GENERATION OF NOON STATES IN TRAPPED IONS AND CAVITIES

by

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Abstract

In the first part of this thesis, by using the two-photon Jaynes - Cummings Model (JCM), we show how NOON states may be generated entangling two cavities, by passing atoms through them. The atoms interact with each cavity via two-photon resonant transitions. We take advantage of the fact that depending on the state the atom enters (excited or ground), it leaves or takes two photons per interaction and leaves the cavities in a pure state.

In the second part of this thesis, we consider the Hamiltonian of a single ion vibrating in two dimensions interacting with a laser, we show how NOON states may be generated in ion traps. We use the individual interaction of light with each of two vibrational modes of the ion to entangle them. This allows us to generate NOON states with N = 8.

Resumen

En la primera parte de estas tesis usando el Modelo de Jaynes - Cummings (JCM) de dos fotones interactuando con un átomo de dos niveles, se muestra como los estados N00N se pueden generar entrelazando dos cavidades, pasando átomos a través de ellas. Los átomos interactúan con cada cavidad via transiciones resonantes de dos fotones. Aprovechamos el hecho de que dependiendo del estado en el que el átomo entra (excitado ó base), deja ó toma dos fotones por interacción y deja las dos cavidades en un estado puro.

En la segunda parte de estas tesis consideramos el Hamiltoniano de un ion vibrando en dos dimensiones interactuando con un láser. Se muestra como los estados N00N se pueden generar en trampas de iones. Usamos la interacción individual de la luz con cada uno de los dos modos vibracionales del ion para entrelazarlos. Esto nos permite generar el estado N00N con N=8.

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Contents

CHAPTER 1

INTRODUCTION

The interaction of matter with radiation has been one of the driving forces of modern physics. The problem of blackbody radiation led Planck [1] to introduce the idea of quantisation at the turn of the previous century. In describing the photoelectric effect [2], Einstein in 1905, introduced the concept of the photon. The problems of spectral radiation from atoms culminated in the development of Quantum Mechanics during the 1920s. One of Physics most accurate theory, Quantum Electrodynamics (QED), describes the interaction of electrons and other leptons with the electromagnetic field. The advent of the laser has not only opened up new perspectives in physics research, but also revolutionised communications.

The field of cavity QED can be said to have been born more than 46 years ago with the prediction by Purcell [3] that the spontaneous emission rate for an atom inside a conducting cavity can be greatly enhanced in comparison with the rate in free space if the dimensions of the cavity are of the order of the transition wavelength. The subject was only of theoretical interest until experimental techniques became available to observe such effects.

A particular cavity QED system, in which we will be interested, consists of one or more two-level atoms coupled to a single mode electromagnetic field of an optical resonator, cavity QED processes engender an intimate correlation between the states of the atom and those of the field, and so their study provides new insights into quantum aspects of the interaction between light and matter.

The first experiments with atoms in a cavity were those of Vaidnaythan, Spencer, and Kleppner [4], Goy *et al.* [5], and Jhe *et al.* [6]. Several later experiments have been carried out to investigate various features of the change in spontaneous emission rates due to the presence of boundaries which alter the electromagnetic field mode density from its free space value [7,8].

Another approach to what is now known as cavity QED had its beginnings in the work of Jaynes and Cummings [9].

The Jaynes - Cummings Model (JCM) is one of the simplest systems in quantum optics. This means that the JCM serves as a useful approximation to more complicated systems.

The JCM consists of a single mode of the quantized electromagnetic field interacting with a two-level atom in a lossless cavity. The JCM exhibits interesting features, namely, the atomic inversion (the probability of being in the excited state minus the probability of being in the ground state) is very sensitive to the statistics of the initial field. If the field is initially prepared in a number state, the inversion will show the usual Rabi oscillations (also shown when the input field is not quantized, i.e., classical). However, if the input field is prepared in a coherent state, the oscillations are modulated in such a way they exhibit collapses and revivals [10]. The JCM can be solved exactly (non-perturbatively) in the rotating wave approximation (RWA) in which the high frequency terms of the Hamiltonian in the interaction picture are neglected while the low frequency terms hold [11], and this allows us to obtain exact expressions for the expectation values of physical quantities.

In more recent years the advent of atom trapping and optical microcavities has opened new opportunities to test fundamental physics and possible links to future technologies, for example Quantum Computing. One of the areas of Quantum Optics that relates to this is known as Cavity Quantum Electrodynamics (CQED). CQED is an area of considerable theoretical and experimental interest. One of the core systems of Cavity QED is that of cold atoms held inside an optical resonator interacting with external laser light fields.

Advances in trapping and cooling of ions and atoms [12] as well as progress in cavity quantum electrodynamics with strong atom-field coupling in the optical regime [13] make an experimental realization of an ion-trap laser possible.

The ions confined in a Paul trap have become a problem of increasing interest over the last few years. Most of the works are focused on classically chaotic motions of the systems with ion numbers greater than one [14-16]. Chacón *et al.* used a semiclassical approximation to show that chaos can occur in the system of a single trapped ion interacting with a sufficiently strong standing laser field [17,18]. The quantum mechanics of a trapped single ion interacting with a weak laser field is described by the Schrödinger equation with harmonic potential and spatially periodical perturbation.

Besides cavity quantum electrodynamics (QED), the system of ultracold ions moving in a harmonic trap has become a prospective physical setting for the preparation and measurement of nonclassical states [19,20], because the model of an ultracold ion strongly coupled to a laser field in the limit where coherent interactions can dominate over dissipative processes is formally similar to cavity QED.

The Jaynes - Cummings model, has been successfully used to describe ion-trap system [21-23], in which the ion is considered to have two levels and moves within a region of space much smaller than the effective wavelength of the laser field, i.e. the Lamb Dicke limit (LDL), the trap's potential is quantized as a harmonic oscillator, and the cooling laser is supposed as the classical form of a standing or traveling wave. Further investigation show that the situation outside the LDL for the ion-trap system just corresponds to the nonlinear JCM which is also a well-developed subject in quantum optics. If the ion is considered to have three levels, the JCM could still be realized for the dipole-forbidden interaction between the ground state and the metastable excited state, driven by a laser [23].

The existence and properties of nonclassical states, such as N00N states [24], defined as maximally path entangled states. These multiphoton entangled states, can be used to obtain high-precision phase measurements, becoming more and more advantageous as the number of photons grows. Many applications in quantum imaging, quantum information and quantum metrology [25] depend on the availability of entangled photon pairs because entanglement is a distinctive feature of quantum mechanics that lies at the core of many new applications. In this thesis, the generation of such states is the main subject.

1.0.1 Organization of This Thesis

In this thesis it is shown, in the first chapter a description of the type Jaynes - Cummings Hamiltonian, which is used to describe the single mode of the quantized electromagnetic field interacting with a two-level atom, as well as the description of the ion-laser interaction. In the second chapter, we discuss the quantization of the radiation field through the vector potential written in terms of creation and annihilation operators and thus we obtain the electric field operator. We also define some of the quantum states of radiation field, i.e. number states, coherent states and N00N states. In the third chapter it is shown the atom-field interaction in the picture of semiclassical theory. In the fourth chapter it is shown a basic theoretical development of the quantized atom-field interaction, using the two-photon Jaynes - Cummings Hamiltonian of a two level atom interacting with two cavities, a N00N state is generated by entangling these cavities. Then we will show how N00N state may be generated in ion-laser interactions. Playing the role of cavity fields (A and B), the ion will be assumed to be trapped in two dimensions *x* (cavity A) and *y* (cavity B). These will allow us the generation of the state $|0\rangle_x |8\rangle_y + |8\rangle_x |0\rangle_y$. This is of interest as the generation of "high" N00N states has not been reported. Up to now, only N00N states with N=5 has been proposed. In Chapter 6 shows the general conclusions.

CHAPTER 2

QUANTUM STATES OF RADIATION FIELD

2.1 Quantization of the Radiation Field

The quantum theory of the radiation field has some similarities with the classical theory. The field vectors (such as $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ and the vector potential $\vec{A}(\vec{r},t)$) in quantum theory must be taken as operators instead of the algebraic quantities of classical theory [26], but both theories are based on Maxwell's equations. It is not possible to derive the quantum theory from the classical equations, but the transition to quantum mechanics can be accomplished most easily if the equations of classical electromagnetic theory are put into a form where the classical harmonic oscillator equation is replaced by the corresponding quantum mechanical equation [27]. The first thing is to write the classical equations in such a form that the harmonic oscillator depends on appropriate field variables for the conversion to quantum mechanics.

We assume there are no sources of radiation, this is, absence of charges and currents, the electric and magnetic fields are fully specified by the vector potential $\vec{A}\,(\vec{r},t)$. Since $\vec{A}\,(\vec{r},t)$ is transverse (perpendicular to the wave vector k), it has only two nonzero components along the directions of two polarization (unit) vectors, $\vec{\varepsilon_k}$ which lie in a plane perpendicular to k. We can thus expand $\vec{A}\,(\vec{r},t)$ in a Fourier series as follows:

$$\vec{A}(\vec{r},t) = \sum\nolimits_k \left\{ A_k \exp\left(-i\omega_k t + i\vec{k}\cdot\vec{r}\right) + A_k^* \exp\left(i\omega_k t - i\vec{k}\cdot\vec{r}\right) \right\} \vec{\varepsilon_k}, \tag{2.1}$$

where A_k and A_k^* are the coefficients of the Fourier series, which represent the amplitudes of the electromagnetic field. We have assumed that the electromagnetic field is confined to a large volume V with periodic boundary conditions. By analogy with the quantization of a classical harmonic oscillator, the quantization of radiation can be achieved by writing the electromagnetic field in terms of creation and annihilation operators.

The classical field energy, or Hamiltonian, of the single mode field is given by:

$$\boldsymbol{H} = \frac{1}{2} \int\limits_{v} \left(\varepsilon_0 \vec{E}\left(\vec{r}, t\right)^2 + \vec{H}\left(\vec{r}, t\right)^2 \right) dV = 2\varepsilon_0 V \omega_k^2 A_k \cdot A_k^*, \tag{2.2}$$

where $\vec{E}(\vec{r}, t) = -\frac{\partial \vec{A}}{\partial t}$ and $\vec{H}(\vec{r}, t) = \frac{1}{\mu_0} \vec{B}(\vec{r}, t) = \frac{1}{\mu_0}(\nabla \times \vec{A})$.

Instead of the two variables $\vec{A}_k$ and $\vec{A}_k^*$, we can introduce a new set of two canonically conjugate variables:

$$Q_k = \frac{1}{\sqrt{4\varepsilon_0 \omega_k^2 V}} \left(A_k + A_k^* \right), \tag{2.3}$$

$$P_k = \frac{i\omega_k}{\sqrt{4\varepsilon_0 \omega_k^2 V}} \left(A_k^* - A_k \right), \tag{2.4}$$

we obtain

$$\boldsymbol{H} = \frac{1}{2} \left(\omega_k^2 Q_k^2 + P_k^2 \right), \tag{2.5}$$

which is the usual form for the energy of a classical harmonic oscillator of unit mass. Thus, the problem of the vector potential associated with cavity modes is completely equivalent to the problem of a classical harmonic oscillator.

2.1.1 Vector Potential

We assume that the physical observables will be associated to operator

$$\begin{array}{l} \hat{P}_k \to \hat{p}_k \\ \hat{Q}_k \to \hat{q}_k \end{array}, \tag{2.6}$$

then, the energy written now in term of operators is:

$$\hat{H} = \frac{1}{2}\left(\hat{p}_k^2 + \omega_k^2 \hat{q}_k^2\right), \tag{2.7}$$

where $\hat{p}_k$ and $\hat{q}_k$ are operators that are related to A_k and A_k^* by means of the Fourier transform, then

$$\begin{array}{l} A_k \rightarrow \hat{a}_k \\ A_k^* \rightarrow \hat{a}_k^\dagger \end{array}. \tag{2.8}$$

We can write to A_k and A_k^* as

$$\hat{a}_k = \left(4\varepsilon_0 V \omega_k^2\right)^{-1/2} \left(\omega_k \hat{q}_k + i\hat{p}_k\right), \tag{2.9}$$

$$\hat{a}_k^\dagger = \left(4\varepsilon_0 V \omega_k^2\right)^{-1/2} \left(\omega_k \hat{q}_k - i\hat{p}_k\right), \tag{2.10}$$

where the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ are called annihilation and of creation operators respectively.

From equations (2.7) and (2.9), the amplitudes A_k have the form

$$A_k = \left(\frac{\hbar}{2\varepsilon_0 V \omega_k}\right)^{1/2} \hat{a}_k \tag{2.11}$$

and equations (2.7) and (2.10) we have

$$A_k^* = \left(\frac{\hbar}{2\varepsilon_0 V \omega_k}\right)^{1/2} \hat{a}_k^\dagger \tag{2.12}$$

So that the vector potential in terms of these operators is:

$$\hat{A}_k = \left(\frac{\hbar}{2\varepsilon_0 V \omega_k}\right)^{1/2} \sum_k \left\{\hat{a}_k \exp\left(-i\omega_k t + i\vec{k}\cdot\vec{r}\right) + \hat{a}_k^\dagger \exp\left(i\omega_k t - i\vec{k}\cdot\vec{r}\right)\right\}\vec{\varepsilon}_k, \tag{2.13}$$

this expression is the quantum mechanical vector potential.

The operators for the Electric and Magnetic fields are:

$$\hat{E}_k = i\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2}\vec{\varepsilon}_k\left\{\hat{a}_k\exp\left(-i\omega_k t + i\vec{k}\cdot\vec{r}\right) - \hat{a}_k^\dagger\exp\left(i\omega_k t - i\vec{k}\cdot\vec{r}\right)\right\}, \quad (2.14)$$

and

$$\hat{H}_k = i\left(\frac{\hbar c^2}{2\mu_0 V\omega_k}\right)^{1/2}\vec{k}\times\vec{\varepsilon}_k\left\{\hat{a}_k\exp\left(-i\omega_k t + i\vec{k}\cdot\vec{r}\right) - \hat{a}_k^\dagger\exp\left(i\omega_k t - i\vec{k}\cdot\vec{r}\right)\right\}. \quad (2.15)$$

The total transverse fields, in the cavity are then

$$\hat{E}_T = \sum_k \hat{E}_k, \quad (2.16)$$

and

$$\hat{H} = \sum_k \hat{H}_k. \quad (2.17)$$

We conclude that the electromagnetic field is quantized by the association of a quantum mechanical harmonic oscillator, with each mode $\vec{k}$ of the radiation field.

2.2 Fock or Number States

Now we will considered a single mode of the radiation field, then adding equations (2.9) and (2.10) we obtain:

$$\hat{a} + \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}2\omega\hat{q},$$

then

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}} \left(\hat{a} + \hat{a}^\dagger\right). \tag{2.18}$$

Again from equations (2.9) and (2.10), we can obtain the generalized momentum

$$\hat{a} - \hat{a}^\dagger = \frac{2i}{\sqrt{2\hbar\omega}} \hat{p},$$

so that

$$\hat{p} = -i\sqrt{\frac{\hbar\omega}{2}} \left(\hat{a} - \hat{a}^\dagger\right). \tag{2.19}$$

The commutator between $\hat{q}$ and $\hat{p}$ is, simply

$$[\hat{q}, \hat{p}] = i\hbar \left(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right) = i\hbar \left[a, \hat{a}^\dagger\right]. \tag{2.20}$$

The annihilation and creation operators written in terms of the position and momentum operators are:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} \left(\omega\hat{q} + i\hat{p}\right), \tag{2.21}$$

and

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}} \left(\omega\hat{q} - i\hat{p}\right). \tag{2.22}$$

The energy of the field becomes the Hamiltonian operator

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{2.23}$$

where we have defined $\hat{n} = \hat{a}^\dagger\hat{a}$, that is the number operator for the harmonic oscillator, so that now we have

$$\hat{H} = \hbar\omega \left(\hat{n} + \frac{1}{2}\right). \tag{2.24}$$

The eigenvalues of $\hat{n}$ are integers and positives, the spectrum is quantized, also it fulfills the eigenvalue equation

$$\hat{n}\left|n\right\rangle = n\left|n\right\rangle, \tag{2.25}$$

If we consider when $n = 0$, we can calculated the energy of the vacuum, this is $E = \frac{1}{2}\hbar\omega$. The mode energy associated to n photons is:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \qquad\qquad n = 0, 1, 2, ... \tag{2.26}$$

The states $\left|n\right\rangle$ are called number states and also are eigenstates of the Hamiltonian

$$\hat{H}\left|n\right\rangle = E_n\left|n\right\rangle. \tag{2.27}$$

These number state are orthogonal according to

$$\left\langle n \,\middle|\, n^{'}\right\rangle = \delta_{nn^{'}}. \tag{2.28}$$

The action of the annihilation operator $\hat{a}$ on $\left|n\right\rangle$ is:

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \tag{2.29}$$

similarly, for the creation operator

$$a^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \tag{2.30}$$

with $\hat{a}\left|0\right\rangle = 0$. All the number states can be generated from the vacuum according to:

$$\left|n\right\rangle = \frac{\left(\hat{a}^{\dagger}\right)^{n}}{\sqrt{n!}}\left|0\right\rangle. \tag{2.31}$$

2.3 Coherent States

The coherent states denoted as $|\alpha\rangle$, are a linear superposition of the number states with an indefinite number of photons. Its electric field approaches that of a classical wave of stable amplitude and stationary phase, that corresponds to a normal mode of oscillation. They are important because are the quantum mechanical states most similar to a classical wave packet which has an analogous uncertainty spread in both position and momentum.
The coherent state, $|\alpha\rangle$, may be obtained as a superposition of the number states

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n\rangle. \tag{2.32}$$

In this expression α can be any complex number and the coherent states so defined form a double continuum corresponding to the continuous ranges of values of the real and imaginary parts of α. The coherent states form an overcomplete set of states for the radiation field, i.e., a set of not orthogonal elements with a major number of elements than the dimension of the vectorial space, that fulfill the following completeness relation:

$$\left(1/\pi\right)\int |\alpha\rangle\langle\alpha|\, d^{(2)}\alpha = 1, \tag{2.33}$$

where $d^{(2)}\alpha$ is an element of real area of the complex plane α .
They are normalized, that is:

$$\langle\alpha \mid \alpha\rangle = 1, \tag{2.34}$$

but, they are not orthogonal

$$\left\langle\alpha \,\middle|\, \alpha'\right\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2 + \left|\alpha'\right|^2\right) + \alpha\alpha'^*\right], \tag{2.35}$$

They are eigenstate of the operator $\hat{a}$, i.e.,

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{2.36}$$

$$\langle\alpha| \hat{a}^{\dagger} = \alpha^{*} \langle\alpha| . \tag{2.37}$$

2.3.1 Photon Number Distribution

The probability of finding a certain number of photons when a measurement is done on coherent states satisfies a Poisson distribution since a series of measures of this nature on this system represents independent events:

$$P(n) = |\langle n \mid \alpha\rangle|^{2} = \frac{|\alpha|^{2n} e^{-|\alpha|^{2}}}{n!}, \tag{2.38}$$

where $|\alpha|^2$ is the mean number of photons [28], the figure (2.1) shows the photon number probability distribution.

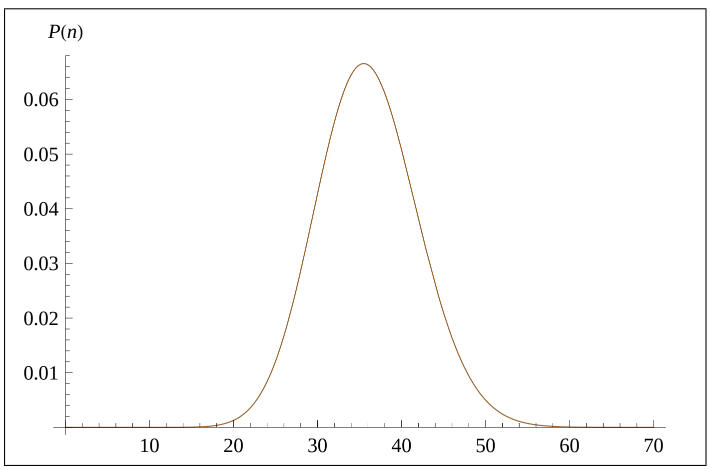

Figure 2.1: *Photon distribution for a coherent state with* $|\alpha|^2 = 36$.

Let us consider a cavity in which the photons are in thermal equilibrium with the radiation field, the probability distribution for the electromagnetic energy of the total radiation field in the cavity satisfies the Boltzmann probability distribution:

$$P\left(n\right) = \exp\left(-\frac{E_{n}}{k_{B}T}\right), \tag{2.39}$$

where k_B is the Boltzmann's constant, T is the tempetature.

which leads to Bose-Einstein probability distribution

$$P(n) = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n, \tag{2.40}$$

where the mean number of photons is:

$$\bar{n} = \frac{1}{\exp\left(\hbar\omega/_{k_B T}\right) - 1}. \tag{2.41}$$

2.4 N00N States

In the previous sections we have studied states of a single mode of the radiation field, now we will introduce states of two modes, such as N00N states. Two mode states have much richer features as it can occur that they may be written in a entangled form, this is a total wave function that can not be written is a product of two individual wave functions: $|\psi\rangle \neq |\psi_1\rangle |\psi_2\rangle$. Whenever this happens, it is said that the state is entangled.

A NOON-state is a maximally entangled states. It is written in the form

$$|N00N\rangle_{A,B} \propto (|N\rangle_A |0\rangle_B + |0\rangle_A |N\rangle_B), \tag{2.42}$$

that makes clear why they are called this way.

It represents, in the case of cavities, a superposition of N photons in the first cavity (*A*) and the vacuum in the second cavity (*B*), and vice versa, where a normalization constant of $\frac{1}{\sqrt{2}}$ has been dropped for convenience (and convention).They are maximally entangled in the sense that the state consists of a superposition with components where all N photons are in one cavity (mode or path) or the other. The state is entangled because in no basis it can be written as a product state of the two separate cavities (modes or paths); i.e.$|N00N\rangle_{A,B} \neq |\psi_A\rangle \otimes |\psi_B\rangle$. According to the Copenhagen interpretation of quantum mechanics, the superposition in equation (2.43) means that the path of the definite number of photons N is objectively indefinite.The N00N states exhibit fascinating quantum interference properties, N00N states are considered the optimal quantum states of light for quantum metrology applications such as quantum lithography and quantum imaging. The quantum properties of a N00N states allow precision measurements far greater than classical measurements. Unlike the coherent states that minimize the uncertainty relation, in N00N states are completely uncertain if all the N photons are in cavity A (and none in B) or if all the N photons are in cavity B (and none in A).

The number-phase uncertainty relation for N00N states is

$$\Delta N \Delta \varphi \geq 1. \tag{2.43}$$

The probability distribution $P\left(n_1, n_2\right)$, inside the interferometer, consider the joint photon number distribution of the N00N state of equation (2.43), its only nonzero probabilities are $P\left(N, 0\right) = 1/2 = P\left(0, N\right)$ indicating a large fluctuation in the number of photons in each of the modes. In fact, for each of the modes, the fluctuations are clearly given by $\Delta N \sim N$ which, from the number-phase uncertainty relation leads to $\Delta \varphi \sim 1/N$, the Heisenberg limit.

CHAPTER 3

ATOM-FIELD INTERACTION SEMICLASSICAL THEORY

In this section we study the atom-field interaction in the framework of semiclassical theory, in which the atom is treated as a quantum two-level system and the field is treated classically, (in units such that $\hbar = 1$).

3.1 Interaction of an Atom with a Classical Field

We consider the interaction of a single-mode radiation field of frequency ω with a two-level atom,

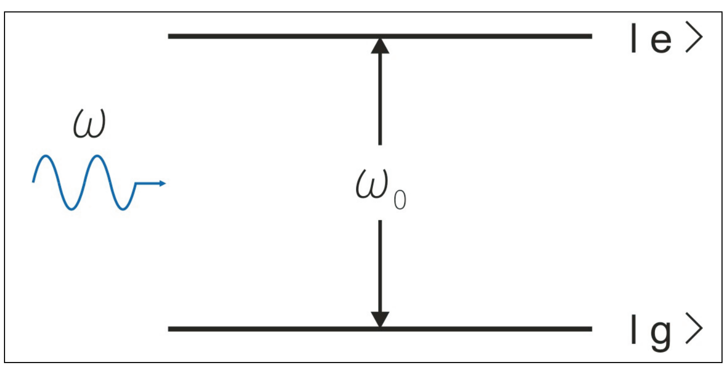

Figure 3.1: *interaction of a two-level atom with a single-mode field. Where ω is the field frequency and ω_0 is the resonant frequency between the two atomic levels.*

where the two energy levels $|e\rangle$ and $|g\rangle$ are the excited and ground states of the atom respectively, i.e. they are eigenstates of the unperturbed part of the Hamiltonian H_A with the eigenvalues ω_e and ω_g, respectively. The wave function of the two-level atom can be written as

$$|\psi(t)\rangle = C_e(t)|e\rangle + C_g(t)|g\rangle \tag{3.1}$$

where C_e and C_g describe the wave function amplitude coefficients for the two states of the atom. The corresponding Schrödinger equation is

$$i\frac{\partial |\psi(t)\rangle}{\partial t} = (H_A + H_I)|\psi(t)\rangle \tag{3.2}$$

where the atomic Hamiltonian is [29]

$$H_A = \frac{\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|), \tag{3.3}$$

with $\omega_0 = \omega_e - \omega_g$ and the interaction Hamiltonian between the atom and the electromagnetic field, in the dipole approximation is [30]

$$H_I = -\hat{\mathbf{d}}\cdot\mathbf{E} = -\left[d|e\rangle\langle g| + d^*|g\rangle\langle e|\right]\mathbf{E}, \tag{3.4}$$

we have assumed definite parity of the states $|e\rangle$ and $|g\rangle$ [31], then the off-diagonal elements of the dipole operator are nonzero and where $d = \langle e|\hat{\mathbf{d}}|g\rangle$ and $d^* = \langle g|\hat{\mathbf{d}}|e\rangle$.

We can now write the Hamiltonian for the atom-field interaction as

$$H_{sc} = \frac{\omega_0}{2}(|e\rangle\langle e| + |g\rangle\langle g|) + \Omega_R\left(e^{i\phi}|e\rangle\langle g| + e^{-i\phi}|g\rangle\langle e|\right)\cos\omega t, \tag{3.5}$$

where we have considered a single-mode field $\mathbf{E}(t) = E_0\cos(\omega t)$, $\Omega_R = |d|E_0$ is the Rabi frequency , ϕ is the phase of the dipole matrix element $d = |d|e^{i\phi}$ and ω is the field frequency.

Passing to a interaction picture, where we are writing the function $cos(\omega t) = \frac{e^{i\omega t}+e^{-i\omega t}}{2}$, we obtain the interaction Hamiltonian

$$\begin{array}{l} H_I = \frac{\Omega_R}{2} e^{-i\phi} \left|e\right\rangle \left\langle g\right| e^{i(\omega_0-\omega)t} + e^{i\phi} \left|g\right\rangle \left\langle e\right| e^{-i(\omega_0-\omega)t} \\ +e^{-i\phi} \left|e\right\rangle \left\langle g\right| e^{-i(\omega_0+\omega)t} + e^{i\phi} \left|g\right\rangle \left\langle e\right| e^{-i(\omega_0+\omega)t} \end{array} \tag{3.6}$$

The terms proportional to $\exp\left[\pm i\left(\omega_0+\omega\right)t\right]$ vary very rapidly and their average over a time scale larger than $1/\omega$ is approximately zero. These terms can therefore be neglected in the so called Rotating Wave Approximation. The simplified interaction picture Hamiltonian is the written as

$$H_I = \frac{\Omega_R}{2}\left(e^{-i\phi}|e\rangle\langle g|e^{i\Delta t} + e^{i\phi}|g\rangle\langle e|e^{-i\Delta t}\right), \tag{3.7}$$

where $\Delta = \omega_0 - \omega$ is the detuning $\Delta = 0$. In the resonance case, we have

$$H_I = \frac{\Omega_R}{2}\left(e^{-i\phi}|e\rangle\langle g| + e^{i\phi}|g\rangle\langle e|\right), \tag{3.8}$$

the evolution operator $U_I(t) = e^{-\frac{i}{\hbar}H_I t}$, may be calculated from the expressions

$$H_I^{2n} = \left(\frac{\Omega_R}{2}\right)^{2n}\left(|e\rangle\langle e| + |g\rangle\langle g|\right), \tag{3.9}$$

$$H_I^{2n+1} = \left(\frac{\Omega_R}{2}\right)^{2n+1}\left(e^{-i\phi}\left|e\right\rangle\left\langle g\right| + e^{i\phi}\left|g\right\rangle\left\langle e\right|\right), \tag{3.10}$$

so that

$$U_I\left(t\right) = \cos\left(\frac{\Omega_R t}{2}\right)\left(|e\rangle\langle e| + |g\rangle\langle g|\right) + i\sin\left(\frac{\Omega_R t}{2}\right)\left(e^{-i\phi}\left|e\right\rangle\left\langle g\right| + e^{i\phi}\left|g\right\rangle\left\langle e\right|\right). \tag{3.11}$$

If we consider that the atom is initially in its excited state $\left(|\psi\left(0\right)\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right.$, we obtain the wave function, by application of the evolution operator to the initial state

$$|\psi(t)\rangle = \cos\left(\frac{\Omega_R}{2}t\right)\left|e\right\rangle + i\sin\left(\frac{\Omega_R}{2}t\right)e^{i\phi}\left|g\right\rangle. \tag{3.12}$$

The probability that the atom is in the excited state, $|e\rangle$, is then

$$P_e(t) = \cos^2\left(\frac{\Omega_R}{2}t\right). \tag{3.13}$$

which we plot in Fig. 3.2. Usual Rabi oscillations may be observed [32]

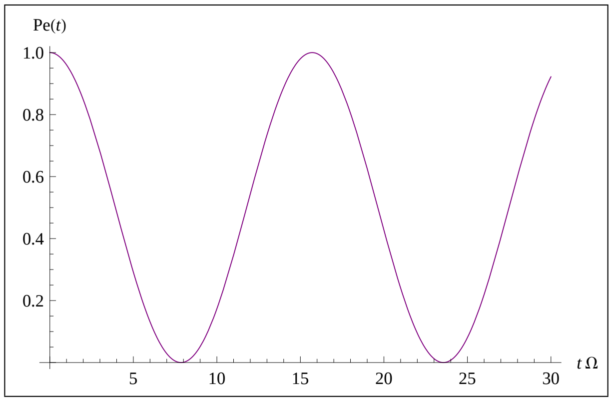

Figure 3.2: *Probability to find the atom in its excited state as a function of the scale time.*

3.1.1 Atomic Inversion

Now, we compute the atomic inversion $W(t)$ defined as the difference of the excited and ground state populations

$$W(t) = P_e(t) - P_g(t) \tag{3.14}$$

which for the case with the atom initially in excited state is simply

$$W(t) = \cos\left(\Omega_R t\right), \tag{3.15}$$

Therefore the Rabi frequency is the oscillation frequency of the atomic inversion.

CHAPTER 4

ATOM - FIELD INTERACTION

This section basically shows how to obtain the Jaynes-Cummings Hamiltonian. We solve the two-photon interaction Hamiltonian between a two-level atom and two photons which corresponds, first to the cavity A and then to a second cavity denoted as B. We conclude generating the N00N state in entangled cavities.

4.1 Interaction of a Two level Atom with a Quantized Field

We consider a single free - space field mode of the form given by

$$\hat{E}(t) = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} \vec{\varepsilon}\left(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}\right), \tag{4.1}$$

where the dipole approximation has been made. The free Hamiltonian $\hat{H}_0$ now must be

$$\hat{H}_0 = \hat{H}_{atom} + \hat{H}_{field}, \tag{4.2}$$

where $\hat{H}_{atom}$ is the free-atom Hamiltonian as before and $\hat{H}_{field}$ is the free-field Hamiltonian $\omega\hat{a}^\dagger\hat{a}$, where the zero-point term has been dropped as it does not contribute to the dynamics. The interaction Hamiltonian is

$$\hat{H}^{(I)} = -\hat{d} \cdot \hat{E} = i \left(\frac{\hbar\omega}{2\varepsilon_0 V} \right)^{1/2} \left(\hat{d} \cdot \vec{\varepsilon} \right) \left(\hat{a} - \hat{a}^{\dagger} \right), \tag{4.3}$$

where $\hat{d}$ is the dipole moment.

We can rewrite it as

$$\hat{H}^{(I)} = -\hat{d} \cdot \epsilon_0 \left(\hat{a} - \hat{a}^{\dagger} \right), \tag{4.4}$$

with $\epsilon_0 = i \left(\frac{\hbar\omega}{2\varepsilon_0 V} \right)^{1/2} \vec{\varepsilon}$.

4.2 Jaynes - Cummings Model

We now consider an atom interacting with a single - mode cavity field of the form

$$\hat{E} = \vec{\varepsilon} \left(\frac{\hbar\omega}{\varepsilon_0 V} \right)^{1/2} \left(\hat{a} + \hat{a}^{\dagger} \right) \sin(kz), \tag{4.5}$$

where $\vec{\varepsilon}$ is an arbitrarily oriented polarization vector. The interaction Hamiltonian is now

$$\hat{H}^{(I)} = \hat{d} g \left(\hat{a} + \hat{a}^{\dagger} \right), \tag{4.6}$$

with

$$g = -\left(\frac{\hbar\omega}{\varepsilon_0 V} \right)^{1/2} \sin(kz), \tag{4.7}$$

It is convenient to introduce the so-called atomic transition operators

$$\hat{\sigma}_+ = |e\rangle \langle g|, \qquad \hat{\sigma}_- = |g\rangle \langle e| = \hat{\sigma}_+^{\dagger}, \tag{4.8}$$

and the inversion operator

$$\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|\,. \tag{4.9}$$

These operators obey the Pauli spin algebra

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z, \tag{4.10}$$

$$[\hat{\sigma}_z, \hat{\sigma}_\pm] = 2\hat{\sigma}_\pm. \tag{4.11}$$

Only the off-diagonal elements of the dipole operator are nonzero, since by parity consideration $\langle e|\,\hat{d}\,|e\rangle = 0 = \langle g|\,\hat{d}\,|g\rangle$, so that we may write

$$\hat{d} = d\,|g\rangle\langle e| + d^*\,|e\rangle\langle g|$$

$$= d\hat{\sigma}_- + d^*\hat{\sigma}_+ = d\,(\hat{\sigma}_+ + \hat{\sigma}_-)\,, \tag{4.12}$$

where we have set $\langle e|\,\hat{d}\,|g\rangle = d$ we have assumed, that *d* is real. Thus the interaction Hamiltonian is

$$\hat{H}^{(I)} = \lambda\,(\hat{\sigma}_+ + \hat{\sigma}_-)\left(\hat{a} + \hat{a}^\dagger\right), \tag{4.13}$$

where $\lambda = dg$.

If we define the level of the energy to be zero halfway between the states $|g\rangle$ and $|e\rangle$, then the free atomic Hamiltonian may be written as

$$\hat{H}_A = \frac{1}{2}\,(E_e - E_g)\,\hat{\sigma}_z = \frac{1}{2}\omega_0\hat{\sigma}_z, \tag{4.14}$$

where $E_e = -E_g = \frac{1}{2}\omega_0$. The free-field Hamiltonian is, after dropping the zero-point energy term,

$$\hat{H}_F = \omega\hat{a}^\dagger\hat{a}. \tag{4.15}$$

Thus the total Hamiltonian is

$$\hat{H} = \hat{H}_A + \hat{H}_F + H^{(I)} \tag{4.16}$$

$$= \frac{1}{2}\omega_0\hat{\sigma}_z + \omega\hat{a}^\dagger\hat{a} + \lambda\left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\left(\hat{a} + \hat{a}^\dagger\right). \tag{4.17}$$

In the free-field case, the operators $\hat{a}$ and $\hat{a}^\dagger$ evolve as:

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}, \qquad a^\dagger(t) = a^\dagger(0)e^{i\omega t}. \tag{4.18}$$

One can show similarly that for the free-atomic case

$$\hat{\sigma}_\pm(t) = \hat{\sigma}_\pm(0)e^{\pm i\omega t}. \tag{4.19}$$

Thus we can see that the approximate time dependences of the operator products in equation (4.17) are as follows:

$$\hat{\sigma}_+\hat{a} \sim e^{i(\omega_0-\omega)t},$$

$$\hat{\sigma}_-\hat{a}^\dagger \sim e^{-i(\omega_0-\omega)t},$$

$$\hat{\sigma}_+\hat{a}^\dagger \sim e^{i(\omega_0+\omega)t},$$

$$\hat{\sigma}_-\hat{a} \sim e^{-i(\omega_0+\omega)t},$$

for $\omega_0 \approx \omega$ the last two terms vary much more rapidly than the first two. Furthermore, the last two terms do not conserve energy in contrast to the first two. The term $\hat{\sigma}_+\hat{a}^\dagger$ corresponds to the emission of a photon as the atom goes from the ground to the excited state, whereas $\hat{\sigma}_-\hat{a}$ corresponds to the absorption of a photon as the atom goes from the excited to the ground state. Making the Rotating Wave Approximation (RWA) our Hamiltonian in this approximation is

$$\hat{H}_{JC} = \frac{1}{2}\omega_0 \hat{\sigma}_z + \omega \hat{a}^\dagger \hat{a} + \lambda \left(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger \right). \tag{4.20}$$

The interaction described by this Hamiltonian is widely referred to as the Jaynes - Cummings model [33]. The Jaynes - Cummings Hamiltonian describes the interaction of one mode of the electromagnetic field with a two - level atom in the electric dipole and rotating wave approximations.

Passing to the interaction picture by means the transformation, [where we are considering the field on resonance with the atom, with frequency ω, (in units such $\hbar = 1$)]

$$T = \exp\left[-i\omega t \left(\hat{a}^\dagger \hat{a} + \frac{\hat{\sigma}_z}{2} \right)\right], \tag{4.21}$$

the Schrödinger equation can be written as

$$i\frac{\partial \left|\psi\left(t\right)\right\rangle}{\partial t} = H_{JC} \left|\psi\left(t\right)\right\rangle, \tag{4.22}$$

with $\left|\psi\left(t\right)\right\rangle = T\left|\varphi\left(t\right)\right\rangle$, substituting in the equation (4.22)

$$i\frac{\partial T}{\partial t}\left|\varphi\left(t\right)\right\rangle + iT\frac{\partial \left|\varphi\left(t\right)\right\rangle}{\partial t} = \left\{ \omega \left(\hat{a}^\dagger \hat{a} + \frac{\hat{\sigma}_z}{2} \right) + \lambda \left(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger \right) \right\} T \left|\varphi\left(t\right)\right\rangle, \tag{4.23}$$

we obtain the interaction Hamiltonian

$$H = \lambda \left(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger \right), \tag{4.24}$$

using the Susskind - Glogower phase operator [34]

$$\hat{V} = \frac{1}{\sqrt{\hat{n}+1}}\hat{a} = \sum_{n=0}^{\infty} \left|\hat{n}\right\rangle \left\langle \hat{n}+1\right| \tag{4.25}$$

and

$$\hat{V}^\dagger = \hat{a}^\dagger \frac{1}{\sqrt{\hat{n}+1}} = \sum_{n=0}^{\infty} \left|\hat{n}+1\right\rangle \left\langle \hat{n}\right|, \tag{4.26}$$

we obtain

$$H = \begin{pmatrix} 0 & \lambda\sqrt{\hat{n}+1}\hat{V} \\ \lambda\hat{V}^{\dagger}\sqrt{\hat{n}+1} & 0 \end{pmatrix}, \tag{4.27}$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}^{\dagger} \end{pmatrix} \begin{pmatrix} 0 & \lambda\sqrt{\hat{n}+1} \\ \lambda\sqrt{\hat{n}+1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{V} \end{pmatrix}, \tag{4.28}$$

where

$$H_I = \begin{pmatrix} 0 & \lambda\sqrt{\hat{n}+1} \\ \lambda\sqrt{\hat{n}+1} & 0 \end{pmatrix}, \tag{4.29}$$

then the evolution operator, $U_I(t) = e^{[-iH_I t]}$ is

$$U_I(t) = \begin{pmatrix} \cos\left(\lambda t\sqrt{n+1}\right) & -i\sin\left(\lambda t\sqrt{n+1}\right) \\ -i\sin\left(\lambda t\sqrt{n+1}\right) & \cos\left(\lambda t\sqrt{n+1}\right) \end{pmatrix}, \tag{4.30}$$

then

$$U(t) = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}^{\dagger} \end{pmatrix} \begin{pmatrix} \cos\left(\lambda t\sqrt{\hat{n}+1}\right) & -i\sin\left(\lambda t\sqrt{\hat{n}+1}\right) \\ -i\sin\left(\lambda t\sqrt{\hat{n}+1}\right) & \cos\left(\lambda t\sqrt{\hat{n}+1}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{V} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & |0\rangle\langle 0| \end{pmatrix} \tag{4.31}$$

$$= \begin{pmatrix} \cos\left(\lambda t\sqrt{\hat{n}+1}\right) & -i\sin\left(\lambda t\sqrt{\hat{n}+1}\right)\hat{V} \\ -i\hat{V}^{\dagger}\sin\left(\lambda t\sqrt{\hat{n}+1}\right) & \hat{V}^{\dagger}\cos\left(\lambda t\sqrt{\hat{n}+1}\right)\hat{V} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & |0\rangle\langle 0| \end{pmatrix}$$

we see that $\hat{V}^{\dagger}\hat{V} = 1$, but $\hat{V}\hat{V}^{\dagger} = 1 - |0\rangle\langle 0|$, then

$$U(t) = \begin{pmatrix} \cos\left(\lambda t\sqrt{n+1}\right) & -i\sin\left(\lambda t\sqrt{n+1}\right)V \\ -iV\sin\left(\lambda t\sqrt{n+1}\right) & \cos\left(\lambda t\sqrt{n}\right) \end{pmatrix}, \tag{4.32}$$

4.2.1 Field in a number state

We consider as initial state of the field a number state $|n\rangle$ and the atom in its excited state, so that

$$W(t) = \cos\left(2\lambda t\sqrt{n+1}\right) \tag{4.33}$$

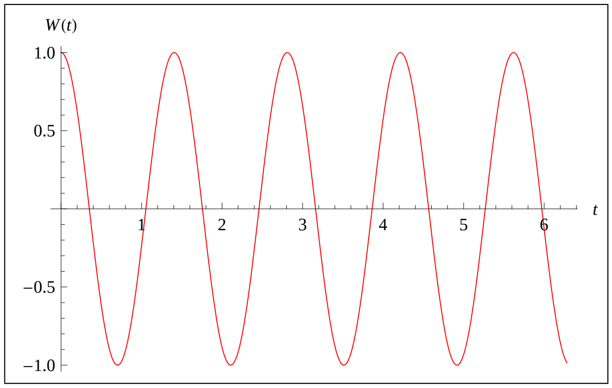

Figure 4.1: *Atomic inversion with the field initially in a number state with n=4 photons.*

The atomic inversion for the field initially in a number state is strictly periodic (see Fig. 4.1), just as in the semiclassical case of Eq.(3.15) except for the fact that in the classical case there must always be a field present initially. But in the quantum mechanical case there are Rabi oscillations even for the case when $n = 0$. These are the vacuum-field Rabi oscillations.

4.2.2 Field in a coherent state

We consider as initial state of the field a coherent state

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{4.34}$$

and the atom in its excited state, we can compute the atomic inversion,

$$W(t) = \sum_{n=0}^{\infty} P_n \cos\left(\lambda t\sqrt{n+1}\right) \tag{4.35}$$

where P_n is the photon distribution for the coherent state. (See eq. (2.39))

We plot the atomic inversion for a coherent state as a function of λt, we can note significant discrepancies between the fully quantization and semiclassical Rabi oscillations. We

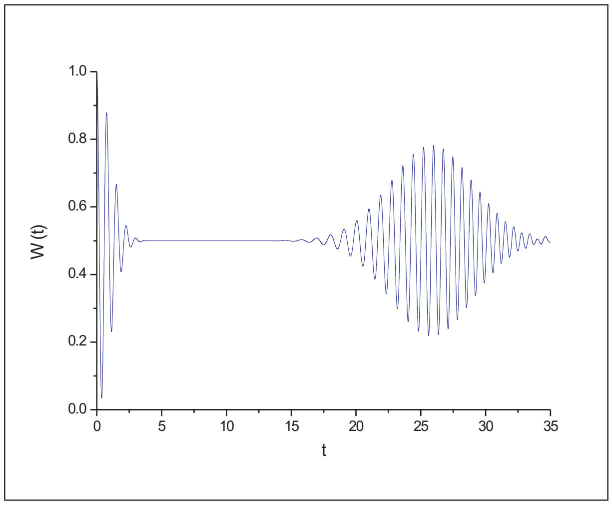

Figure 4.2: *Atomic inversion for a coherent state with* $\alpha = 4$.

observe first that the Rabi oscillations initially appear to damp out, or collapse. At longer times one finds a sequence of collapses and revivals, the revivals becoming less distinct as time increases. This collapse and revival behavior of the Rabi oscillations in the fully quantized model is strikingly different to the semiclassical case where the oscillations have constant amplitude.

4.3 Two - Photon Interaction

The interaction Hamiltonian between a two level atom and two photons is of the form (in units such that $\hbar = 1$)

$$H = \omega \left(\hat{a}^{\dagger}\hat{a} + \hat{\sigma}_z\right) + \lambda \left(\hat{a}^2\hat{\sigma}_+ + \hat{a}^{\dagger 2}\hat{\sigma}_-\right), \tag{4.36}$$

where $\hat{a}$ and $\hat{a}^{\dagger}$ are the operator of annihilation and creation for the field mode (cavity A), respectively. We consider the field on resonance with the atom, with frequency ω, $\hat{\sigma}_+$, $\hat{\sigma}_-$ and $\hat{\sigma}_z$ are Pauli spin-flip operators, λ is the coupling constant. In this case, it must be noticed that the interaction Hamiltonian contains the following terms: $\hat{a}^2\hat{\sigma}_+$ and $\hat{a}^{\dagger 2}\hat{\sigma}_-$ which produce that the atom goes from ground to excited state by absorbing 2 photons, and viceversa. This Hamiltonian is a first generalization of the JCM Hamiltonian to account for multiphoton processes.

Passing to a interaction picture by means the transformation

$$T = \exp\left[-i\omega t\left(\hat{a}^\dagger\hat{a} + \hat{\sigma}_z\right)\right] \tag{4.37}$$

we obtain the interaction Hamiltonian, (we will add a subscript a, because later in this thesis, we will look at different cavities)

$$H_a = \lambda\left(\hat{a}^2\hat{\sigma}_+ + \hat{a}^{\dagger 2}\hat{\sigma}_-\right). \tag{4.38}$$

Now, making use of Susskind - Glogower phase operator [34], $\hat{V}_a = \frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}\hat{a}$, we obtain

$$\hat{V}_a^{\dagger 2} = \hat{a}^{\dagger 2}\frac{1}{\sqrt{(\hat{n}+1)(\hat{n}+2)}} \approx \hat{a}^{\dagger 2}\frac{1}{\left(\hat{n}+\frac{3}{2}\right)}, \tag{4.39}$$

we write H_a in matrix form

$$H_a = \begin{pmatrix} 0 & \lambda\left(\hat{n}+\frac{3}{2}\right)\hat{V}_a^2 \\ \lambda\hat{V}_a^{\dagger 2}\left(\hat{n}+\frac{3}{2}\right) & 0 \end{pmatrix}, \tag{4.40}$$

$$H_a = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_a^{\dagger 2} \end{pmatrix}\begin{pmatrix} 0 & \lambda\left(\hat{n}+\frac{3}{2}\right) \\ \lambda\left(\hat{n}+\frac{3}{2}\right) & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_a^2 \end{pmatrix}, \tag{4.41}$$

where

$$H_I^{(a)} = \begin{pmatrix} 0 & \lambda\left(\hat{n}+\frac{3}{2}\right) \\ \lambda\left(\hat{n}+\frac{3}{2}\right) & 0 \end{pmatrix}, \tag{4.42}$$

so that the evolution operator $U_I^{(a)}(t) = e^{-iH_I^{(a)}t}$

$$U_I^{(a)}(t) = e^{-iH_I^{(a)}t} = \sum_m \frac{(-it)^{2m}}{m!}\lambda_1^{2m}\left(\hat{n}+\frac{3}{2}\right)^{2m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+\sum_{m}\frac{(-it)^{2m+1}}{(m+1)!}\lambda^{2m+1}\left(\hat{n}+\frac{3}{2}\right)^{2m+1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$U_I^{(a)}(t)=\begin{pmatrix}\cos\lambda t\left(\hat{n}+\frac{3}{2}\right) & -i\sin\lambda t\left(\hat{n}+\frac{3}{2}\right)\\ -i\sin\lambda t\left(\hat{n}+\frac{3}{2}\right) & \cos\lambda t\left(\hat{n}+\frac{3}{2}\right)\end{pmatrix}, \tag{4.43}$$

then

$$U_a(t)=\begin{pmatrix}1 & 0\\ 0 & \hat{V}_a^{\dagger 2}\end{pmatrix}\begin{pmatrix}\cos\lambda t\left(\hat{n}+\frac{3}{2}\right) & -i\sin\lambda t\left(\hat{n}+\frac{3}{2}\right)\\ -i\sin\lambda t\left(\hat{n}+\frac{3}{2}\right) & \cos\lambda t\left(\hat{n}+\frac{3}{2}\right)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \hat{V}_a^{2}\end{pmatrix} \tag{4.44}$$

$$=\begin{pmatrix}\cos\lambda t\left(\hat{n}+\frac{3}{2}\right) & -i\sin\lambda t\left(\hat{n}+\frac{3}{2}\right)\hat{V}_a^2\\ -i\hat{V}_a^{\dagger 2}\sin\lambda t\left(\hat{n}+\frac{3}{2}\right) & \hat{V}_a^{\dagger 2}\cos\lambda t\left(\hat{n}+\frac{3}{2}\right)\hat{V}_a^2\end{pmatrix}.$$

We consider as initial state of the field a coherent state

$$|\alpha\rangle=e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{4.45}$$

and the atom initially in its excited state, we may write the probability of finding the atom in its initial state

$$P_e(t)=\sum_{n=0}^{\infty}P_n\cos^2\left[\lambda t(n+3/2)\right], \tag{4.46}$$

(See Fig. 4.3),where $P_n=\langle n|\,\rho\,|n\rangle$ is the photon distribution for the coherent state, (see equation (2.36)).

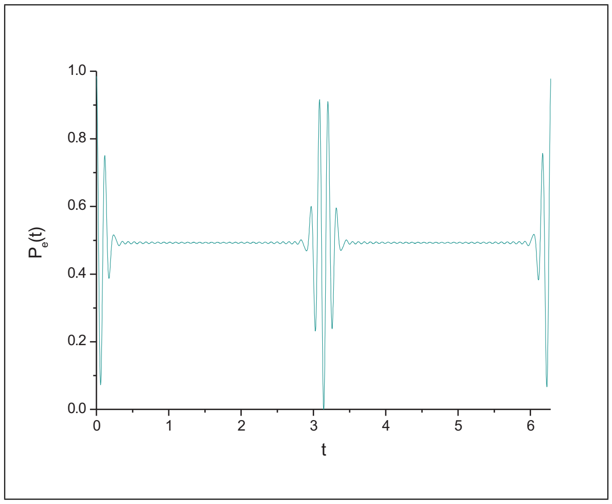

Figure 4.3: *Probability to find the atom in its excited state provided it enters the cavity in the excited state, with* $\alpha = 4$.

We note that the probability of finding the atom in the excited state goes to zero for a time $t = \pi/\lambda$, and then the atom is leaving two photons in the cavity in a clean way, this is, leaving both systems in pure states after their interaction. In case we consider the atom initially in its ground state, the probability to find it in the ground state at time $t = \pi/\lambda$ is also zero. In this case the atom removes in a clean form two photons from the cavity.

Now we consider the interaction time such that $\tau \equiv \lambda t = \pi$ to rewrite the evolution operator as

$$U_a(t) = \begin{pmatrix} 0 & -i\sin\lambda t\left(\hat{n}+\frac{3}{2}\right)\hat{V}_a^2 \\ -i\hat{V}_a^{\dagger 2}\sin\lambda t\left(\hat{n}+\frac{3}{2}\right) & 0 \end{pmatrix}, \tag{4.47}$$

where

$\sin\lambda t\left(\hat{n}+\frac{3}{2}\right) = \sin\pi\hat{n}\cos\frac{3\pi}{2} + \sin\frac{3\pi}{2}\cos\pi\hat{n} = -\left(-1\right)^{\hat{n}}$, is the parity operator, then

$$U_a\left(\frac{\pi}{\lambda}\right) = \begin{pmatrix} 0 & i\left(-1\right)^{\hat{n}}\hat{V}_a^2 \\ i\left(-1\right)^{\hat{n}}\hat{V}_a^{\dagger 2} & 0 \end{pmatrix}. \tag{4.48}$$

Now we consider a second cavity, denoted as B, then the interaction Hamiltonian is:

$$H_b = \lambda \left(\hat{b}^2 \hat{\sigma}_+ + \hat{b}^{\dagger 2} \hat{\sigma}_- \right), \tag{4.49}$$

where $\hat{b}$ and $\hat{b}^\dagger$ are the annihilation and creation operators for the cavity B.

We write this Hamiltonian in matrix form as

$$H_b = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_b^{\dagger 2} \end{pmatrix} \begin{pmatrix} 0 & \lambda \left(\hat{N} + \frac{3}{2} \right) \\ \lambda \left(\hat{N} + \frac{3}{2} \right) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_b^2 \end{pmatrix}, \tag{4.50}$$

where $\hat{N} = \hat{b}^\dagger \hat{b}$ is the number operator. And the evolution operator is

$$U_b = \begin{pmatrix} \cos \lambda t \left(\hat{N} + \frac{3}{2} \right) & -i \sin \lambda t \left(\hat{N} + \frac{3}{2} \right) \hat{V}_b^2 \\ -i \hat{V}_b^{\dagger 2} \sin \lambda t \left(\hat{N} + \frac{3}{2} \right) & \hat{V}_b^{\dagger 2} \cos \lambda t \left(\hat{N} + \frac{3}{2} \right) \hat{V}_b^2 \end{pmatrix}. \tag{4.51}$$

Considering the same interaction time $\tau \equiv \lambda t = \pi$

$$U_b \left(\frac{\pi}{\lambda} \right) = \begin{pmatrix} 0 & i (-1)^{\hat{N}} \hat{V}_b^2 \\ i (-1)^{\hat{N}} \hat{V}_b^{\dagger 2} & 0 \end{pmatrix}. \tag{4.52}$$

4.3.1 Generation of the State $|2\rangle_a |2\rangle_b$

We can generate the state $|2\rangle_a |2\rangle_b$, by starting from two empty cavities and pass an initially excited atom through cavity A, let it interact with the vacuum field a time (π/λ), then the atom leaves the cavity in its ground state (See Fig. 3.1). After it exits cavity A, a classical field is turn on (second Ramsey zone in Fig. 3.2), which produces a rotation in the atom that takes it again to its excited state. Then it passes through cavity B leaving again two photons in it as it exits (at the same interaction time). In this way we pass from the state $|0\rangle_a |0\rangle_b$ to $|2\rangle_a |2\rangle_b$.

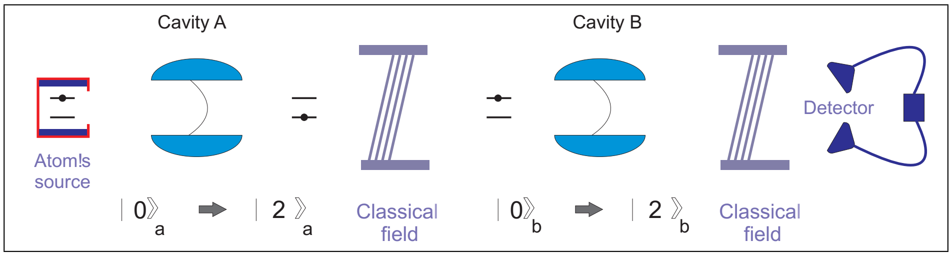

Figure 4.4: Diagram

4.4 NOON States by Entangling the Cavities

We now consider the atom to be in a superposition of its ground an excited states, i.e.

$$|\psi_{atom}\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle + |g\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.53}$$

After interaction with cavity A (which is in a number state with 2 photons in it), we obtain the field-atom entangled state

$$|\psi_{a-f}\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix} |0\rangle_a \\ |4\rangle_a \end{pmatrix}, \tag{4.54}$$

without disturbing the atom, it passes now through the second cavity, which produces the field-atom-field entangled state

$$|\psi_{f-a-f}\rangle = -\frac{1}{2}\begin{pmatrix} |4\rangle_a |0\rangle_b \\ |0\rangle_a |4\rangle_b \end{pmatrix}, \tag{4.55}$$

When the atom exits cavity B, the last Ramsey zone is turn on, rotating the atom, and therefore producing the state

$$|\psi_{f-a-f}\rangle = -\frac{1}{2\sqrt{2}}\begin{pmatrix} |4\rangle_a |0\rangle_b - |0\rangle_a |4\rangle_b \\ |0\rangle_a |4\rangle_b + |4\rangle_a |0\rangle_b \end{pmatrix}. \tag{4.56}$$

Finally, by detecting the atom in its ground state, the wave function is collapsed to an entangled states of the two separate cavities, i.e. to the NOON state

$$|\psi_{f-f}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_a |4\rangle_b + |4\rangle_a |0\rangle_b \right). \tag{4.57}$$

If we consider the detection of the atom in its excited state would have produced a NOON state with a different sign in between.

CHAPTER 5

ION - LASER INTERACTION

In this chapter we consider the Jaynes Cummings model, to describe the ion-trap system in which the ion is considered to have two levels and moves within a region of space much smaller than the effective wavelength of the laser field [35], i.e. the Lamb Dicke limit, the trap's potential is quantized as a harmonic oscillator, and the cooling laser field is supposed as a classical plane wave.

5.1 Ion - laser Interaction

We consider only two energy levels of ion (ground state $|g\rangle$, excited state $|e\rangle$) across the spectrum. The single ion is trapped in a quadrupole electric field vibrating harmonically in the direction x, later we will assume it vibrating in two dimensions x and y. This vibrational motion is coupled to the internal dynamics of the ion, which is accomplished via the interaction with a laser. Then the Hamiltonian that describes the ion-laser interaction is given by

$$H = H_{vib} + H_{ion} + H^{(I)}, \tag{5.1}$$

where H_{vib} is the vibrational energy of the center of mass ion, $\nu_x \hat{a}_x \hat{a}_x^\dagger$, with ν_x the trap frequency, H_{ion} is the internal energy $\frac{1}{2}\omega_{21}\hat{\sigma}_z$ with ω_{21},the transition of frequency. The interaction Hamiltonian is

$$H^{(I)} = -\hat{d}\hat{E} = \lambda \left| E_0 \right| \left[e^{i(k_x\hat{x}-\omega t)}\hat{\sigma}_+ + e^{-i(k_x\hat{x}-\omega t)}\hat{\sigma}_+ + e^{i(k_x\hat{x}-\omega t)}\hat{\sigma}_- + e^{-i(k_x\hat{x}-\omega t)}\hat{\sigma}_- \right], \tag{5.2}$$

where λ is the electric coupling matrix element, we assume that the light field induces coupling between the internal and vibrational states of the ion is a plane wave, whose negative part of the classical electric field is $E^{(-)}(\hat{x}, t) = E_0 e^{i(k_x\hat{x}-\omega t)} + E_0^* e^{-i(k_x\hat{x}-\omega t)}$.

The Hamiltonian of a single ion trapped in a harmonic potential in interaction with laser light in the Rotating Wave Approximation (again we are considering $\hbar = 1$) is

$$H = \nu\hat{a}_x^\dagger\hat{a}_x + \omega_{21}\frac{\hat{\sigma}_z}{2} + \Omega_x \left[e^{-i(k_x\hat{x}-\omega t)}\hat{\sigma}_+ + e^{i(k_x\hat{x}-\omega t)}\hat{\sigma}_- \right], \tag{5.3}$$

where $\Omega_x = \lambda \left| E_0 \right|$ is the Rabi frequency.

Now, we can write

$$k_x\hat{x} = \eta_x \left(\hat{a}_x^\dagger + \hat{a}_x \right), \tag{5.4}$$

where η_x is the Lamb - Dicke parameter, which for a single ion of mass m in the trap frequency ν_x interacting with laser field of wave vector k_x is

$$\eta_x = k_x \sqrt{\frac{1}{2m\nu_x}}. \tag{5.5}$$

This allow us to rewrite the Hamiltonian of the equation (5.3) as

$$H = \nu_x\hat{a}_x^\dagger\hat{a}_x + \omega_{21}\frac{\hat{\sigma}_z}{2} + \Omega_x \left[e^{-i\left[\eta_x\left(\hat{a}_x + a_x^\dagger\right)+\omega t\right]}\hat{\sigma}_+ + e^{i\left[\eta_x\left(\hat{a}_x+\hat{a}_x^\dagger\right)+\omega t\right]}\hat{\sigma}_- \right]. \tag{5.6}$$

5.1.1 Ion Vibrating in Two Dimensions

The Hamiltonian describing a two-dimensional vibrating ion may be easily generalized to the form

$$H = \nu_x\hat{a}_x^\dagger\hat{a}_x + \nu_y\hat{a}_y^\dagger\hat{a}_y + \frac{\omega_{21}}{2}\hat{\sigma}_z + \Omega_x \left[e^{-i\left[\eta_x\left(\hat{a}_x+\hat{a}_x^\dagger\right)+\omega t\right]}\hat{\sigma}_+ + h.c. \right]$$

$$+\Omega_y \left[e^{-i\left[\eta_y\left(\hat{a}_y+\hat{a}_y^\dagger\right)+\omega t\right]}\hat{\sigma}_+ + h.c.\right]. \tag{5.7}$$

We will consider the Hamiltonian of the vibrating ion in two dimensions, because we will generate the N00N state by entangling the vibrational motion of the ion trapped in two dimensions.

5.2 Ion Trapped in the *x* - axis

We consider that the ion is trapped en the $x - axis$, i.e. $\Omega_x \neq 0$ and $\Omega_y = 0$, then

$$H_x = \nu_x \hat{a}_x^\dagger \hat{a}_x + \frac{\omega_{21}}{2}\hat{\sigma}_z + \Omega_x \left[e^{-i\left[\eta_x\left(\hat{a}_x+\hat{a}_x^\dagger\right)+\omega t\right]}\hat{\sigma}_+ + h.c.\right], \tag{5.8}$$

we consider that $\omega_{21} = \omega + \delta$,where δ is the so - called detuning, we obtain

$$H_x = \nu_x \hat{a}_x^\dagger \hat{a}_x + \frac{(\omega+\delta)}{2}\hat{\sigma}_z + \Omega_x \left[e^{-i\left[\eta_x\left(\hat{a}_x+\hat{a}_x^\dagger\right)+\omega t\right]}\hat{\sigma}_+ + h.c.\right], \tag{5.9}$$

we transform to a frame rotating at ω by means of the transformation

$$T = e^{-i\frac{\omega t}{2}}, \tag{5.10}$$

we consider that the Hamiltonian (5.9) corresponds to a wave function $|\psi(t)\rangle$, the Schrödinger equation can be written as

$$i\frac{\partial |\psi(t)\rangle}{\partial t} = H_x |\psi(t)\rangle, \tag{5.11}$$

with $|\psi(t)\rangle = T|\varphi(t)\rangle$, substituting in the equation (5.11)

$$i\frac{\partial T}{\partial t}|\varphi(t)\rangle + iT\frac{\partial |\varphi(t)\rangle}{\partial t} = \left\{ \nu_x \hat{a}_x^\dagger \hat{a}_x + \frac{(\omega+\delta)}{2}\hat{\sigma}_z + \Omega_x \left[e^{-i\left[\eta_x\left(\hat{a}_x+\hat{a}_x^\dagger\right)+\omega t\right]}\hat{\sigma}_+ + h.c.\right]\right\} T|\varphi(t)\rangle,$$

we now multiply by $T^\dagger$

$$e^{-i\omega t}e^{-i\frac{\omega t}{2}\hat{\sigma}_z}\hat{\sigma}_+e^{i\frac{\omega t}{2}\hat{\sigma}_z} = e^{-i\omega t}\left[\hat{\sigma}_+ + i\frac{\omega t}{2}\left(2\hat{\sigma}_+\right) + ...\right] = e^{-i\omega t}\hat{\sigma}_+e^{i\omega t} = \hat{\sigma}_+,$$

we obtain

$$i\frac{\partial\left|\varphi(t)\right\rangle}{\partial t} = \left\{\nu_x\hat{a}_x^\dagger\hat{a}_x + \frac{\delta}{2}\hat{\sigma}_z + \Omega_x\left[e^{-i\left[\eta_x\left(\hat{a}_x+\hat{a}_x^\dagger\right)\right]}\hat{\sigma}_+ + h.c.\right]\right\}\left|\varphi(t)\right\rangle, \tag{5.12}$$

then

$$H_x = \nu_x\hat{a}_x^\dagger\hat{a}_x + \frac{\delta}{2}\hat{\sigma}_z + \Omega_x\left[e^{-i\left[\eta_x\left(\hat{a}_x+\hat{a}_x^\dagger\right)\right]}\hat{\sigma}_+ + e^{i\left[\eta_x\left(\hat{a}_x+\hat{a}_x^\dagger\right)\right]}\hat{\sigma}_-\right], \tag{5.13}$$

using the Baker - Hausdorff formula [36], we factorize the exponentials in the interaction Hamiltonian term

$$e^{-\eta_x\left(\hat{a}_x+\hat{a}_x^\dagger\right)} = e^{-\frac{\eta_x^2}{2}}e^{-i\eta_x\hat{a}_x^\dagger}e^{-i\eta_x\hat{a}_x},$$

then the exponentials can be written in Taylor series:

$$e^{-i\eta_x\hat{a}_x^\dagger} = \sum_n\frac{\left(-i\eta_x\right)^n}{n!}\hat{a}_x^{\dagger n} \qquad and \qquad e^{-i\eta_x\hat{a}_x} = \sum_m\frac{\left(-i\eta_x\right)^m}{m!}\hat{a}_x^m$$

so that we obtain

$$H_x = \nu_x\hat{a}_x^\dagger\hat{a}_x + \frac{\delta}{2}\hat{\sigma}_z + \Omega_x\left[e^{-\frac{\eta_x^2}{2}}\sum_{n,m}\frac{\left(-i\eta_x\right)^n}{n!}\frac{\left(-i\eta_x\right)^m}{m!}\hat{a}_x^{\dagger n}\hat{a}_x^m\hat{\sigma}_+ + h.c.\right], \tag{5.14}$$

we apply the following transformation to eliminate the free terms

$$T_1 = e^{-i\nu_x\left(\hat{a}_x^\dagger \hat{a}_x + k\frac{\hat{\sigma}_z}{2}\right)t} \qquad with \qquad k = \frac{\delta}{\nu_x}, \quad an \quad integer \quad number,$$

so that the Schrödinger equation is

$$i\frac{\partial \left|\varphi(t)\right\rangle}{\partial t} = H_x \left|\varphi(t)\right\rangle, \qquad \left|\varphi(t)\right\rangle = T_1 \left|\phi(t)\right\rangle,$$

so that substituting and then multiplying by $T_1^\dagger$ and by

$$e^{i\nu_x \hat{a}_x^\dagger \hat{a}_x t} \hat{a}_x e^{-i\nu_x \hat{a}_x^\dagger \hat{a}_x t} = \hat{a}_x e^{-i\nu_x t},$$

$$e^{i\nu_x \hat{a}_x^\dagger \hat{a}_x t} \hat{a}_x^\dagger e^{-i\nu_x \hat{a}_x^\dagger \hat{a}_x t} = \hat{a}_x^\dagger e^{i\nu_x t},$$

we obtain

$$i\frac{\partial \left|\varphi(t)\right\rangle}{\partial t} = \Omega_x \left[e^{-\frac{\eta_x^2}{2}} \sum_{n,m} \frac{(-i\eta_x)^n}{n!} \frac{(-i\eta_x)^m}{m!} \left(\hat{a}_x^\dagger e^{i\nu_x t}\right)^n \left(\hat{a}_x e^{-i\nu_x t}\right)^m \hat{\sigma}_+ e^{ik\nu_x t} + h.c. \right] \left|\varphi(t)\right\rangle, \tag{5.15}$$

so the interaction Hamiltonian is

$$H_{Ix} = \Omega_x \left[e^{-\frac{\eta_x^2}{2}} \sum_{n,m} \frac{(-i\eta_x)^n}{n!} \frac{(-i\eta_x)^m}{m!} \hat{a}_x^{\dagger n} \hat{a}_x^m \hat{\sigma}_+ e^{i\nu_x t(n-m+k)} + h.c. \right], \tag{5.16}$$

now we apply the rotating wave approximation, this approach is valid for $\Omega_x << \nu_x$, as Ω_x is proportional to the amplitude of the laser electric field, we say that this approximation is valid for a low-intensity regimen, so all the terms of the Hamiltonian that do not oscillate, it is when $m = \kappa + n$, then

$$H_{Ix} = \Omega_x \left[e^{-\frac{\eta_x^2}{2}} \sum_{n=0}^{\infty} \frac{(-i\eta_x)^n}{n!} \frac{(-i\eta_x)^{k+n}}{(k+n)!} \hat{a}_x^{\dagger n} \hat{a}_x^{k+n} \hat{\sigma}_+ + h.c. \right], \tag{5.17}$$

so that

$$H_{Ix} = \Omega_x \left[e^{-\frac{\eta_x^2}{2}} (-i\eta_x)^k \sum_{n=0}^{\infty} \frac{(-\eta_x)^{2n}}{n!\,(k+n)!} \hat{a}_x^{\dagger n} \hat{a}_x^{k+n} \hat{\sigma}_+ + h.c. \right] \tag{5.18}$$

and by substituting $\hat{a}_x^{\dagger n} \hat{a}_x^n = \frac{\hat{n}!}{(\hat{n}-n)!}$ we obtain

$$H_{Ix} = \Omega_x \left[e^{-\frac{\eta_x^2}{2}} (-i\eta_x)^k \sum_{n=0}^{\hat{n}} \frac{(-1)^n (\eta_x^2)^n}{n!\,(k+n)!} \frac{\hat{n}!}{(\hat{n}-n)!} \hat{a}_x^k \hat{\sigma}_+ + h.c. \right]. \tag{5.19}$$

Now, multiplying by $\frac{(\hat{n}+k)!}{(\hat{n}+k)!}$ and rearranging terms

$$H_{Ix} = \Omega_x \left[e^{-\frac{\eta_x^2}{2}} (-i\eta_x)^k \frac{\hat{n}!}{(\hat{n}+k)!} L_{\hat{n}}^k (\eta_x^2)\, \hat{a}_x^k \hat{\sigma}_+ + h.c. \right], \tag{5.20}$$

where we have identified $L_{\hat{n}}^k (\eta_x^2) = \sum_{n=0}^{\hat{n}} \frac{(-1)^n (\eta_x^2)^n}{n!} \frac{(\hat{n}+k)!}{(n+k)!(\hat{n}-n)!}$, with the associated Laguerre polynomials, so that

$$H_{Ix} = \Omega_x \left(f_x^k (\hat{n})\, \hat{a}_x^k \hat{\sigma}_+ + \hat{a}_x^{\dagger k} f_x^{*k} (\hat{n})\, \hat{\sigma}_- \right), \tag{5.21}$$

where $f_x^k (\hat{n}) = e^{-\frac{\eta_x^2}{2}} (-i\eta_x)^k \frac{\hat{n}!}{(\hat{n}+k)!} L_{\hat{n}}^k (\eta_x^2)$.

If we consider $k = 1$ and $\eta << 1$, which means that the ion oscillation amplitude is much smaller than the laser frequency, we can see that this Hamiltonian is the The Jaynes - Cummings Hamiltonian

$$H_{Ix} = -i\eta_x \Omega_x \left(\hat{a}\hat{\sigma}_+ - \hat{a}^\dagger \hat{\sigma}_- \right). \tag{5.22}$$

This Hamiltonian describes the emission and absorption of a quantum vibrational excitation when the atom undergoes electronic transitions. The first term corresponds to the creation of an excited internal state of the ion and the decay, at the same time, of the vibrational motion in a quantum. The second term involves the ascent in a quantum of the vibrational motion of the ion and the transition, at the same time, of the excited internal state to the ground state.

We write the equation (5.21) in matrix form

$$H_{Ix} = \begin{pmatrix} 0 & \Omega_x f_x^k(\hat{n})\, \hat{a}_x^k \\ \Omega_x \hat{a}_x^{\dagger k} f_x^{*k}(\hat{n}) & 0 \end{pmatrix}, \tag{5.23}$$

which, by using the Susskind - Glogower phase operator, we can write as:

$$H_{Ix} = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_x^{\dagger k} \end{pmatrix} \begin{pmatrix} 0 & \Omega_x f_x^k(\hat{n}) \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \\ \Omega_x f_x^{*k}(\hat{n}) \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_x^k \end{pmatrix} \tag{5.24}$$

or

$$H_{1x} = \begin{pmatrix} 0 & \Omega_x f_x^k(\hat{n}) \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \\ \Omega_x f_x^{*k}(\hat{n}) \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} & 0 \end{pmatrix}. \tag{5.25}$$

The evolution operator is $U_{1x} = e^{-iH_{1x}t}$, may be calculate easily. For this we need

$$H_{1x}^{2m} = \begin{pmatrix} \Omega_x^{2m} \left| f_x^k \left(\hat{n} \right) \right|^{2m} \left(\sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right)^{2m} & 0 \\ 0 & \Omega_x^{2m} \left| f_x^k \left(\hat{n} \right) \right|^{2m} \left(\sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right)^{2m} \end{pmatrix},$$

and

$$H_{1x}^{2m+1} = \begin{pmatrix} \Omega_x^{2m+1} \frac{\left| f_x^k(\hat{n}) \right|^{2m+1} f_x^{*k}(\hat{n})}{\left| f_x^k(\hat{n}) \right|} \left(\sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right)^{2m+1} & 0 \\ 0 & \Omega_x^{2m+1} \frac{\left| f_x^k(\hat{n}) \right|^{2m+1} f_x^{*k}(\hat{n})}{\left| f_x^k(\hat{n}) \right|} \left(\sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right)^{2m+1} \end{pmatrix}.$$

Then

$$U_{1x}\left(t \right) = \sum_m \frac{\left(-it \right)^{2m}}{(2m)!} H_{1x}^{2m} + \sum_m \frac{\left(-it \right)^{2m+1}}{(2m+1)!} H_{1x}^{2m+1},$$

or

$$U_{1x}\left(t \right) = \begin{pmatrix} \cos\left(\Omega_x t \left| f_x^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right) & -i \frac{f_x^k(\hat{n})}{\left| f_x^k(\hat{n}) \right|} \sin\left(\Omega_x t \left| f_x^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right) \\ -i \frac{f_x^{*k}(\hat{n})}{\left| f_x^k(\hat{n}) \right|} \sin\left(\Omega_x t \left| f_x^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right) & \cos\left(\Omega_x t \left| f_x^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right) \end{pmatrix}. \tag{5.26}$$

Therefore the evolution operator $U_{Ix}(t)$ is:

$$U_{Ix}\left(t \right) = \begin{pmatrix} \cos\left(\Omega_x t \left| f_x^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right) & -i \left(-i \right)^k \sin\left(\Omega_x t \left| f_x^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right) \hat{V}_x^k \\ -i \hat{V}_x^{\dagger k} \left(i \right)^k \sin\left(\Omega_x t \left| f_x^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right) & \hat{V}_x^{\dagger k} \cos\left(\Omega_x t \left| f_x^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right) \hat{V}_x^k \end{pmatrix}, \tag{5.27}$$

for the element (22) $\hat{V}_x^{\dagger k} Cos \left(\Omega_x t \left| f_x^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}} \right) \hat{V}_x^k$ must have

$$= \cos\left(\Omega_x t \left|f_x^k(\hat{n}-k)\right| \sqrt{\frac{\hat{n}!}{(\hat{n}-k)!}} \left(1 - |0\rangle\langle 0| - \cdots - |k-1\rangle\langle k-1|\right)\right)$$

If the ion is initially in the ground state and the vibrational motion state has elements whose number states are less than k, this element can be ignored as there will be no contributions of it $\hat{V}_x^k(|k-1\rangle, |k-2\rangle, etc.) = 0$.

Now we consider as initial state of the vibrational motion a coherent state and the ion initially in its excited state, we may write the probability of finding the ion in its initial state

$$W_x(t) = \sum_{n=0}^{\infty} P_n \cos^2\left(\Omega_x t \left|f_x^k(\hat{n})\right| \sqrt{\hat{a}_x^k \hat{a}_x^{\dagger k}}\right). \tag{5.28}$$

Giving values for k.

1) For $k = 1$,

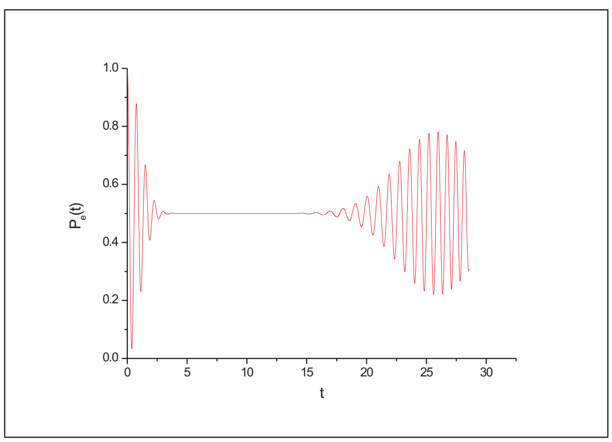

Figure 5.1: *We plot the ion-laser interaction, with the ion is in excited state and the ionic vibration is initially in a coherent state, with $\alpha = 4$. We can observe that this plot is similar for the case atom-field interaction.*

5.3 Ion Trapped in the *y* - axis

Now we consider that the ion is trapped in the $y-axis$, i.e. $\Omega_x = 0$ y $\Omega_y \neq 0$, so

$$H_{Iy} = \Omega_y \left(f_y^k \left(\hat{n} \right) \hat{a}_y^k \hat{\sigma}_+ + \hat{a}_y^{\dagger k} f_y^{*k} \left(\hat{n} \right) \hat{\sigma}_- \right), \tag{5.29}$$

where$f_y^k \left(\hat{n} \right) = e^{-\frac{\eta_y^2}{2}} \left(-i\eta_y \right)^k \frac{\hat{n}!}{(\hat{n}+k)!} L_{\hat{n}}^k \left(\eta_y^2 \right)$ with $L_{\hat{n}}^k \left(\eta_y^2 \right) = \sum_{n=0}^{\hat{n}} \frac{(-1)^n \left(\eta_y^2 \right)^n}{n!} \frac{(\hat{n}+k)!}{(n+k)!(\hat{n}-n)!}$

we again use Susskind - Glogower phase operators:

$$H_{Iy} = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_y^{\dagger k} \end{pmatrix} \begin{pmatrix} 0 & \Omega_y f_y^k \left(\hat{n} \right) \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}} \\ \Omega_y f_y^{*k} \left(\hat{n} \right) \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{V}_y^k \end{pmatrix} \tag{5.30}$$

and

$$H_{1y} = \begin{pmatrix} 0 & \Omega_y f_y^k \left(\hat{n} \right) \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}} \\ \Omega_y f_y^{*k} \left(\hat{n} \right) \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}} & 0 \end{pmatrix} \tag{5.31}$$

where the evolution operator $U_{1y} = e^{-iH_{1y}t}$ is

$$U_{1y}\left(t \right) = \begin{pmatrix} \cos\left(\Omega_y t \left| f_y^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}} \right) & -i \frac{f_y^k(\hat{n})}{\left| f_y^k(\hat{n}) \right|} \sin\left(\Omega_y t \left| f_y^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}} \right) \\ -i \frac{f_y^{*k}(\hat{n})}{\left| f_y^k(\hat{n}) \right|} \sin\left(\Omega_y t \left| f_y^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}} \right) & \cos\left(\Omega_y t \left| f_y^k \left(\hat{n} \right) \right| \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}} \right) \end{pmatrix}, \tag{5.32}$$

therefore the evolution operator U_{Iy} is

$$U_{Iy}(t) = \begin{pmatrix} \cos\left(\Omega_y t \left|f_y^k(\hat{n})\right| \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}}\right) & -i(-i)^k \sin\left(\Omega_y t \left|f_y^k(\hat{n})\right| \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}}\right) \hat{V}_y^k \\ -i\hat{V}_y^{\dagger k}(i)^k \sin\left(\Omega_y t \left|f_y^k(\hat{n})\right| \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}}\right) & \hat{V}_y^{\dagger k} \cos\left(\Omega_y t \left|f_y^k(\hat{n})\right| \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}}\right) \hat{V}_y^k \end{pmatrix}. \tag{5.33}$$

Now, we consider that the ion is in an excited state and the ionic vibration is in a coherent state, the probability for the excited state is:

$$W_y(t) = \sum_{n=0}^{\infty} P_n \cos^2\left(\Omega_y t \left|f_y^k(\hat{n})\right| \sqrt{\hat{a}_y^k \hat{a}_y^{\dagger k}}\right). \tag{5.34}$$

5.3.1 Particular Case

We consider the ion trapped in the $x - axis$ and also that $\eta_x << 1$ with $k = 4$ phonons, so that $f_x^4(\hat{n}) = \frac{(-i\eta_x)^4}{4!}$, the interaction Hamiltonian is:

$$H_{Ix} = \Omega_x \left(f_x^4(\hat{n})\, \hat{a}_x^4 \hat{\sigma}_+ + \hat{a}_x^{\dagger 4} f_x^{*4}(\hat{n})\, \hat{\sigma}_-\right), \tag{5.35}$$

we note that this Hamiltonian is similar to the Hamiltonian for the cavity A, where the corresponding evolution operator is:

$$U_{Ix}(t) = \begin{pmatrix} \cos\left(\Omega_x t \left(\frac{\eta_x^4}{4!}\right) \sqrt{\hat{a}_x^4 \hat{a}_x^{\dagger 4}}\right) & -i \sin\left(\Omega_x t \left(\frac{\eta_x^4}{4!}\right) \sqrt{\hat{a}_x^4 \hat{a}_x^{\dagger 4}}\right) \hat{V}_x^4 \\ -i\hat{V}_x^{\dagger 4} \sin\left(\Omega_x t \left(\frac{\eta_x^4}{4!}\right) \sqrt{\hat{a}_x^4 \hat{a}_x^{\dagger 4}}\right) & \hat{V}_x^{\dagger 4} \cos\left(\Omega_x t \left(\frac{\eta_x^4}{4!}\right) \sqrt{\hat{a}_x^4 \hat{a}_x^{\dagger 4}}\right) \hat{V}_x^4 \end{pmatrix} \tag{5.36}$$

and replacing

$$\sqrt{\hat{a}_x^4\hat{a}_x^{\dagger 4}} = \sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}} = \sqrt{(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)(\hat{n}+4)}.$$

So that, the evolution operator is

$$U_{Ix}(t) = \begin{pmatrix} \cos\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\right)\sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}}\right) & -i\sin\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\right)\sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}}\right)\hat{V}_x^4 \\ -i\hat{V}_x^{\dagger 4}\sin\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\right)\sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}}\right) & \hat{V}_x^{\dagger 4}\cos\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\right)\sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}}\right)\hat{V}_x^4 \end{pmatrix}. \tag{5.37}$$

We consider the ion in its excited state and the ionic vibration is initially prepared in a coherent state, the probability of finding the ion in its initial state is

$$W_x(t) = \cos^2\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\sqrt{\frac{n!}{(n-4)!}}\right)\right). \tag{5.38}$$

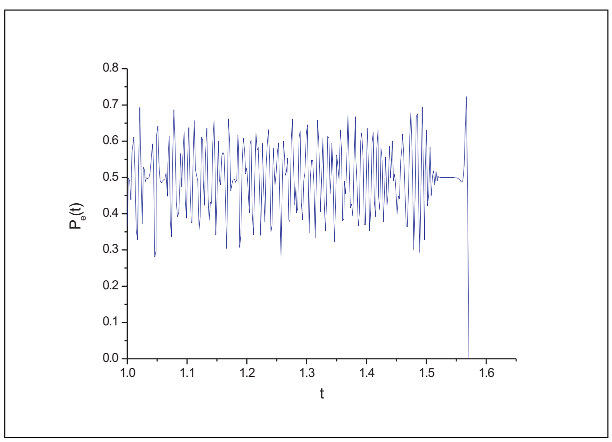

Figure 5.2: *Probability to find the ion in its excited state, with* $\alpha = 4$.

We observe that the probability of finding the ion in the excited state for a time $t = \pi/2\Omega$ goes to zero, where $\Omega = \Omega_x\left(\frac{\eta_x^4}{4!}\right)$, and then the ion is giving four phonons to the vibrational motion. Now, if we consider the ion initially in its ground state, the probability to find it in the ground state at the same time $t = \pi/2\Omega$ is also zero. In this case the ion removes four phonons of the vibrational motion.

Now we consider the interaction time such that $\tau \equiv \Omega t = \pi/2$ to rewrite the evolution operator as

$$U_{Ix}\left(\frac{\pi}{2\Omega}\right) = \begin{pmatrix} 0 & -i\left(-1\right)^{\hat{n}}\hat{V}_x^4 \\ -i\left(-1\right)^{\hat{n}}\hat{V}_x^{\dagger 4} & 0 \end{pmatrix}. \tag{5.39}$$

We consider the ion trapped in the $y-axis$, the interaction Hamiltonian is

$$H_{Iy} = \Omega_y\left(f_y^4\left(\hat{n}\right)\hat{a}_y^4\hat{\sigma}_+ + \hat{a}_y^{\dagger 4}f_y^{*4}\left(\hat{n}\right)\hat{\sigma}_-\right). \tag{5.40}$$

And the evolution operator considering the same interaction time $\tau \equiv \Omega t = \pi/2$ is:

$$U_{Iy}\left(\frac{\pi}{2\Omega}\right) = \begin{pmatrix} 0 & -i\left(-1\right)^{\hat{N}}\hat{V}_y^4 \\ -i\left(-1\right)^{\hat{N}}\hat{V}_y^{\dagger 4} & 0 \end{pmatrix}. \tag{5.41}$$

5.3.2 Generation of the State $\left|4\right\rangle_x\left|4\right\rangle_y$

We generate the state $\left|4\right\rangle_x\left|4\right\rangle_y$, we consider the ion trapped in the $x-axis$ and initially in excited state and the vibrational motion in a vacuum state, after of a time interaction $\left(\frac{\pi}{2\Omega}\right)$, the ion passes to its ground state losing 4 phonons. After it interacts with the vacuum field, which produces a rotation in the ion leaving again in its excited state. Then we consider the ion trapped in the $y-axis$ losing once again 4 phonons (at the same time interaction). In this way we pass from the state $\left|0\right\rangle_x\left|0\right\rangle_y$ to $\left|4\right\rangle_x\left|4\right\rangle_y$.

5.4 N00N States by Trapping Ions

We consider the ion to be in a superposition of its ground an excited states, i.e.

$$\left|\psi_{ion}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|e\right\rangle + \left|g\right\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.42}$$

We are considering the vibrational motion en pure state with 4 phonons, using the conditions for the ion trapped in the $x - axis$, and the time interaction $\frac{\pi}{2}$, we obtain the entangled state

$$|\psi_{i-vm}\rangle = \frac{i}{\sqrt{2}} \begin{pmatrix} |0\rangle_x \\ |8\rangle_x \end{pmatrix}, \tag{5.43}$$

Now we are considering the ion trapped in the $y - axis$, at the same time intarection, we obtain the vm-ion-vm entangled state

$$|\psi_{vm-ion-vm}\rangle = -\frac{1}{2} \begin{pmatrix} |8\rangle_x |0\rangle_y \\ |0\rangle_x |8\rangle_y \end{pmatrix}, \tag{5.44}$$

after the ion interacts with a vacuum field, rotating the ion, we obtain the state

$$|\psi_{vm-ion-vm}\rangle = -\frac{1}{2\sqrt{2}} \begin{pmatrix} |8\rangle_x |0\rangle_y - |0\rangle_x |8\rangle_y \\ |0\rangle_x |8\rangle_y + |8\rangle_x |0\rangle_y \end{pmatrix}. \tag{5.45}$$

Finally, by detecting the ion in its ground state, we obtain the NOON state

$$|\psi_{vm-vm}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_x |8\rangle_y + |8\rangle_x |0\rangle_y \right). \tag{5.46}$$

CHAPTER 6

CONCLUSIONS

It has been presented the basic elements of the quantization of the radiation field, which is written in terms of creation $\hat{a}^{\dagger}$ and annihilation $\hat{a}$ operators.

It has been shown some of the states, single-mode radiation field, such as number states and coherent states, as well as states of two modes of radiation field as N00N states.

It has been analyzed the interaction of an atom with a single mode field in the picture of semiclassical theory.

Making use of transformations and the Susskind - Glogower operators, we solve the Hamiltonian of interaction of an atom of two levels with two photons.

It has been shown that by controlling the interaction time between of atoms passing through two cavities, the cavity fields may be entangled. A final detection of the atom in its excited or ground state yields a NOON state, i.e. and entangled state of both cavities.

It has been shown the generation of NOON states by entangling the vibrational motion of an ion trapped in two dimensions. In particular we had generated the N00N state $\frac{1}{\sqrt{2}}\left(|0\rangle_x |8\rangle_y + |8\rangle_x |0\rangle_y\right)$.

We note that one can build, in both cases (cavities entangled and trapped ions) any N00N state, this is accomplished by generalizing the respective interaction Hamiltonians;

a)$H_I = \lambda \left(\hat{a}^k \hat{\sigma}_+ + \hat{a}^{\dagger k} \hat{\sigma}_- \right)$,

b)$H_I = \Omega \left(\hat{a}^k \hat{\sigma}_+ + \hat{a}^{\dagger k} \hat{\sigma}_- \right)$.

6.1 Papers

∗ D. Rodríguez-Méndez and H. Moya-Cessa, **"VERY HIGH NOON STATES IN TRAPPED IONS"**, accepted, (2011).

∗ B.M. Rodríguez-Lara, D. Rodríguez-Méndez, H. Moya-Cessa, **"SOLUTION TO THE LANDAU-ZENER PROBLEM VIA SUSSKIND-GLOGOWER OPERATORS"**, Physics Letters A (2011), in press arXiv:1105.4013 [quant-ph].

∗ D. Rodríguez-Méndez, O. Aguilar-Loreto, H. Moya-Cessa, **"MIRROR-FIELD-ATOM INTERACTION: HAMILTONIAN DIAGONALIZATION"**. Rev. Mex. Fís. S 57, (2011).

∗ D. Rodríguez-Méndez and H. Moya-Cessa, **"NOON STATES IN ENTANGLED CAVITIES"**. Optics Communications, 284, 3345-3347, (2011).

∗ D. Rodríguez-Méndez, H. Moya-Cessa and O. Aguilar-Loreto, **"Estudio de la interacción entre un campo cuantizado, un espejo y un átomo de dos niveles"**. XXIII Reunión Anual de Óptica, http://mictlan.utm.mx/rao23/iso/mautor.html Puebla, LIII Congreso Nacional de Física, Boca del Río, Veracruz and Undécimo Encuentro de Investigación, pp. 51-53, INAOE, Puebla, (2010).

CHAPTER 7

RESUMEN EN EXTENSO

7.1 Estados NOON en cavidades

En el siguiente trabajo de tesis en la primera parte, mostramos como los estados NOON pueden ser generados entrelazando dos cavidades pasando los átomos a través de ellas. Los átomos interactuan con cada cavidad via transiciones resonantes de dos fotones. Tomamos ventaja del hecho que dependiendo en el estado en que el átomo entra (excitado ó base), deja o toma dos fotones por interacción, dejando a las cavidades en estado puro.

Un estado NOON es un estado maximalmente entrelazado. Se escriben de la forma

$$|N00N\rangle_{A,B} \propto (|N\rangle_A |0\rangle_B + |0\rangle_A |N\rangle_B), \tag{7.1}$$

En el caso de cavidades [3], lo cuál se estudia en una parte de esta tésis, representa una superposición de N fotones en la primera cavidad (*A*) y vacía la segunda cavidad (*B*), ó viceversa, donde una constante de normalización de $\frac{1}{\sqrt{2}}$ no se escribió por conveniencia (y por convención). Son estados maximalmente entrelazados en el sentido de que el estado consiste de una superposición con componentes donde todos los N fotones estan en una cavidad (modo ó camino) ó en la otra. Los estados son entrelazados porque de ninguna manera se puede escribir como un producto de estados de las dos cavidades (modos o caminos) separadas; es decir $|N00N\rangle_{A,B} \neq |\psi_A\rangle \otimes |\psi_B\rangle$. Los estados N00N exhiben fascinantes propiedades de interferencia cuántica, los estados N00N son considerados los estados cuánticos óptimos de la luz para aplicaciones de metrología cuántica tales como litografía cuántica.

7.2 Dinámica de dos - fotones

En esta sección se estudia el comportamiento atómico cuando la luz interactúa con la materia en una transición de resonancia de dos fotones. Consideramos el Hamiltoniano de interacción de dos fotones [33]

$$H_I = \lambda \left(a^2 \sigma_+ + a^{\dagger 2} \sigma_- \right) \tag{7.2}$$

donde λ es la constante de acoplamiento, a y $a^\dagger$ son los operadores de aniquilación y creación para el modo del campo (cavidad A), respectivamente, $\sigma_+ = |e\rangle \langle g|$ y $\sigma_- = |g\rangle \langle e|$, son las matrices de spin de Pauli para las transiciones de dos fotones, aquí ($|e\rangle$ $|g\rangle$) significa estado atomico excitado (base). Consideramos que el estado intermedio está tan lejos de la resonancia que puede ser eliminado adiabáticamente para dar un acoplamiento efectivo de dos fotones de la forma anterior.

El operador de evolución está dado por

$$U_I^{(a)}(t) = e^{-iH_I t} = \begin{pmatrix} \cos\left(\lambda t \sqrt{a^2 a^{\dagger 2}}\right) & -i \sin\left(\lambda t \sqrt{a^2 a^{\dagger 2}}\right) V_a^2 \\ -i V_a^{\dagger 2} \sin\left(\lambda t \sqrt{a^2 a^{\dagger 2}}\right) & \cos\left(\lambda t \sqrt{a^{\dagger 2} a^2}\right) \end{pmatrix}, \tag{7.3}$$

con $V_a = \frac{1}{\sqrt{aa^\dagger}} a$, es el llamado operador de fase de Susskind-Glogower [34].

Si consideramos como estado inicial del campo un estado coherente

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{7.4}$$

y el atomo inicialmente en su estado excitado, podemos escribir la probabilidad de encontrar el átomo en su estado inicial, como

$$P_e(t) = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} Q_n \cos\left[2\lambda t\sqrt{(n+1)(n+2)}\right], \tag{7.5}$$

donde $Q_n = \langle n|\,\rho\,|n\rangle$ es la distribución de fotones para el estado que se esta considerando. Graficamos en la Fig. 1 la probabilidad de encontrar el átomo en el estado escitado. Podemos notar que la probabilidad de encontrar el atomo en el estado excitado tiende a cero para un tiempo $t = {}^{\pi}/_{\lambda}$, entonces el atomo esta dejando dos fotones en la cavidad de manera *limpia*, esto es, dejando ambos sistemas en estados puros después de su interacción.

En el caso que consideremos el átomo inicialmente en su estado base, se muestra que la probabiliad de encontrarlo en el estado base en el tiempo $t = {}^{\pi}/_{\lambda}$ también es cero. En este caso el átomo quita en una forma limpia dos fotones de la cavidad.

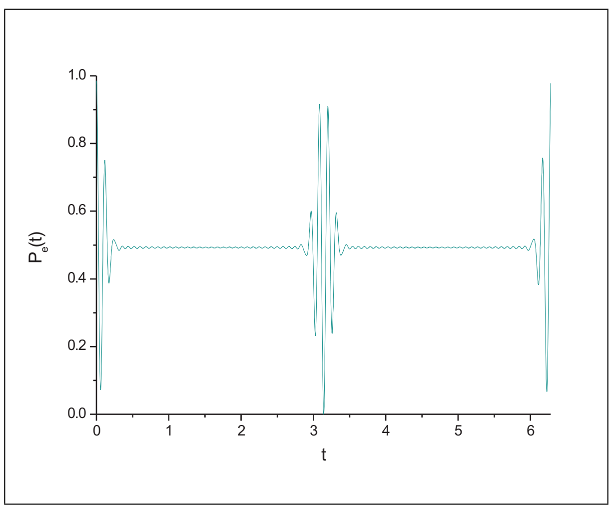

Figure 7.1: *Probabilidad de encontrar el átomo en su estado excitado siempre que entra en la cavidad en el estado excitado, con* $\alpha = 4$.

usaremos la siguiente aproximación $\sqrt{a^2 a^{\dagger 2}} = \sqrt{(\hat{n}+1)(\hat{n}+2)} \approx \hat{n} + {}^{3}/_{2}$, que si bien es para $n's$ grandes, es también buena para $n = 0$ y considerando el tiempo de interacción de tal manera que $\tau \equiv \lambda t = \pi$ reescribimos el operador de evolución como

$$U_I^{(a)}\left(\frac{\pi}{\lambda}\right) \approx \begin{pmatrix} 0 & i\left(-1\right)^{\hat{n}} V_a^2 \\ iV_a^{\dagger 2}\left(-1\right)^{\hat{n}} & 0 \end{pmatrix}. \tag{7.6}$$

Si consideramos una segunda cavidad, denotadda como B en Fig. (7.2). Denotamos los operadores de aniquilación y creación por b y $b^{\dagger}$ y el operador de número $\hat{N} = b^{\dagger}b$ de manera que escribimos el operador de evolución para la segunda cavidad también para un tiempo de interacción (π/λ) como

$$U_I^{(b)}\left(\frac{\pi}{\lambda}\right) \approx \begin{pmatrix} 0 & i\left(-1\right)^{\hat{N}} V_b^2 \\ iV_b^{\dagger 2}\left(-1\right)^{\hat{N}} & 0 \end{pmatrix} \tag{7.7}$$

7.2.1 Generación del estado $|2\rangle_a |2\rangle_b$

Podemos generar el estado $|2\rangle_a |2\rangle_b$ si partimos de dos cavidades vacias y pasa inicialmente un átomo en estado excitado a través de la cavidad A, se deja interactuar con el campo vacío en un tiempo (π/λ), enotnces el átomo sale de la cavidad en su estado base (Ver Fig. 7.2). Después de salir de la cavidad A, un campo clasico es encendido, el cuál produce una rotación en el átomo y regresa una vez más a su estado excitado. Entonces para a través de la cavidad B dejando una vez más dos fotones en ella y sale (en el mismo tiempo de interacción). De esta manera pasamos de el estado $|0\rangle_a |0\rangle_b$ al estado $|2\rangle_a |2\rangle_b$.

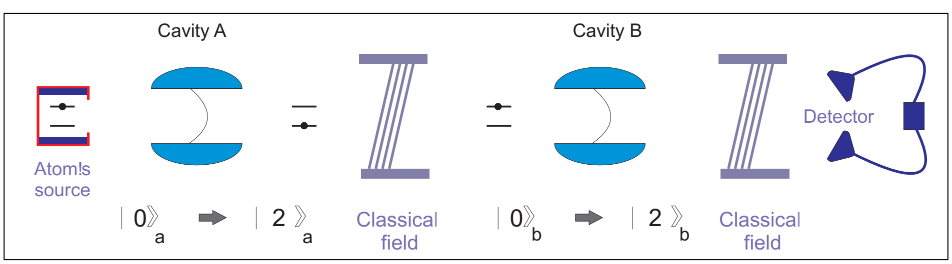

Figure 7.2: Diagrama

7.3 Estados NOON por entrelazamiento de cavidades

Ahora consideramos que el átomo esta en una superposición de sus estados base y excitado, es decir

$$|\psi_{atomo}\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle + |b\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{7.8}$$

Después de la interacción con la cavidad A (la cuál esta enun estado de número con 2 fotones), obtenemos el estado entrelazado campo-atomo

$$|\psi_{a-c}\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix} |0\rangle_a \\ |4\rangle_a \end{pmatrix}, \tag{7.9}$$

sin perturbar el átomo, éste ahora pasa a través de la segunda cavidad, produciendo el estado entrelazado campo-atomo-campo

$$|\psi_{c-a-c}\rangle = -\frac{1}{2}\begin{pmatrix} |4\rangle_a |0\rangle_b \\ |0\rangle_a |4\rangle_b \end{pmatrix}, \tag{7.10}$$

Cuando el átomo sale de la cavidad B, la última zona de Ramsey se enciende, rotando el átomo y por lo tanto produce el estado

$$|\psi_{c-a-c}\rangle = -\frac{1}{2\sqrt{2}}\begin{pmatrix} |4\rangle_a |0\rangle_b - |0\rangle_a |4\rangle_b \\ |0\rangle_a |4\rangle_b + |4\rangle_a |0\rangle_b \end{pmatrix}. \tag{7.11}$$

Finalmente, detectando el átomo en su estado base, la función de onda se colapsa a un estado entrelazado de las dos cavidades separadas, es decir a el estado NOON.

$$|\psi_{c-c}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_a |4\rangle_b + |4\rangle_a |0\rangle_b\right). \tag{7.12}$$

7.4 Estados NOON en Iones

En la segunda parte de estas tesis consideramos el Hamiltoniano de un ion vibrando en dos dimensiones interactuando con un láser. Se muestra como los estados N00N se pueden generar en trampas de iones. Usamos la interacción individual de la luz con cada uno de los dos modos vibracionales del ion para entrelazarlos. Esto nos permite generar el estado N00N con N=8.

7.5 Caso Particular

Consideramos el ion atrapado en el eje x y con $\eta_x << 1$ y $k = 4$ fotones, y además que $f_x^4(\hat{n}) = \frac{(-i\eta_x)^4}{4!}$, el Hamiltoniano de interacción es:

$$H_{Ix} = \Omega_x \left(f_x^4(\hat{n})\, \hat{a}_x^4 \hat{\sigma}_+ + \hat{a}_x^{\dagger 4} f_x^{*4}(\hat{n})\, \hat{\sigma}_- \right), \tag{7.13}$$

Notamos que el Hamiltoniano es similar al Hamiltoniano para la cavidad A, donde el correspondiente operador de evolución es:

$$U_{Ix}(t) = \begin{pmatrix} \cos\left(\Omega_x t \left(\frac{\eta_x^4}{4!}\right) \sqrt{\hat{a}_x^4 \hat{a}_x^{\dagger 4}}\right) & -i \sin\left(\Omega_x t \left(\frac{\eta_x^4}{4!}\right) \sqrt{\hat{a}_x^4 \hat{a}_x^{\dagger 4}}\right) \hat{V}_x^4 \\ -i\hat{V}_x^{\dagger 4} \sin\left(\Omega_x t \left(\frac{\eta_x^4}{4!}\right) \sqrt{\hat{a}_x^4 \hat{a}_x^{\dagger 4}}\right) & \hat{V}_x^{\dagger 4} \cos\left(\Omega_x t \left(\frac{\eta_x^4}{4!}\right) \sqrt{\hat{a}_x^4 \hat{a}_x^{\dagger 4}}\right) \hat{V}_x^4 \end{pmatrix} \tag{7.14}$$

y reemplazando

$$\sqrt{\hat{a}_x^4 \hat{a}_x^{\dagger 4}} = \sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}} = \sqrt{(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)(\hat{n}+4)}.$$

Asi que, el operador de evolución es

$$U_{Ix}(t) = \begin{pmatrix} \cos\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\right)\sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}}\right) & -i\sin\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\right)\sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}}\right)\hat{V}_x^4 \\ -i\hat{V}_x^{\dagger 4}\sin\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\right)\sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}}\right) & \hat{V}_x^{\dagger 4}\cos\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\right)\sqrt{\frac{\hat{n}!}{(\hat{n}-4)!}}\right)\hat{V}_x^4 \end{pmatrix}. \tag{7.15}$$

Consideramos el ion en su estado excitado y la vibración ionica esra inicialmente preparada en un estado coherente, la probabilidad de encontrar el ion en su estado inicial es

$$W_x(t) = \cos^2\left(\Omega_x t\left(\frac{\eta_x^4}{4!}\sqrt{\frac{n!}{(n-4)!}}\right)\right). \tag{7.16}$$

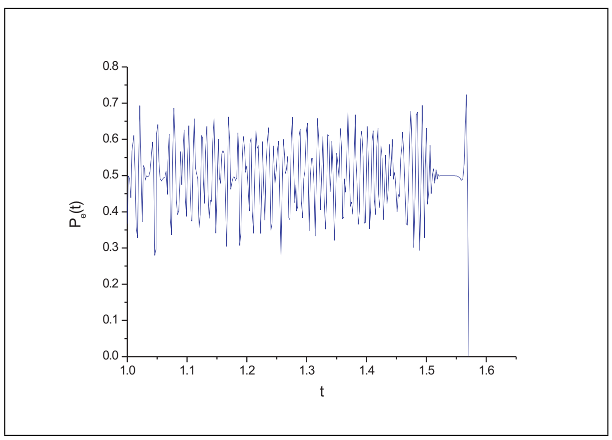

Figure 7.3: *Probabilidad de encontrar el ion en su estado excitado, con $\alpha = 4$.*

Observamos que la probabilidad de encontrar el ion en el estado excitado para un tiempo $t = \pi/2\Omega$ tiende a cero, donde $\Omega = \Omega_x\left(\frac{\eta_x^4}{4!}\right)$, entonces el ion esta dando cuatro fonones a el movimiento vibracional. Ahora, si consideramos el ion inicialmente en su estado base, la probabilidad de encontrarlo en el estado base al mismo tiempo $t = \pi/2\Omega$ es también cero. En este caso el ion quita 4 fonones a el movimiento vibracional.

Ahora consideramos el tiempo de interacción de manera que $\tau \equiv \Omega t = \pi/2$ reescribimos el operador de evolución como:

$$U_{Ix}\left(\frac{\pi}{2\Omega}\right)=\left(\begin{array}{cc} 0 & -i\left(-1\right)^{\hat{n}}\hat{V}_x^4 \\ -i\left(-1\right)^{\hat{n}}\hat{V}_x^{\dagger 4} & 0 \end{array}\right). \tag{7.17}$$

Consideramos el ion atrapado en el eje y, el Hamiltoniano de interacción es

$$H_{Iy}=\Omega_y\left(f_y^4\left(\hat{n}\right)\hat{a}_y^4\hat{\sigma}_+ + \hat{a}_y^{\dagger 4}f_y^{*4}\left(\hat{n}\right)\hat{\sigma}_-\right). \tag{7.18}$$

Y el operador de evolución considerando el mismo tiempo de interacción $\tau\equiv\Omega t=\pi/2$ es:

$$U_{Iy}\left(\frac{\pi}{2\Omega}\right)=\left(\begin{array}{cc} 0 & -i\left(-1\right)^{\hat{N}}\hat{V}_y^4 \\ -i\left(-1\right)^{\hat{N}}\hat{V}_y^{\dagger 4} & 0 \end{array}\right). \tag{7.19}$$

7.5.1 Generación del estado $|4\rangle_x|4\rangle_y$

Generamos el estado $|4\rangle_x|4\rangle_y$, consideramos el ion atrapado en el eje x e inicialmente en estado excitado y el movimiento vibracional en estado vacío, después de un tiempo de interacción $(\frac{\pi}{2\Omega})$, el ion pasa de su estado excitado a su estado base perdiendo 4 fonones. Después este interactúa con un campo vacío, el cuál produce una rotación en el ion dejandolo una vez más en su estado excitado. Después consideramos el ion atrapado en el eje y perdiendo una vez mas 4 fonones (en el mismo tiempo de interacción). De ésta forma pasamos del estado $|0\rangle_x|0\rangle_y$ al estado $|4\rangle_x|4\rangle_y$.

7.6 Estados N00N en Ions Atrapados

Consideramos que el ion esta en una superposicion de sus estado excitado y base, es decir

$$|\psi_{ion}\rangle=\frac{1}{\sqrt{2}}\left(|e\rangle+|g\rangle\right)=\frac{1}{\sqrt{2}}\left(\begin{array}{c}1\\1\end{array}\right). \tag{7.20}$$

Estamos considerando el movimiento vibracional en un estado puro con 4 fonones, usando las condiciones para el ion atrapado en el eje x, y el tiempo de interacción $\frac{\pi}{2}$, obtenemos el estado entrelazado

$$|\psi_{i-vm}\rangle = \frac{i}{\sqrt{2}} \begin{pmatrix} |0\rangle_x \\ |8\rangle_x \end{pmatrix}, \tag{7.21}$$

Ahora estamos considerando el ion atrapado en el eje y, en el mismo tiempo de interacción, obtenemos el estado entrelazado mv-ion-mv

$$|\psi_{vm-ion-vm}\rangle = -\frac{1}{2} \begin{pmatrix} |8\rangle_x |0\rangle_y \\ |0\rangle_x |8\rangle_y \end{pmatrix}, \tag{7.22}$$

después el ion interactúa con un campo vacío, rotando el ion, obteniendo el estado

$$|\psi_{mv-ion-mv}\rangle = -\frac{1}{2\sqrt{2}} \begin{pmatrix} |8\rangle_x |0\rangle_y - |0\rangle_x |8\rangle_y \\ |0\rangle_x |8\rangle_y + |8\rangle_x |0\rangle_y \end{pmatrix}. \tag{7.23}$$

Finalmente, detectando el ion en su estado base, obtenemos el estado NOON

$$|\psi_{mv-mv}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_x |8\rangle_y + |8\rangle_x |0\rangle_y\right). \tag{7.24}$$

7.7 Conclusiones

Se ha mostrado que controlando el tiempo de interacción entre los atomos pasando a través de las dos cavidades, los campos puedes ser entrelazados. Una detección final de el átomo en su estado excitado o base nos conduce al estado NOON, es decir u estado entrelazado de ambas cavidades.

Se ha mostrado que la generación de estados NOON por entrelazamiento del movimiento vibracional de un ion atrapado en dos dimensiones. En particular hemos generado el estado N00N $\frac{1}{\sqrt{2}}\left(\left|0\right\rangle_x\left|8\right\rangle_y+\left|8\right\rangle_x\left|0\right\rangle_y\right)$.

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