Periodic and quasi-periodic non-diffracting wave fields generated by superposition of multiple Bessel beams

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Abstract: We discuss a computer generated hologram whose transmittance is defined in terms of the Jacobi-Anger identity. If the hologram is implemented with a continuous phase spatial light modulator it generates integer-order non-diffracting Bessel beams, with a common asymptotic radial frequency, at separated propagation axes. On the other hand, when the hologram is implemented with a low-resolution pixelated phase modulator, it is possible to generate multiple Bessel beams with a common propagation axis. We employ this superposition of multiple Bessel beams to generate non-diffracting periodic and quasi-periodic wave fields.

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1. Introduction

Non-diffracting Bessel beams have attracted the attention of Optics researchers through the last three decades due to their invariant properties and their applications [1-6]. Classic techniques to synthesize Bessel beams are based on the use of axicons [7] and annular slits [8]. Unfortunately, the use of these approaches restricts the order of the generated beams. In addition, the efficiency of the second approach is extremely low. A different method, based on the use of computer generated holograms (CGHs) [9-12], allows the generation of arbitrary complex fields. A particular interesting task is the simultaneous generation of multiple non-diffracting beams. Obvious methods that perform this task are based on the sampling of an annular slit [13] and in the use of an array of microaxicons [14]. A different method, proposed by Kotlyar et al [15], employs an iterative algorithm to encode multiple diffraction-free Bessel beams into a single phase CGH. Another approach consists in the generation of multiple Bessel beams implementing an array of CGHs in a single spatial light modulator (SLM) [9].

We discuss a special phase CGH, whose transmittance is analytically specified in terms of the Jacobi-Anger identity [16], which is useful for the optical generation of multiple non-diffracting Bessel beams. We firstly specify the hologram transmittance as a continuous phase function that enables the spatial separation of the different encoded beams. However, we focus our attention on the performance of the hologram when a low-resolution phase SLM is employed for its implementation. We show that certain under-sampled versions of the CGH, implemented with a pixelated SLM, allow the generation of multiple Bessel beams with a common asymptotic radial frequency propagating in a common axis. The spatial superposition of these multiple Bessel beams produces special types of non-diffracting, periodic and quasi-periodic wave fields.

2. Hologram defined in terms of the Jacobi-Anger identity

The complex amplitude of the *q-th* order non-diffracting Bessel beam can be expressed as

$$b_q(x,y) = J_q(2\pi\,\rho_0\, r)\exp(iq\theta)\ , \tag{1}$$

where J_q denotes the Bessel function of integer order q, (r,θ) represent the radial and azimuthal polar coordinates, respectively, and ρ_0 is the asymptotic radial frequency of the beam. The explicit dependence of the Bessel function $b_q(x,y)$ on the rectangular coordinates (x,y), is obtained by means of the relations $r=(x^2+y^2)^{1/2}$ and $\theta=arctan(y/x)$. According to the Jacobi-Anger identity [16] the sum of all the non-diffracting Bessel beams is given by

$$\exp[i2\pi\rho_0 r\sin(\theta)]=\sum_{q=-\infty}^{\infty} b_q(x,y)\ . \tag{2}$$

At first sight the function at the left side of Eq. (2) can be interpreted as a phase hologram that encodes the complete set of non-diffracting Bessel beams, without modulation error. However, employing the coordinate conversion $y=r\sin(\theta)$, this function is identified as the tilted plane wave $\exp[i2\pi\rho_0 y]$. Therefore, the Bessel beams can not be physically isolated (by spatial frequency filtering) from the Fourier spectrum of this phase modulation, which is given by an off-axis Dirac delta.

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A phase hologram that enables the generation and isolation of the Bessel beams is obtained by adding the linear carrier function $2\pi(u_0x+v_0y)$, with spatial frequencies (u_0,v_0), to the coordinate θ of the phase function in Eq. (2). Performing this modification, the hologram transmittance is transformed into

$$h(x,y) = \exp\{i2\pi\rho_0 r \sin[\theta + 2\pi(u_0 x + v_0 y)]\} . \tag{3}$$

Considering the Jacobi-Anger identity again, the new hologram transmittance can be expressed by the Fourier series

$$h(x,y) = \sum_{q=-\infty}^{\infty} b_q(x,y) \exp[i2\pi(qu_0 x + qv_0 y)] . \tag{4}$$

For this hologram, the spatial separation of the encoded Bessel beam $b_q(x,y)$ is enabled by the linear phase modulation $exp[i2\pi(qu_0x+qv_0y)]$. Indeed, the Fourier spectrum of the hologram $h(x,y)$ is given by

$$H(u,v) = \sum_{q=-\infty}^{\infty} B_q(u - qu_0, v - qv_0) , \tag{5}$$

where $B_q(u,v)$ is the Fourier transform of the Bessel beam $b_q(x,y)$, and (u,v) denote the spatial frequency coordinates, respectively associated to the spatial variables (x,y). Thus, the function in Eq. (3) represents a phase CGH that encodes the complete set of non-diffracting Bessel beams. The encoded beams occupy different regions at the hologram Fourier spectrum domain. Specifically, the spectrum of the q-th order beam is centered at the spatial frequencies (qu_0,qv_0). The Bessel beams encoded by the CGH can be spatially separated if at least one of the carrier frequencies (u_0,v_0) is greater than $2\rho_0$.

The CGH presents the Fourier spectrum structure of Eq. (5) when it is implemented with a continuous phase SLM, or with a high resolution pixelated SLM. On the other hand, if the hologram transmittance $h(x,y)$ is implemented with a low-resolution pixelated SLM, its Fourier spectrum is formed by laterally shifted replicas of the function $H(u,v)$. If the CGH transmittance implemented with the pixelated SLM is under-sampled, these spectrum replicas show some degree of overlapping. We prove below that an extreme overlapping of the replicas of the spectrum function $H(u,v)$, may be applied for the generation of multiple Bessel beams at a common propagation axis.

Let us assume that the CGH defined by Eq. (3), is implemented with a low-resolution pixelated phase SLM. For simplicity we consider that the SLM pixel pitch, δx, is the same in the horizontal and the vertical axes, and that the pixels are squares of width b. Under such conditions, the Fourier spectrum of the pixelated CGH is given by

$$H_p(u,v) = E(u,v) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} H(u - n\Delta u, v - m\Delta u) , \tag{6}$$

where $\Delta u=1/\delta x$ is the SLM bandwidth, $H(u,v)$ is the Fourier spectrum of the continuous CGH, given by Eq. (5), and $E(u,v)=b^2 sinc(bu)\, sinc(bv)$ is the Fourier transform of the square pixel window. In the expression for $E(u,v)$ we employed the definition $sinc(\alpha)\equiv(\pi\alpha)^{-1}sin(\pi\alpha)$. A strong under-sampling of the CGH transmittance propitiates a significant overlapping among the different terms $H(u-n\Delta u,v-m\Delta u)$ in the CGH spectrum. We can take advantage of this overlapping to obtain multiple Bessel beams propagating at a common axis. In particular, these beams are generated if the CGH carrier frequencies are given as $u_0=v_0=\Delta u/Q$, where Q is a positive integer (greater than 1). In this case it is straightforward to prove that the beams spectra $B_{p+jQ}(u,v)$, which are specified by two fixed integer indices p and Q and a variable index j, appear centered at the spatial frequency coordinates (pu_0,pu_0). The superposition of these spectra terms is expressed as

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$$S_{pQ}(u,v) = E(u,v) \sum_{j=-\infty}^{\infty} B_{p+jQ}(u - pu_0, v - pu_0) \; . \tag{7}$$

The spectrum function $S_{pQ}(u,v)$ can be spatially isolated, with an appropriate pupil, from other field contributions that are present at the CGH spectrum. Performing this spatial filtering and neglecting the influence of the factor $E(u,v)$, one obtains the non-diffracting wave field

$$s_{pQ}(x,y) = \exp[i2\pi\, pu_0(x+y)] \sum_{j=-\infty}^{\infty} b_{p+jQ}(x,y) \; , \tag{8}$$

by means of the Fourier transformation of the spectrum function $S_{pQ}(u,v)$. The field $s_{pQ}(x,y)$ generated with the above procedure is formed by the infinite set of non-diffracting Bessel beams of orders $p+jQ$. This set, which is characterized by the fixed integer indices p and Q, includes the beams of orders p, $p\pm Q$, $p\pm 2Q$, and so on. Different sets of Bessel beams $s_{pQ}(x,y)$, corresponding to different values of the index p, are generated by the CGH with carrier frequencies $u_0=v_0=\Delta u/Q$.

3. Computational implementation of under-sampled holograms

In a first numerical simulation, we specified a pixelated CGH with carrier frequencies $u_0=v_0=\Delta u/Q$, with $Q=6$, to encode Bessel beams of radial frequency $\rho_0=u_0/4$. In addition the CGH limiting aperture is assumed to be a circle of diameter $D=6/\rho_0$. For this combination of parameters the diameter of the CGH support is equivalent to *144* pixels of the phase SLM. The phase distribution of the computed CGH, depicted in Fig. 1(a), shows a periodicity that is originated in a severe under-sampling of the CGH transmittance. The normalized intensity of the CGH Fourier spectrum is partially shown in Fig. 1(b). In this figure the separated arrays of spots correspond to 3 Fourier spectra $S_{p6}(u,v)$ with $p=0, 1, 2$ (from top-left to bottom-right). It is remarkable that each Fourier spectrum function $S_{p6}(u,v)$ is formed by 6 symmetrically arranged spots.

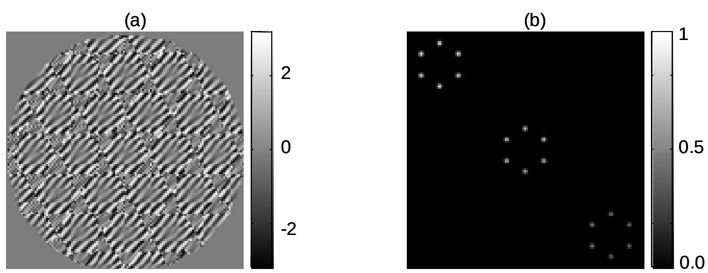

Fig. 1. (a) Phase distribution of a under-sampled CGH designed with carrier frequencies $u_0=v_0=\Delta u/6$, and (b) arrays of spots $S_{p6}(u,v)$, with $p=0, 1, 2$, in the CGH Fourier spectrum domain.

In other numerical simulations we obtained that, in general, the Fourier spectrum $S_{pQ}(u,v)$ is formed by Q symmetric spots, placed at the corners of a regular polygon of Q sides. This fact is consequence of a general significant result, namely that the infinite series of Bessel beams in Eq. (8) is equivalent to the sum of Q plane waves. This equivalence is explicitly given by

$$\sum_{j=-\infty}^{\infty} b_{p+jQ}(x,y) = Q^{-1} \sum_{m=0}^{Q-1} \exp(ipm\Delta) \exp\left[i2\pi\rho_0 r \sin(\theta - m\Delta)\right] , \tag{9}$$

where $\Delta=2\pi/Q$. A prove of this relation is obtained by developing and reducing the exponential at the right side of Eq. (9), by means of the Jacobi-Anger identity. The obtained formula represents an interesting generalization of the Jacobi-Anger identity. Indeed, it is straightforward to show that Eq. (9) reduces to Eq. (2) assuming $Q=1$ and $p=0$. It must be

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noted that the plane wave corresponding to the index m in Eq. (9) is modulated by a phase shift $exp(ipm\Delta)$. This phase shift provides a topological charge of order p to the array of spots in the spectrum function $S_{pQ}(u,v)$.

Continuing with the simulation, the wave fields $s_{06}(x,y)$ and $s_{16}(x,y)$ are obtained performing the Fourier transformation of the hexagonal spectra spots arrays $S_{06}(u,v)$ and $S_{16}(u,v)$, respectively. Each one of these spectra functions is isolated by spatial filtering in the CGH Fourier domain. The computed intensities and phases of the wave fields $s_{06}(x,y)$ and $s_{16}(x,y)$ are shown in Fig. 2. The field $s_{06}(x,y)$, which corresponds to the sum of 6 plane waves without phase shifts [Eq. (9)], presents the binary phase of a real function [Fig. 2(b)]. On the other hand, the phase shifts for the 6 plane waves that form the field $s_{16}(x,y)$ increases monotonically with the plane wave index m. In this case a remarkable result is the field intensity distribution with the form of a honeycomb [Fig. 2(c)]. In addition, each cell of this array is centered at a perfect vortex of topological charge 1 [Fig. 2(d)], originated in the topological charge of the array of spots in the spectrum function $S_{16}(u,v)$. For the displayed phase of the field $s_{16}(x,y)$ we have omitted the linear phase factor $exp[i2\pi u_0(x+y)]$.

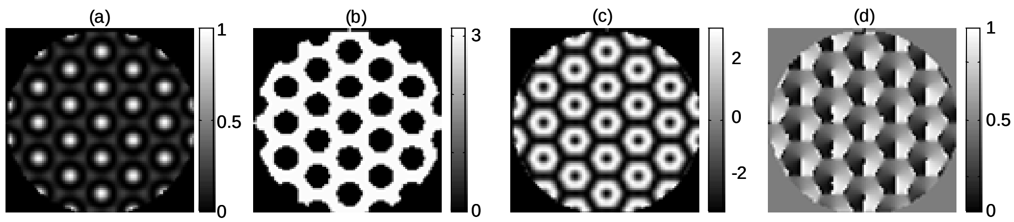

Fig. 2. (a) Intensity and (b) phase of the wavefield $s_{06}(x,y)$ obtained from the CGH depicted in Fig. 1. The intensity and phase of the field $s_{16}(x,y)$, generated with the same CGH, are respectively shown in (c), and (d).

Extending our numerical simulations we obtained that the periodic non-diffracting fields that can be generated with under-sampled CGHs correspond to the functions $s_{pQ}(x,y)$, with indices $Q=2$, 3, 4, and 6. For other values of Q, the field $s_{pQ}(x,y)$ presents a quasi-periodic structure.

4. Experimental implementation of holograms

In order to confirm the theory developed above we employed the reflective phase modulator SLM512, of Boulder Nonlinear Systems. The pixel count of this SLM is 512×512 and the pixel pitch is p=15μm. The phase modulation depth of 2π radians is obtained (with approximately 70 unequal steps) illuminating the SLM with a He-Ne laser beam (633 nm). We employed this SLM to implement the CGH described in the first numerical simulation presented in section 3, and experimentally generated the non-diffracting field $s_{16}(x,y)$. The phase distribution of this CGH was shown in Fig. 1(a). The experimentally recorded intensities of the Fourier spectrum $S_{16}(u,v)$, and its corresponding periodic field $s_{16}(x,y)$, are shown in Fig. 3. We also implemented other CGHs for generation of quasi-periodic non-diffracting fields $s_{1Q}(x,y)$ with $Q=5$ and $Q=12$. The parameters that we employed for the second CGH [that generates $s_{15}(x,y)$] are $u_0=v_0=\Delta u/5$, $\rho_0=u_0/4$, and $D=12/\rho_0$ (D is the CGH pupil diameter, equivalent to 240 SLM pixels). The case with $Q=12$ was implemented with parameters $u_0=v_0=\Delta u/12$, $\rho_0=u_0/3$, and $D=14/\rho_0$. The intensities of these quasi-periodic fields, obtained experimentally, are shown in Fig. 4.

5. Concluding remarks

We have discussed a phase CGH, whose transmittance is defined in terms of the Jacobi-Anger identity, that simultaneously generates multiple Bessel beams with a common asymptotic radial frequency. This CGH, implemented with a continuous or with a high resolution

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pixelated SLM, generates Bessel beams of different orders at different propagation axes. A different result occurs when we implement a severely under-sampled version of the CGH transmittance, with a low-resolution SLM. If the carrier frequencies of this under-sampled CGH are $u_0=v_0=\Delta u/Q$, with an integer Q (greater than 1), then a collection of Bessel beams of orders $p+jQ$ (characterized by the fixed integer numbers p and Q, and a variable integer

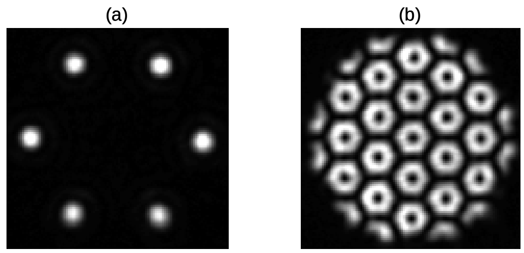

Fig. 3. (a) Experimentally recorded intensity of the spectrum spots array $S_{16}(u,v)$ generated with the CGH displayed in Fig. 1, and (b) intensity of the generated field $s_{16}(x,y)$.

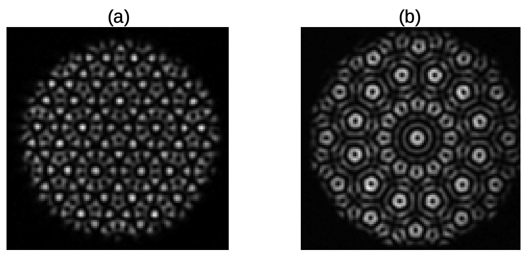

Fig. 4. (a) Experimentally recorded intensities of the quasi-periodic fields $s_{1Q}(x,y)$, generated with CGHs designed with parameters (a) $Q=5$ and (b) $Q=12$.

number j), share the spatial frequency coordinates (pu_0,pv_0), at the CGH Fourier domain. The superposition of beams that share these frequency coordinates, denoted $s_{pQ}(x,y)$, is equivalent to the sum of Q plane waves, whose spectrum spots appear at the corners of a regular polygon of Q-sides. This result, expressed in Eq. (9), represents a generalization of the Jacobi-Anger identity. These Q plane waves, numbered by the index m (with values from 0 to Q-1), are modulated by the phase factors $exp(ipm\Delta)$, with $\Delta=2\pi/Q$, which represents a topological charge of order p.

The superposition of the infinite set of Bessel beams $s_{pQ}(x,y)$, equivalent to the Q-plane waves specified above, generates special types of non-diffracting fields. We have established, in numerical simulations, that the non-diffracting fields $s_{pQ}(x,y)$ are transversally periodic (when $Q=2$, 3, 4, and 6) or quasi-periodic (for other values of Q). This kind of fields can be used to create dynamic photonic lattices in nonlinear media [17,18]. Its possible application in free-space optical communications [19] also deserves future consideration.

The developed theory and the results in numerical simulations were experimentally supported by the implementation of CGHs for the generation of the periodic field $s_{16}(x,y)$, and the quasi-periodic fields $s_{1Q}(x,y)$ with $Q=5$, 12. The experimentally generated fields (Figs. 3 and 4) reproduced with remarkable high fidelity the features of the numerically generated fields. This high fidelity is illustrated in the case of the field $s_{16}(x,y)$ by comparison of Fig. 2(c) with Fig. 3(b). The high fidelity of the generated fields is propitiated by a unique feature of the CGH, namely that the high order diffraction field contributions transmitted by this hologram, form part of the encoded fields. This is not the case of conventional CGHs, for which the high order diffraction contributions represent sources of noise.

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