}^\dagger - \hat{V}\right)^7 &= :\left(\hat{V}^\dagger - \hat{V}\right)^7: + \binom{7}{2}\left(|1\rangle\langle 0| - |0\rangle\langle 1|\right), \\ &- \binom{7}{1}\left(|3\rangle\langle 0| - |2\rangle\langle 1| + |1\rangle\langle 2| - |0\rangle\langle 3|\right), \\ &+ \binom{7}{0}(|5\rangle\langle 0| - |4\rangle\langle 1| + |3\rangle\langle 2| - |2\rangle\langle 3|, \\ &+ |1\rangle\langle 4| - |0\rangle\langle 5|),\end{aligned} \tag{5.10}$$

where : : means to arrange terms in such a way that the powers of the operator $\hat{V}$ are always at the left of the powers of the operator $\hat{V}^\dagger$.

From the definition of the **SG** coherent states (5.1) we can write

$$\begin{aligned}|\alpha\rangle_{SG} &= e^{-x\hat{V}}e^{x\hat{V}^\dagger}|0\rangle + \sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{n=0}^{\left[\frac{k}{2}-1\right]}(-1)^n\binom{k}{n}|k-2n-2\rangle, \\ &= \hat{V}^2\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{n=0}^{\left[\frac{k}{2}-1\right]}(-1)^n\binom{k}{n}\hat{V}^{2n}\hat{V}^{\dagger k}|0\rangle,\end{aligned} \tag{5.11}$$

where the square brackets in the sum stand for the floor function [27].

We can rewrite the above equation as

$$|\alpha\rangle_{SG} = e^{-x\hat{V}}e^{x\hat{V}^\dagger}|0\rangle + \hat{V}^2\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{n=0}^{\infty}(-1)^n\binom{k}{n}\hat{V}^{2n}\hat{V}^{\dagger k}|0\rangle, \tag{5.12}$$

where we have taken the second sum to $\infty$ as we would add only zeros.

We now exchange the order of the sums

$$|\alpha\rangle_{SG} = e^{-x\hat{V}} e^{x\hat{V}^\dagger} |0\rangle + \hat{V}^2 \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} (-1)^n \frac{x^k}{(k-n)!n!} \hat{V}^{2n} \hat{V}^{\dagger k} |0\rangle . \tag{5.13}$$

By setting $m = k - n$ we write

$$\begin{aligned}
|\alpha\rangle_{SG} &= e^{-x\hat{V}} e^{x\hat{V}^\dagger} |0\rangle + \hat{V}^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n \frac{x^{m+n}}{m!n!} \hat{V}^{2n} \hat{V}^{\dagger m+n} |0\rangle , \\
&= e^{-x\hat{V}} e^{x\hat{V}^\dagger} |0\rangle + \hat{V}^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n \frac{x^{m+n}}{m!n!} \hat{V}^{n} \hat{V}^{\dagger m} |0\rangle , \\
&= e^{-x\hat{V}} e^{x\hat{V}^\dagger} |0\rangle + \hat{V}^2 \sum_{n=0}^{\infty} \frac{\left(-x\hat{V}\right)^n}{n!} \sum_{m=0}^{\infty} \frac{\left(x\hat{V}^\dagger\right)^m}{m!} |0\rangle , \\
&= e^{-x\hat{V}} e^{x\hat{V}^\dagger} |0\rangle + \hat{V}^2 e^{-x\hat{V}} e^{x\hat{V}^\dagger} |0\rangle , 
\end{aligned} \tag{5.14}$$

to finally obtain

$$|\alpha\rangle_{SG} = \left(1 + \hat{V}^2\right) e^{-x\hat{V}} e^{x\hat{V}^\dagger} |0\rangle . \tag{5.15}$$

Applying the exponential terms on the vacuum state, we have

$$\begin{aligned}
|\alpha\rangle_{SG} &= \left(1 + \hat{V}^2\right) \sum_{n=0}^{\infty} J_n(2x) |n\rangle , \\
&= \sum_{n=0}^{\infty} J_n(2x) |n\rangle + \sum_{n=2}^{\infty} J_n(2x) |n-2\rangle , 
\end{aligned} \tag{5.16}$$

making $m = n - 2$ in the second sum we write

$$\begin{aligned}
|\alpha\rangle_{SG} &= \sum_{n=0}^{\infty} J_n(2x) |n\rangle + \sum_{m=0}^{\infty} J_{m+2}(2x) |m\rangle , \\
&= \sum_{n=0}^{\infty} \left[J_n(2x) + J_{n+2}(2x)\right] |n\rangle , 
\end{aligned} \tag{5.17}$$

performing another index change by making $n = k - 1$ we obtain

$$|\alpha\rangle_{SG} = \sum_{k=1}^{\infty} \left[J_{k-1}(2x) + J_{k+1}(2x)\right] |k-1\rangle . \tag{5.18}$$

Using the recurrence relation [25]

$$xJ_{n-1}(x) + xJ_{n+1}(x) = 2nJ_n(x) , \tag{5.19}$$

we have that

$$|\alpha\rangle_{SG} = \frac{1}{x}\sum_{k=1}^{\infty} k J_k\left(2x\right)|k-1\rangle\,, \tag{5.20}$$

and by making $n = k - 1$ we finally write

$$|\alpha\rangle_{SG} = \frac{1}{x}\sum_{n=0}^{\infty}\left(n+1\right) J_{n+1}\left(2x\right)|n\rangle\,. \tag{5.21}$$

Eq. (5.21) is an important result for our thesis work because it constitutes a new expression for nonlinear coherent states. It remains for us to analyze the behavior of the constructed states in order to determine the nonclassical features that nonlinear coherent states may exhibit.

Before we proceed with the analysis and since we will need this result in the next chapter, we make clear that, as we can verify from Eq. (5.1), for $x = 0$, we have

$$|\alpha\left(x=0\right)\rangle_{SG} = |0\rangle\,. \tag{5.22}$$

## 5.3 SG coherent states analysis

There are different ways to find out if the state we are constructing resembles one that we already know. Here, we will use three different methods; the Husimi $\mathcal{Q}$ function [26], the Photon number distribution (3.32) and the Mandel Q-parameter [28].

### 5.3.1 $\mathcal{Q}$ function

$\mathcal{Q}$ function, introduced by Husimi [26], corresponds to a quasiprobability function that helps us to determine the behavior of a quantum state in phase space. The $\mathcal{Q}$ function is defined as the coherent state expectation value of the density operator and is given by

$$\mathcal{Q} = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle \, . \tag{5.23}$$

Using Eq. (3.31) we plot the $\mathcal{Q}$ function for a coherent state $|\alpha\rangle$ and for a number state $|n\rangle$ (Figure 5.1).

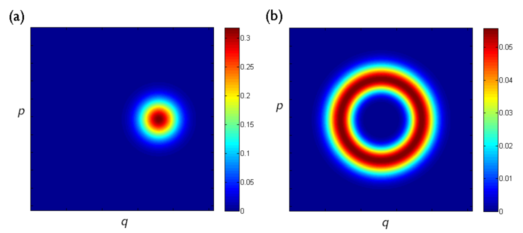


Figure 5.1: $\mathcal{Q}$ function for (a) coherent state $|\alpha\rangle$ with $\alpha = 2$ and (b) number state $|n\rangle$ with $n = 5$.

If we substitute Eq. (3.31) for the **SG** coherent states in Eq. (5.23) we obtain

$$\mathcal{Q}_{SG}(\alpha, t) = \frac{1}{\pi x^2} e^{-|\alpha|^2} \left| \sum_{n=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} (n+1) J_{n+1}(2x) \right|^2 \tag{5.24}$$

Figure 5.2 shows the SG coherent states $\mathcal{Q}$ function for different values of the parameter $x$.

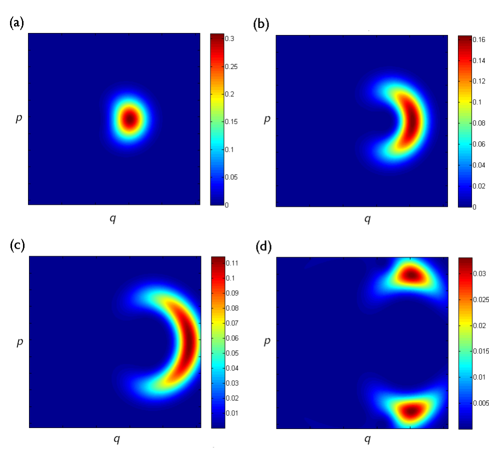


Figure 5.2: Exact **SG** coherent states $\mathcal{Q}$ function for (a) $x = 1$; (b) $x = 5$; (c) $x = 10$ and (d) $x = 20$.

From Figure 5.2, we observe that the initial coherent state squeezes. Later, we will find the value of $x$ for which we obtain the maximum squeezing of the coherent state. We can also see that, as the parameter $x$ gets bigger, the state splits into two coherent-like states, this produces quantum interferences, as we will see in the next section.

### 5.3.2 Photon number distribution

When one studies a quantum state, it is important to know about its photon statistics. The photon number probability distribution (3.32) is useful to determine amplitude squeezing. We should refer to amplitude squeezed light as light for which the photon number distribution is usually narrower than the one of a coherent state of the same amplitude. The photon number distribution is also useful to analyze if there exist effects due to quantum interferences. Using Eq. (3.32) we write the **SG** coherent states photon number distribution as

$$P\left(n\right)=\left|\frac{1}{x}\left(n+1\right)J_{n+1}\left(2x\right)\right|^{2}. \tag{5.25}$$

Figure 5.3 shows the **SG** coherent states photon number distribution for different values of the amplitude parameter $x$.

Figure 5.3 helps us to understand the effect of quantum interferences; for instance, consider Figure 5.3(c). We see that it is not a uniform distribution of photons, the distribution has “holes”, these holes are the consequence of the interference between the two states arising from the splitting of the initial one. Comparing Figure 5.3(a) and Figure 3.2(a) we can see that the photon number distribution for the **SG** coherent states is narrower than the one for a coherent state of the same amplitude, this suggests that we are, indeed, obtaining an amplitude squeezed state. It is interesting to know when the state is maxi-

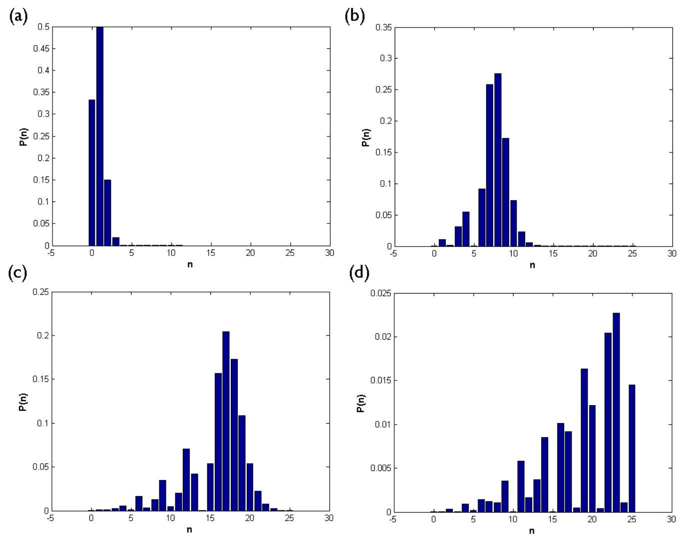

Figure 5.3: **SG** coherent states photon number probability distributions for (a) $x = 1$; (b) $x = 5$; (c) $x = 10$ and (d) $x = 20$.

mally squeezed but we need another tool to obtain the value of the parameter $x$ for which this occurs.

### 5.3.3 Mandel Q-parameter

There has been an extensive argument about which is the better way to determine the quantumness of a given state; nevertheless, there exists a very useful tool for determining the nature of the states we have constructed. This tool is called the Mandel Q-parameter [28]. We will use it not only because it represents a good parameter to define the quantumness of **SG** coherent states but also, because it will allow us to find the domain of $x$ for which they exhibit a nonclassical behavior; moreover, it will help us to find the value of $x$ for which the state is maximally squeezed.

Mandel Q-parameter is defined by

$$\mathrm{Q} = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle} - 1, \tag{5.26}$$

where

$$If \quad Q \begin{cases} > 0 & , super - Poissonian\ distribution \\ = 0 & , Poissonian\ distribution\ (coherent\ state) \\ < 0 & , sub - Poissonian \\ = -1 & , number\ state \end{cases} \tag{5.27}$$

For the **SG** coherent states we have that

$$\langle \hat{n} \rangle = \frac{1}{x^2} \left[ \sum_{k=0}^{\infty} k^3 J_k^2 (2x) - \sum_{k=0}^{\infty} k^2 J_k^2 (2x) \right], \tag{5.28}$$

$$\left\langle \hat{n}^2 \right\rangle = \frac{1}{x^2} \left[ \sum_{k=0}^{\infty} k^4 J_k^2 (2x) - 2 \sum_{k=0}^{\infty} k^3 J_k^2 (2x) + \sum_{k=0}^{\infty} k^2 J_k^2 (2x) \right]. \tag{5.29}$$

In order to obtain the values for the sums in Eq. (5.28) and Eq. (5.29) we make use of Parseval's theorem in the following way.
Consider the function

$$F(y) = e^{ixsin(y)} = \sum_{n=-\infty}^{\infty} J_n (x) e^{iny}, \tag{5.30}$$

taking its first derivative we have

$$F^{'}(y) = \sum_{n=-\infty}^{\infty} in J_n (x) e^{iny}, \tag{5.31}$$

and, by means of the Parseval's theorem, we finally obtain

$$\frac{1}{2\pi} \int_0^{2\pi} dy \left| F^{'}(y) \right|^2 = \sum_{n=-\infty}^{\infty} \left| in J_n (x) e^{iny} \right|^2 = \sum_{n=-\infty}^{\infty} n^2 J_n^2 (x). \tag{5.32}$$

For the next even powers we perform the second derivative of $F(y)$ and we use the Parseval's theorem in the same way as in Eq. (5.32).
As we need to obtain odd powers sums we have to perform a numerical fitting to an appropriate polynomial.

Using the method described above and numerical fitting we obtain the following results

$$\sum_{k=0}^{\infty} k^2 J_k^2(2x) = x^2, \tag{5.33}$$

$$\sum_{k=0}^{\infty} k^4 J_k^2(2x) = 3x^4 + x^2, \tag{5.34}$$

$$\sum_{k=0}^{\infty} k^3 J_k^2(2x) = 1.698x^3 + 0.0001x^2 + 0.1579x + 0.0042, \tag{5.35}$$

where in Eq. (5.35), after testing different options, we realized that a fitting to a third order polynomial rapidly converges.

Substituting the values of the sums into Eq. (5.26) we obtain the plot shown in Figure 5.4.

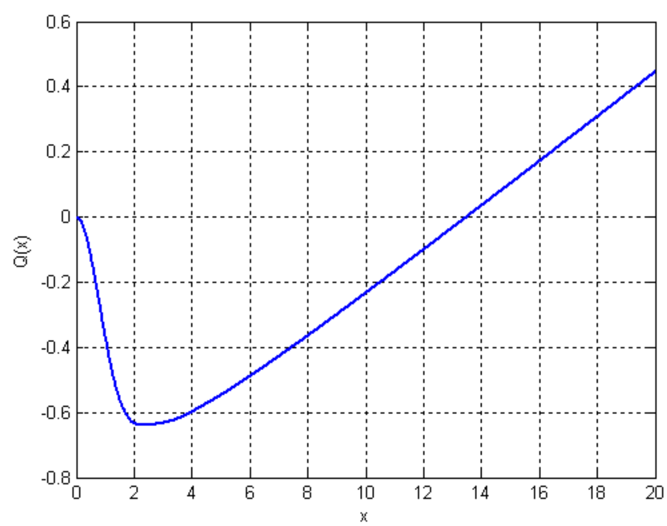


Figure 5.4: Mandel Q-parameter for the **SG** coherent states.

From Figure 5.4 we can see that, depending on the parameter $x$, the photon distribution of the constructed states is sub-poissonian, $Q < 0$, meaning that amplitude squeezing states may be find for a value of $x$ within the domain $0 < x \leq 13.5$. Also, we find that the most squeezed state may be obtained at

$x = 2.32$.

At this point, all the comments on the results have been made considering $x$ as a parameter; however, in the next chapter, we will see that $x$ corresponds to an expression that depends explicitly on what we should refer to as interaction time.

# Chapter 6

# Eigenfunctions of the SG Hamiltonian

In the previous chapter we managed to construct nonlinear coherent states applying the displacement operator on the vacuum state. However, even when the obtained results are very interesting, we cannot avoid to wonder how these states could be physically interpreted. Physical interpretations are given by operators representing observables, this is, quantities that can be measured in the lab. The most important observable is the Hamiltonian, this operator helps us to find the energy distribution of an state via its eigenvalues.

Here, we construct **SG** coherent states as eigenfunctions of a Hamiltonian that we propose and which represents the fundamental coupling to the radiation field via the **SG** operators. In section 6.1 we construct time-dependent **SG** coherent states with the vacuum state $|0\rangle$ as initial condition, this is, we construct states that satisfy Eq. (5.22). In section 6.2 we generalize the eigenfunction problem for an arbitrary $|m\rangle$ initial condition, we also show that previous results correspond to the particular case $m = 0$ and in section 6.3 we make use of the three methods presented in the previous chapter in order to analyze the properties and time-evolution of the constructed states.

## 6.1 Solution for $|0\rangle$ as initial condition

As we mentioned before, it is possible to construct **SG** coherent states as eigenfunctions of the interaction Hamiltonian

$$\hat{H} = \eta \left( \hat{V} + \hat{V}^{\dagger} \right), \tag{6.1}$$

where $\eta$ is the coupling coefficient.

Hamiltonian proposed in (6.1) corresponds to a variation of the Hamiltonian used in [29] to model physical couplings to the radiation field. Here, physical couplings take place via the **SG** operators.

To obtain the eigenfunctions of the Hamiltonian (6.1) let us use the time-independent Schrödinger equation, this because we already know that the time solution corresponds to an exponential function of the form $e^{-iEt}$, where $E$ is the eigenvalue of $\hat{H}$ and where we have taken $\hbar = 1$

$$\hat{H} \left|\psi\right\rangle = E \left|\psi\right\rangle, \tag{6.2}$$

where $|\psi\rangle$ is the state vector in the interaction picture and, in general, it may be written as

$$\left|\psi\right\rangle = \sum_{n=0}^{\infty} C_n \left|n\right\rangle. \tag{6.3}$$

Substituting Eq. (6.1) and Eq. (6.3) in Eq. (6.2) we have

$$\begin{aligned}
\hat{H}\left|\psi\right\rangle &= \eta\left(V+V^{\dagger}\right)\left|\psi\right\rangle, \\
&= \eta\left(V+V^{\dagger}\right)\sum_{n=0}^{\infty}C_n\left|n\right\rangle, \\
&= \eta\sum_{n=0}^{\infty}C_nV\left|n\right\rangle+\eta\sum_{n=0}^{\infty}C_nV^{\dagger}\left|n\right\rangle, \\
&= \eta\sum_{n=1}^{\infty}C_n\left|n-1\right\rangle+\eta\sum_{n=0}^{\infty}C_n\left|n+1\right\rangle. \qquad (6.4)
\end{aligned}$$

Now, performing an index change of the coefficients in the following way

$$\begin{aligned}
\text{first sum: } n &\rightarrow n+1, \\
\text{second sum: } n &\rightarrow n-1,
\end{aligned}$$

we obtain

$$\begin{aligned}
\hat{H}\left|\psi\right\rangle &= \eta\sum_{n=0}^{\infty}C_{n+1}\left|n\right\rangle+\eta\sum_{n=1}^{\infty}C_{n-1}\left|n\right\rangle, \\
&= \eta C_1\left|0\right\rangle+\eta\sum_{n=1}^{\infty}C_{n+1}\left|n\right\rangle+\eta\sum_{n=1}^{\infty}C_{n-1}\left|n\right\rangle, \\
&= \eta C_1\left|0\right\rangle+\eta\sum_{n=1}^{\infty}\left(C_{n+1}+C_{n-1}\right)\left|n\right\rangle. \qquad (6.5)
\end{aligned}$$

Equating Eq. (6.5) with Eq. (6.2) we obtain

$$\eta C_1\left|0\right\rangle+\sum_{n=1}^{\infty}\eta\left(C_{n+1}+C_{n-1}\right)\left|n\right\rangle = EC_0\left|0\right\rangle+\sum_{n=1}^{\infty}EC_n.\left|n\right\rangle \qquad (6.6)$$

Comparing coefficients with same number states of the sum we have

$$C_1=\frac{E}{\eta}C_0, \qquad (6.7)$$

$$\eta\left(C_{n+1}+C_{n-1}\right)=EC_n. \qquad (6.8)$$

Eq. (6.7) and Eq. (6.8) remember us to the recurrence relation of the Chebyshev polynomials of the second kind given by [25]

$$\begin{aligned} U_0(\xi) &= 1, \\ U_1(\xi) &= 2\xi, \\ U_{n+1}(\xi) &= 2\xi U_n(\xi) - U_{n-1}(\xi). \end{aligned} \tag{6.9}$$

We see that Eq. (6.7) and Eq. (6.8) satisfy the recurrence relation (6.9), which implies that

$$|\psi(t;\xi)\rangle = \sum_{n=0}^{\infty} e^{-iEt} U_n(\xi) |n\rangle, \tag{6.10}$$

where

$$E = 2\eta\xi. \tag{6.11}$$

From Eq. (6.10) we see that it does not satisfy Eq. (5.2), this is, for $t = 0$ it does not collapse to the vacuum state $|0\rangle$. Moreover, the solution (6.10) has the parameter $\xi$ and, as we see from (5.1), we should not have another parameter, except for the time $t$.

A way to construct a solution as the one we previously obtained (5.21) is by looking at the exponential term in Eq. (6.10), we notice that it has the form of the Fourier transform kernel, so we need to propose a $\xi$-dependent function and integrate it over all $\xi$, in order to obtain a solution where time is the only variable

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\xi P(\xi) U_n(\xi) e^{-i2\eta\xi t} |n\rangle. \tag{6.12}$$

We see from Eq. (6.12) that $|\psi(t)\rangle$ corresponds to a sum of Fourier transforms of Chebyshev polynomials with respect to a weight function $P(\xi)$. This kind of Fourier transforms may be solved by using the following result [30]

$$\mathscr{F}\left\{\frac{J_n(\omega)}{\omega}\right\} = \sqrt{\frac{2}{\pi}} \frac{i}{n} (-i)^n U_{n-1}(\xi) \sqrt{1-\xi^2} rect\left(\frac{\xi}{2}\right), \tag{6.13}$$

where $\mathscr{F}\{\}$ is the Fourier transform and

$$rect\left(\frac{\xi}{2}\right) = \begin{cases} 1 & , -1 \leq \xi \leq 1 \\ \\ 0 & , \quad otherwise. \end{cases} \tag{6.14}$$

From (6.13) we write

$$\frac{J_n(\omega)}{\omega} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{i}{n} (-i)^n U_{n-1}(\xi) \sqrt{1-\xi^2} rect\left(\frac{\xi}{2}\right) e^{i\omega\xi} d\xi. \tag{6.15}$$

Using the definition (6.14) we have

$$\frac{J_n(\omega)}{\omega} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sqrt{\frac{2}{\pi}} \frac{i}{n} (-i)^n U_{n-1}(\xi) \sqrt{1-\xi^2} e^{i\omega\xi} d\xi. \tag{6.16}$$

Making $k = n-1$, we finally obtain

$$\frac{J_{k+1}(\omega)}{\omega} = \int_{-1}^{1} \left[\frac{1}{\pi} \frac{i}{k+1} (-i)^{k+1} \sqrt{1-\xi^2}\right] U_k(\xi) e^{i\omega\xi} d\xi. \tag{6.17}$$

Now that we have obtained Eq. (6.17) it is possible to solve the integral in Eq. (6.12).

Writing the weight function as

$$P(\xi) = \frac{2}{\pi} \sqrt{1-\xi^2} rect\left(\frac{\xi}{2}\right), \tag{6.18}$$

and substituting it into Eq. (6.12), we obtain

$$\begin{aligned} |\psi(t)\rangle &= \sum_{n=0}^{\infty} \int_{-1}^{1} \frac{\frac{1}{\pi}\frac{i}{n+1}(-i)^{n+1}}{\frac{1}{\pi}\frac{i}{n+1}(-i)^{n+1}} \frac{2}{\pi} \sqrt{1-\xi^2} U_n(\xi) e^{i(-2\eta t)\xi} d\xi \, |n\rangle, \\ &= \sum_{n=0}^{\infty} \frac{2}{\pi} \frac{\pi(n+1)}{i(-i)^{n+1}} \int_{-1}^{1} \left[\frac{1}{\pi} \frac{i}{n+1} (-i)^{n+1} \sqrt{1-\xi^2}\right] U_n(\xi) e^{i(-2\eta t)\xi} d\xi \, |n\rangle, \end{aligned}$$

Using Eq. (6.17) we write

$$|\psi(t)\rangle = \frac{2}{-2\eta t} \sum_{n=0}^{\infty} i^n (n+1) J_{n+1}(-2\eta t) |n\rangle. \tag{6.19}$$

Considering the odd parity of the Bessel functions, we finally obtain

$$|\psi(t)\rangle = \frac{1}{\eta t}\sum_{n=0}^{\infty} i^n (n+1) J_{n+1}(2\eta t)|n\rangle . \tag{6.20}$$

We see that Eq. (6.20) depends only on $t$ and, for $t = 0$, considering that

$$\lim_{2\eta t\to 0} \frac{J_n(2\eta t)}{2\eta t} = \begin{cases} 0 & , \; n = 2,3,4,... \\ \\ \frac{1}{2} & , \quad n = 1, \end{cases} \tag{6.21}$$

we can verify that

$$|\psi(t=0)\rangle = |0\rangle . \tag{6.22}$$

Eq. (6.20) corresponds to the expression for **SG** coherent states that we obtained in the previous chapter.

The solution presented in this section allows us to notice that, while in the previous chapter $x$ was only a parameter, now it represents something physical. It may be related to an interaction time, for example, in the motion of a trapped atom [29]. We have managed to construct the same expression for the **SG** coherent states as the one obtained by the application of the displacement operator on the vacuum state; however, we will see that the formalism presented in this section may be used to generalize the solution for an arbitrary initial condition $|m\rangle$.

## 6.2 Solution for $|m\rangle$ as initial condition

At this point, we have managed to construct **SG** coherent states, first as those obtained by the application of the displacement operator on the vacuum state and later, as eigenfunctions of the Hamiltonian (6.1); however, they are a particular case of a more general expression.

Using the recurrence relation for the Chebyshev polynomials (6.9) and the result (6.17), it is possible to generalize **SG** coherent states to an arbitrary $|m\rangle$ initial condition, where $m = 0, 1, 2, ....$

From Eq. (6.12) we have that

$$|\psi(t=0)\rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\xi P(\xi) U_n(\xi) |n\rangle . \tag{6.23}$$

We know that Chebyshev polynomials of the second kind are orthogonal with respect to the weight $\sqrt{1-x^2}$ on the interval $[-1, 1]$, i.e. ,

$$\frac{2}{\pi} \int_{-1}^{1} U_n(\xi) U_m(\xi) \sqrt{1-\xi^2} d\xi = \delta_{nm}. \tag{6.24}$$

Using Eq. (6.24) we are able to define $P(\xi)$ in order to obtain $|\psi\rangle$ from different initial conditions.

Consider the function

$$P_m(\xi) = \frac{2}{\pi} \sqrt{1-\xi^2} U_m(\xi) \, rect\left(\frac{\xi}{2}\right). \tag{6.25}$$

Substituting Eq. (6.25) in Eq. (6.23) we obtain

$$|\psi(t=0)\rangle = \sum_{n=0}^{\infty} \delta_{nm} |n\rangle . \tag{6.26}$$

To obtain the solution for the $m$ - state let us consider the particular cases $m = 0$ and $m = 1$.

- *For* $m = 0$ *...*

$$\begin{aligned}
|\psi(t)\rangle_{m=0} &= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} P_0(\xi) U_n(\xi) e^{-i2\eta\xi t} d\xi |n\rangle , \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{2}{\pi} \sqrt{1-\xi^2} U_0(\xi) U_n(\xi) e^{-i2\eta\xi t} rect\left(\frac{\xi}{2}\right) d\xi |n\rangle , \\
&= \sum_{n=0}^{\infty} \int_{-1}^{1} \frac{2}{\pi} \sqrt{1-\xi^2} U_0(\xi) U_n(\xi) e^{-i2\eta\xi t} d\xi |n\rangle , \\
&= \sum_{n=0}^{\infty} \int_{-1}^{1} \frac{2}{\pi} \sqrt{1-\xi^2} U_n(\xi) e^{-i2\eta\xi t} d\xi |n\rangle .
\end{aligned}$$

Using Eq. (6.16) and making $n = n + 1$ we have

$$\left|\psi\left(t\right)\right\rangle_{m=0} = \sum_{n=0}^{\infty} i^{n}\left(n+1\right)\frac{J_{n+1}\left(2\eta t\right)}{\eta t}\left|n\right\rangle . \tag{6.27}$$

Using the recurrence relation

$$xJ_{n-1}\left(2x\right) + xJ_{n+1}\left(2x\right) = nJ_{n}\left(2x\right), \tag{6.28}$$

and performing the index change $n = n + 1$, we finally obtain

$$\left|\psi\left(t\right)\right\rangle_{m=0} = \sum_{n=0}^{\infty}\left[i^{n}J_{n}\left(2\eta t\right) + i^{n}J_{n+2}\left(2\eta t\right)\right]\left|n\right\rangle . \tag{6.29}$$

- *For* $m = 1$ ...

$$\begin{aligned}
\left|\psi\left(t\right)\right\rangle_{m=1} &= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}P_{1}\left(\xi\right)U_{n}\left(\xi\right)e^{-i2\eta\xi t}d\xi\left|n\right\rangle , \\
&= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{2}{\pi}\sqrt{1-\xi^{2}}U_{1}\left(\xi\right)U_{n}\left(\xi\right)e^{-i2\eta\xi t}rect\left(\frac{\xi}{2}\right)d\xi\left|n\right\rangle , \\
&= \sum_{n=0}^{\infty}\int_{-1}^{1}\frac{2}{\pi}\sqrt{1-\xi^{2}}U_{1}\left(\xi\right)U_{n}\left(\xi\right)e^{-i2\eta\xi t}d\xi\left|n\right\rangle , \\
&= \sum_{n=0}^{\infty}\int_{-1}^{1}\frac{2}{\pi}\sqrt{1-\xi^{2}}\left(2\xi\right)U_{n}\left(\xi\right)e^{-i2\eta\xi t}d\xi\left|n\right\rangle .
\end{aligned}$$

Using Eq. (6.9) we write

$$\left|\psi\left(t\right)\right\rangle_{m=1} = \sum_{n=0}^{\infty}\int_{-1}^{1}\frac{2}{\pi}\sqrt{1-\xi^{2}}\left(U_{n-1}\left(\xi\right)+U_{n+1}\left(\xi\right)\right)e^{-i2\eta\xi t}d\xi\left|n\right\rangle . \tag{6.30}$$

Considering Eq. (6.16) for the first integral and making $n = n + 2$ for the second one, we have

$$\left|\psi\left(t\right)\right\rangle_{m=1} = \sum_{n=0}^{\infty}\frac{1}{\eta t}\left[i^{n+1}\left(n+2\right)J_{n+2}\left(2\eta t\right) + i^{n-1}nJ_{n}\left(2\eta t\right)\right]\left|n\right\rangle . \tag{6.31}$$

We now construct a different expression for (6.31).

Let us write (6.31) in the following way

$$\left|\psi\left(t\right)\right\rangle_{m=1} = \sum_{n=0}^{\infty}\left\{-i^{n+2-1}\frac{1}{\eta t}nJ_{n}\left(2\eta t\right) + i^{n+2-1}\frac{1}{\eta t}\left(n+2\right)J_{n+2}\left(2\eta t\right)\right\}, \tag{6.32}$$

making $k = n + 2$

$$|\psi(t)\rangle_{m=1} = \sum_{n=0}^{\infty} \left\{ -i^{k-1} \frac{1}{\eta t} (k-2) J_{k-2}(2\eta t) + i^{k-1} \left[ \frac{1}{\eta t} k J_k (2\eta t) \right] \right\}, \quad (6.33)$$

using the recurrence relation (6.28) we write

$$\begin{aligned}
|\psi(t)\rangle_{m=1} &= \sum_{n=0}^{\infty} \left\{ -i^{k-1} \frac{1}{\eta t} (k-2) J_{k-2}(2\eta t) + i^{k-1} \left[ J_{k-1}(2\eta t) + J_{k+1}(2\eta t) \right] \right\}, \\
&= \sum_{n=0}^{\infty} \left\{ -i^{k-1} \left[ \frac{1}{\eta t} (k-2) J_{k-2}(2\eta t) - J_{k-1}(2\eta t) \right] + i^{k-1} J_{k+1}(2\eta t) \right\}, \\
&= \sum_{n=0}^{\infty} \left\{ i^{k-3} \left[ \frac{1}{\eta t} (k-2) J_{k-2}(2\eta t) - J_{k-1}(2\eta t) \right] + i^{k-1} J_{k+1}(2\eta t) \right\}, \\
&= \sum_{n=0}^{\infty} \left\{ i^{k-2-1} \left[ \frac{1}{\eta t} (k-2) J_{k-2}(2\eta t) - J_{k-2+1}(2\eta t) \right] + i^{k-2+1} J_{k-2+3}(2\eta t) \right\}.
\end{aligned}$$

Making $n = k - 2$ we obtain

$$|\psi(t)\rangle_{m=1} = \sum_{n=0}^{\infty} \left\{ i^{n-1} \left[ \frac{1}{\eta t} n J_n(2\eta t) - J_{n+1}(2\eta t) \right] + i^{n+1} J_{n+3}(2\eta t) \right\}. \quad (6.34)$$

Using the recurrence relation (6.28) we finally write

$$|\psi(t)\rangle_{m=1} = \sum_{n=0}^{\infty} \left[ i^{n-1} J_{n-1}(2\eta t) + i^{n+1} J_{n+3}(2\eta t) \right] |n\rangle. \quad (6.35)$$

Following the same procedure it is easy to prove that for $m = 2$

$$|\psi(t)\rangle_{m=2} = \sum_{n=0}^{\infty} \left[ i^{n-2} J_{n-2}(2\eta t) + i^{n+2} J_{n+4}(2\eta t) \right] |n\rangle. \quad (6.36)$$

Rewriting results (6.29), (6.35) and (6.36)

$$
\begin{aligned}
|\psi(t)\rangle_{m=0} &= \sum_{n=0}^{\infty}\left[i^{n-0}J_{n-0}(2\eta t)+i^{n+0}J_{n+0+2}(2\eta t)\right]|n\rangle\,,\\
|\psi(t)\rangle_{m=1} &= \sum_{n=0}^{\infty}\left[i^{n-1}J_{n-1}(2\eta t)+i^{n+1}J_{n+1+2}(2\eta t)\right]|n\rangle\,,\\
|\psi(t)\rangle_{m=2} &= \sum_{n=0}^{\infty}\left[i^{n-2}J_{n-2}(2\eta t)+i^{n+2}J_{n+2+2}(2\eta t)\right]|n\rangle\,,\\
&.\\
&.\\
&.
\end{aligned}
$$

it is easy to see that the solution for the m-initial condition is

$$
|\psi(t)\rangle_{m} = \sum_{n=0}^{\infty}\left[i^{n-m}J_{n-m}(2\eta t)+i^{n+m}J_{n+m+2}(2\eta t)\right]|n\rangle\,. \tag{6.37}
$$

Eq. (6.37) represents the final result of this thesis. We have constructed a new expression for nonlinear coherent states and that we call **SG** coherent states. We have managed to construct an expression that allows us to study the time evolution of **SG** coherent states for an arbitrary $|m\rangle$ initial condition. We also found the physical interpretation of the parameter $x$ (used in the previous chapter) as a normalized interaction time with respect to the coupling strength, i.e., $x = \eta t$. Additionally, as a future work, we may experimentally study **SG** coherent states in the motion of a trapped atom via the proposed Hamiltonian (6.1) and by following the procedure presented in [29].

As we mentioned in the previous chapter, we need to study the nonclassical features of the constructed states, so we have to make use of the methods previously mentioned, these are the $\mathcal{Q}$ function, the Photon number distribution and the Mandel Q-parameter.

## 6.3 Time-dependent SG coherent states analysis

To perform a complete description of the constructed states (6.37), we have to verify if they present the nonclassical features that nonlinear coherent states may exhibit. In order to study these nonclassical features, we propose to use three methods. First, we analyze their behavior in phase space via the $\mathcal{Q}$ function; then, because we want to analyze amplitude squeezing and quantum interferences, we show the Photon number distribution of the constructed states and, finally, as we want to know when the constructed states are maximally squeezed, we show the Mandel Q-parameter.

### 6.3.1 $\mathcal{Q}$ function

Considering Eq. (5.23) and writing $\hat{\rho}$ in terms of the constructed states (6.37) we write the $\mathcal{Q}$ function for the time-dependent **SG** coherent states as

$$\mathcal{Q}_{SG}\left(\alpha\right)=\frac{1}{\pi}e^{-|\alpha|}\left|\sum_{n=0}^{\infty}\frac{\alpha^{*^{n}}}{\sqrt{n!}}\left[i^{n-m}J_{n-m}\left(2\eta t\right)+i^{n+m}J_{n+m+2}\left(2\eta t\right)\right]\right|^{2} \tag{6.38}$$

Figure 6.1 shows the time-evolved $\mathcal{Q}$ function of **SG** coherent states for different initial conditions. The figure is structured as follows. The time evolution is shown in each row, for example, (a) represents the time evolution of **SG** coherent states for $|0\rangle$ as initial condition and the index (i) represents the normalized interaction time $\eta t$. We have chosen (ii) to be $\eta t = 2.32$ because, as we showed in the previous chapter, it is the time when the state with initial condition $|0\rangle$ is maximally squeezed.

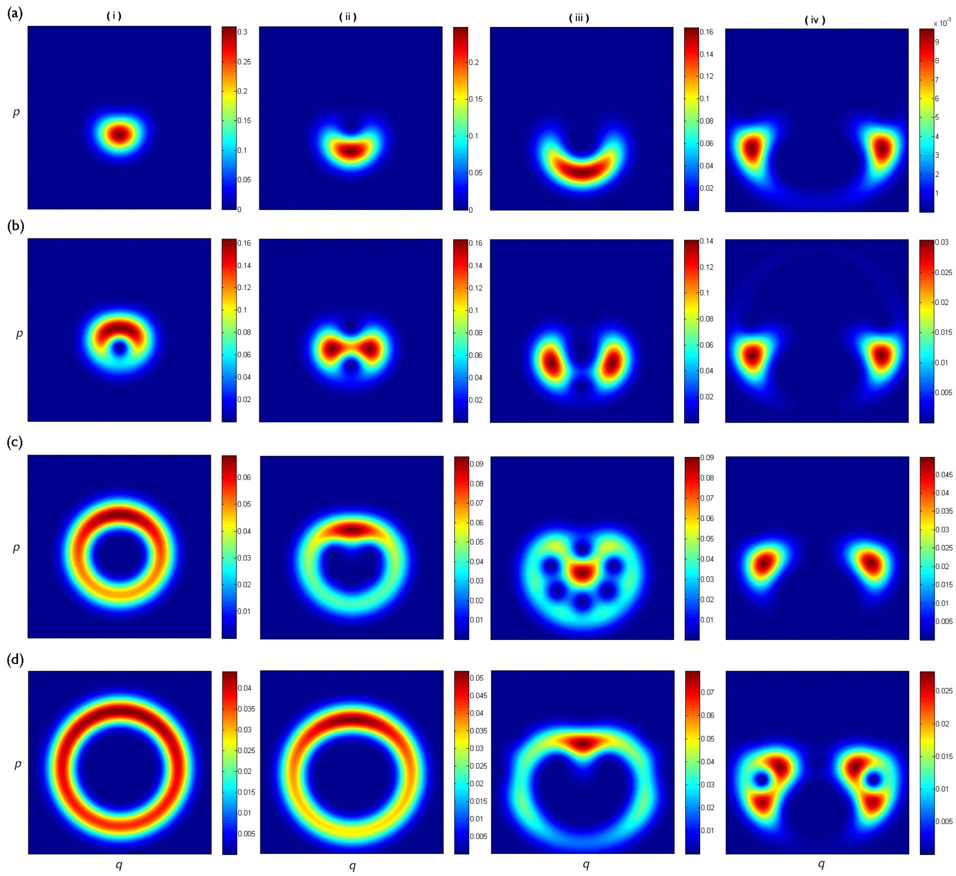


Figure 6.1: **SG** coherent states $\mathcal{Q}$ function for (i) $\eta t = 1$; (ii) $\eta t = 2.32$; (iii) $\eta t = 5$ and (iv) $\eta t = 20$, with initial conditions (a) $|0\rangle$; (b) $|1\rangle$; (c) $|5\rangle$ and (d) $|10\rangle$.

The nonclassical features of the constructed states are summarized in Figure 6.1. **SG** coherent states present a strong amplitude squeezing (Figure 6.1 (a,ii)), splitting into two coherent-like states (Figure 6.1 (b,iii)) and, as a consequence of the splitting, pronounced quantum interferences (Figure 6.1 (c,iii)).

An interesting result is that, no matter what initial condition we choose, **SG** coherent states, eventually, split into two coherent-like states. This is very interesting result because such a distribution corresponds to states called Schrödinger's cat states, which are very useful in quantum information processing [31].

### 6.3.2 Photon number distribution

To complement the description of the nonclassical features that we observed from the $\mathcal{Q}$ function, we show the photon number distribution of the **SG** coherent states considering the same conditions of Figure 6.1.

Time-dependent **SG** coherent states photon number distribution for the $m$-initial condition is given by

$$P_m(n,t) = \left| i^{n-m} J_{n-m}(2\eta t) + i^{n+m} J_{n+m+2}(2\eta t) \right|^2 \tag{6.39}$$

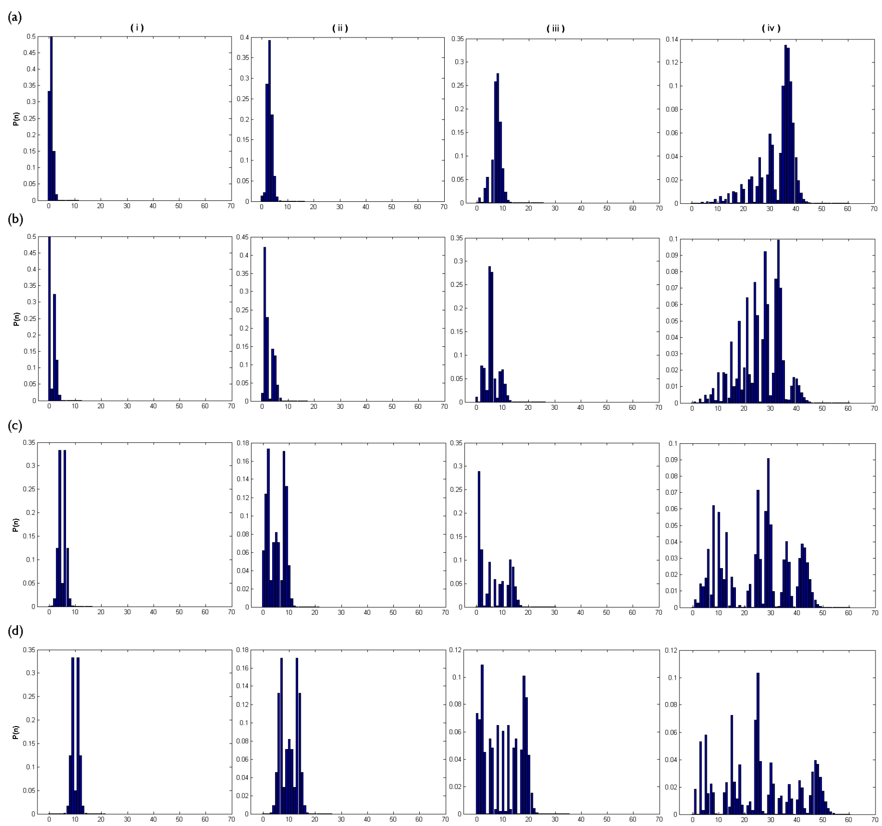

Figure 6.2: **SG** coherent states photon number probability distributions for (i) $\eta t = 1$; (ii) $\eta t = 2.32$; (iii) $\eta t = 5$ and (iv) $\eta t = 20$, with initial conditions (a) $|0\rangle$; (b) $|1\rangle$; (c) $|5\rangle$ and (d) $|10\rangle$.

### 6.3.3 Mandel Q-parameter

As we want to know the domain of time for which the constructed states exhibit amplitude squeezing and; moreover, we want to know when the states are maximally squeezed, we obtain the Mandel Q-parameter for the time-dependent **SG** coherent states.

We have that

$$\langle \hat{n} \rangle = \sum_{n=0}^{\infty} n \left[ J_{n-m}\left(2\eta t\right) + \left(-1\right)^{m} J_{n+m+2}\left(2\eta t\right) \right]^{2}, \tag{6.40}$$

$$\left\langle \hat{n}^{2} \right\rangle = \sum_{n=0}^{\infty} n^{2} \left[ J_{n-m}\left(2\eta t\right) + \left(-1\right)^{m} J_{n+m+2}\left(2\eta t\right) \right]^{2}. \tag{6.41}$$

Substituting in Eq. (5.26) we obtain the plot shown in Figure 6.3 From Figure

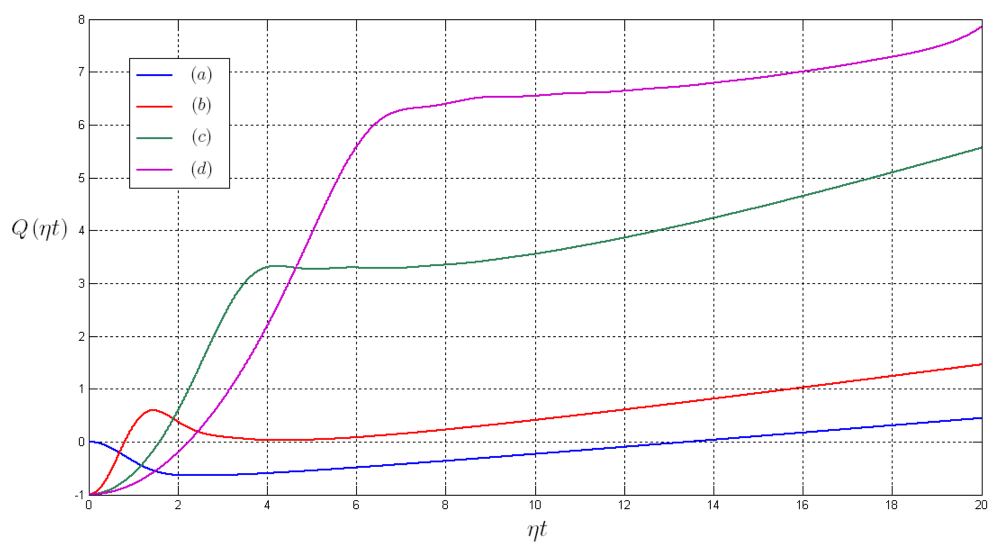


Figure 6.3: **SG** states Mandel Q-parameter with initial conditions (a) $|0\rangle$; (b) $|1\rangle$; (c) $|5\rangle$ and (d) $|10\rangle$.

6.3 we see that (a) shows the Q-parameter for the **SG** coherent states where the maximum squeezing happens when $\eta t = 2.32$. From the others, we see that we cannot observe squeezing because the initial condition is a full squeezed state, this is, a number state; however, we obtain that an initial number state eventually transforms into two coherent-like states that we may identify as Schrödinger's cat states and, due to the splitting, we observe quantum interferences.

The generalization (6.37) does not give us different effects from the ones that we may obtain from (6.20); however, as we will see in the next section, Eq. (6.37) helps us to show that nonlinear coherent states may be modelled by propagating light in semi-infinite arrays of optical fibers.

## 6.4 Classical quantum analogies

The modelling of quantum mechanical systems with classical optics is a topic that has attracted interest recently. Along these lines Man'ko *et al.* [32] have proposed to realize quantum computation by quantum like systems and Crasser *et al.* [33] have pointed out the similarities between quantum mechanics and Fresnel optics in phase space. Following these cross-applications, here we show that nonlinear coherent states may be modelled by propagating light in semi-infinite arrays of optical fibers.

Makris *et al.* [34] have shown that for a semi-infinite array of optical fibers, the normalized modal amplitude in the $n$th optical fiber (after the $m$th has been initially excited) is written as

$$a_n(Z) = A_0\left[i^{n-m}J_{n-m}(2Z) + i^{n+m}J_{n+m+2}(2Z)\right], \tag{6.42}$$

where $Z = cz$ is the normalized propagation distance with respect to the coupling coefficient $c$.

We see that, for $A_0 = 1$ the normalized intensity distribution is

$$I_n(Z) = \left|i^{n-m}J_{n-m}(2Z) + i^{n+m}J_{n+m+2}(2Z)\right|^2. \tag{6.43}$$

Rewriting Eq. (6.39) and using the normalized interaction time with respect to the coupling coefficient $\eta$, i.e., $x = \eta t$ we have

$$P_m\left(n, x\right) = \left| i^{n-m} J_{n-m}\left(2x\right) + i^{n+m} J_{n+m+2}\left(2x\right) \right|^2 . \tag{6.44}$$

We have that Eq. (6.43) and Eq. (6.44) are the same. This result allows us to conclude that the photon number distribution for the **SG** coherent states may be modelled by the intensity distribution of propagating light in semi-infinite arrays of optical fibers, this is, we have found a new relation between quantum mechanical systems and classical optics.

# Chapter 7

# Conclusions

The results of this thesis are presented in two parts. In the first part (chapter 5), we showed the construction of nonlinear coherent states via the **SG** operators. It was possible to propose a construction of such states because the mathematical definition fitted exactly in the way Man'ko [8] and, later, Récamier and collaborators [10] proposed. Following Récamier it was possible to define a displacement operator for the **SG** operators. We managed to construct a new expression for nonlinear coherent states, that we called **SG** coherent states, by developing the displacement operator in a Taylor series. The importance of nonlinear coherent states resides in the nonclassical features that they may exhibit and, in order to analyze the nonclassical behavior of the constructed states, we presented three methods: the $\mathcal{Q}$ function, the Photon number distribution and the Mandel Q-parameter.
The expression for the **SG** coherent states depends on a parameter $x$ and, considering the values of $x$, we found that

- For a certain domain of $x$, **SG** coherent states exhibit amplitude squeezing, this is, their photon number distribution is narrower than the one of a coherent state of the same amplitude. Using the Mandel Q-parameter we determined that maximum squeezing occurs when $x = 2.32$.

- For larger values of $x$, **SG** coherent states split into two coherent-like states, this superposition of coherent states is called Schrödinger's cat states and are very useful in quantum information processing [31]. We also found that the splitting of the states gives rise to quantum interference effects.

For the second part (chapter 6) of this project, we constructed **SG** coherent states as eigenfunctions of a Hamiltonian for the **SG** operators representing physical couplings to the radiation field. We managed to obtain the solution as the one obtained before except for a phase term that rotates by $\pi/2$ the $\mathcal{Q}$ function but it does not change the results that we have already found. The obtained expression for **SG** coherent states helped us to physically interpret the parameter $x$ as a normalized interaction time with respect to the coupling coefficient $\eta$.

The formalism presented in this part allowed us to construct a solution for an arbitrary initial condition $|m\rangle$. Eq. (6.37) is the most important result of our thesis because it represents a new expression for nonlinear coherent states; moreover, it represents a general expression for an arbitrary $|m\rangle$ initial condition where the particular case $m = 0$ corresponds to the solution obtained in Chapter 5.

Analyzing Eq.(6.37), we found that a multiple splitting of the state occurs, this gives rise to interesting structures of the phase-space distribution as the one shown in Figure 6.1(c,iii). Except for that result, the generalized solution presented the same effects as the particular case $m = 0$; however, it turns out to be very useful to establish an interesting relation between quantum and classical optics. We showed that **SG** coherent states photon number distribution may be modelled by the intensity distribution of propagating light in semi-infinite arrays of optical fibers. With this finding, we have presented a new analogy between quantum mechanical systems and classical optics.

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