Long-distance propagation of periodic patterns in weakly nonlinear Kerr medium

Nikolai Korneev and Francisco Marroquín

Instituto Nacional de Astrofísica, Óptica y Electrónica, Apt. Postal 51 y 216, CP 72000, Puebla, Puebla, Mexico

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We investigate the propagation of periodic patterns in one and two dimensions for weak Kerr-type nonlinearity. Nonlinear amplitudes are introduced, which are related to the Fourier harmonics of a wave by polynomials of third and fifth degree. These amplitudes evolve in a particularly simple way and permit easy reconstruction of waveform after propagation. For the one-dimensional case, solutions are quasiperiodic, and solitonlike structures can be identified. For the two-dimensional case, recurrent and chaotic regimes exist depending on lattice type.

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1. INTRODUCTION

Light propagation in a medium with Kerr-type nonlinearity described by the nonlinear Schrödinger (NLS) equation is interesting both from the theoretical point of view and for applications in nonlinear pulse propagation, waveguiding, switching, etc.[1] The NLS equation describes propagation of pulses in one spatial dimension and one temporal dimension and propagation of patterns in two or three spatial dimensions. We will use here the terms of spatial propagation, which is realized in photorefractive and other nonlinear media.

The long-distance behavior for a single pulse in the 1+1-dimensional (1+1D) case is well studied; it is known that spatial solitons and a weak background are formed in this case. The solution can be obtained by the inverse scattering transform method or equivalent procedures.[2,3] For a periodic pattern, the behavior is different: the waveform tends to repeat itself after some propagation distance. This is related to the fact that the spatial spectrum width remains limited upon propagation by a constant, which depends on nonlinearity strength.[4,5] Thus, for an initially narrow spectrum, only a finite number of spatial harmonics is necessary to calculate a propagation. The solution based on inverse scattering is not practical for the periodic case because it includes finding the band structure for a scattering problem with periodic potential and integration on an infinite genus Riemann surface[3,6]; thus the calculation is very complicated, except for specially chosen initial conditions that give a small number of bands.

In this paper we use the complete integrability of the NLS equation to build a simple approximate solution for small nonlinearity parameter $\nu = L^2|\varkappa||\psi|^2 \ll 1$, where L is a half spatial period of the pattern and $|\varkappa||\psi|^2$ is a nonlinear coefficient and intensity product. We introduce new, to our knowledge, nonlinear diffraction-order amplitudes S_k, which are polynomials of the wave spatial Fourier spectrum ψ_m. The nonlinear amplitudes S_k evolve with a propagation path t similar to a linear propagation $S_k(t) = S_k(0)\exp(-iv_k t)$, with real t-independent functions of initial conditions v_k. After a given propagation length, one can reconstruct the Fourier spectrum of a wave by inverting the expressions for $S(\psi)$. Mathematically, this is equivalent to finding approximate angle-action variables for the weakly nonlinear case.

The advantage of the approximation is that it directly takes into account the solution structure, and there are no unphysical terms growing to infinity with propagation length. It permits the determination of characteristic propagation frequencies, distances of self-repetition, and maximal intensities transferred to higher orders, which otherwise can be found only with numerics. Structures similar to free-propagating orders and solitons can be identified. It is also possible to obtain an approximation for a nonrectangular lattice in the 2+1D case. The 2+1D case is not exactly integrable, but for weak nonlinearity one can expect behavior similar to the 1+1D case. We discuss the main differences between the 1+1D and the 2+1D cases.

The method is intended as a practical tool; thus we report the explicit equations for two lower approximations and compare them with numerics to demonstrate their validity. The abstract procedure for obtaining the reported and higher terms is outlined in Appendix A.

2. NONLINEAR AMPLITUDES

The 1+1D NLS equation is written as[3]

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + 2\varkappa|\psi|^2\psi. \quad (1)$$

We consider first the one transversal dimension case, for which $\Delta = \partial^2/\partial x^2$. The complex wave function ψ is x periodic in transverse coordinate x, $\psi(x+2L,t) = \psi(x,t)$. We attribute to t the meaning of the propagation spatial coordinate. The nonlinearity strength $\varkappa$ is positive for the self-defocusing situation and negative for self-focusing. We take the Fourier series for the wave function with t-dependent amplitudes of diffraction orders $\psi_k(t)$,

$$\psi(x,t) = \sum_{k=-\infty}^{\infty} \exp(ikKx)\psi_k(t), \tag{2}$$

and $K=\pi/L$ is a wave vector. An equation is written as an infinite system of ordinary differential equations for order amplitudes. In fact, only a finite number of equations is sufficient for describing a propagation because higher orders are exponentially small for any propagation length and uniformly limited[5]:

$$i\frac{\partial \psi_k}{\partial t} = k^2K^2\psi_k + 2\varkappa\sum_{p,q} \bar{\psi}_{k+p+q}\psi_{k+p}\psi_{k+q}. \tag{3}$$

We are interested in the weakly nonlinear case $\nu = |\varkappa|L^2|\psi|^2 \ll 1$. The new, to our knowledge, nonlinear amplitudes S_k are introduced as

$$S_k = \psi_k + \frac{\varkappa}{K^2}\sum_{p,q\neq 0} \frac{\bar{\psi}_{k+p+q}\psi_{k+p}\psi_{k+q}}{pq}. \tag{4}$$

The moduli square

$$W_k = S_k\bar{S}_k \tag{5}$$

are approximately conserved. The evolution of W_k can be calculated using the NLS equation (3). Terms up to those linear with $\varkappa$ disappear, and $\partial W_k/\partial t = O(\varkappa^2L^4I^3)$. The calculation of $\partial W_k/\partial t$ by itself does not guarantee the approximate conservation of W_k for long propagation distances, but this can be established by one's comparing the equations with exact order-related integrals of motion reported in Ref. 5 (see Appendix A).

The t dependence of the S_k phase can be found by calculating $i\partial S_k/S_k$ with Eq. (3) and neglecting terms proportional to $\varkappa^2$. One finds that the phase velocity is

$$v_k = k^2K^2 + 4\varkappa(I_W - W_k/2) + O(\varkappa^2I^2/K^2), \tag{6}$$

where $I_W = \Sigma W_k$ is total nonlinear intensity conserved upon propagation. After calculating the evolution with

$$S_k(t) = S_k(0)\exp(-iv_kt), \tag{7}$$

one can reconstruct the Fourier spectrum with

$$\psi_k = S_k - \frac{\varkappa}{K^2}\sum_{p,q\neq 0} \frac{\bar{S}_{k+p+q}S_{k+p}S_{k+q}}{pq}. \tag{8}$$

This gives a complete solution.

To analyze the exactitude of the approximation, we separate the diffraction orders ψ_k into three categories. The first category is strong orders with initial intensities $\psi_k\bar{\psi}_k > \nu I$. The conservation of W_k for these orders means that their intensity variation from average level upon propagation is of an order of $\pm\nu I$. The second category includes orders that are initially weak, $\psi_k\bar{\psi}_k < \nu I$, but have combinations of only strong orders in the second term of Eq. (4). We will call them secondary orders. The rest of the orders are of the third category. For example, if only orders ψ_0, ψ_1 are strong, the secondary orders are ψ_2 and ψ_{-1}, and all others are of the third category.

The conservation of W_k gives that if secondary-order intensities are initially small, their characteristic intensities during propagation are in the $\nu^2 I$ range. The error in the integral of motion value in the same range would give a correction comparable with the order intensity itself. Closer inspection reveals, nevertheless, that for secondary orders $\partial W_k/\partial t = O(\varkappa^3L^6I^4)$. This happens because, for weak orders, the six-amplitude terms, which include only strong orders, are compensated as well.

The errors in nonlinear order intensities W_k are uniformly limited with propagation, and the accumulating errors are related to the errors in phase velocities.

3. NEXT APPROXIMATION

The equation for nonlinear amplitudes valid to the fifth degree in Fourier amplitudes is

$$\begin{aligned} S_k = \psi_k &+ \frac{\varkappa}{K^2}\sum_{p,q} \frac{\bar{\psi}_{k+p+q}\psi_{k+p}\psi_{k+q}}{pq} \\ &+ \frac{2\varkappa^2}{K^4}\sum_{p,q,r,s} \frac{\bar{\psi}_{k+p+q}\bar{\psi}_{k+q+r+s}\psi_{k+p}\psi_{k+q+r}\psi_{k+q+s}}{pq(pq+rs)} \\ &- \frac{\varkappa^2}{K^4}\sum_{p,q,r,s} \frac{\bar{\psi}_{k+p+q+r}\bar{\psi}_{k+p+q+s}\psi_{k+p}\psi_{k+q}\psi_{k+p+q+r+s}}{pq(pq-rs)} \\ &- \frac{2\varkappa^2}{K^4}\sum_{p,q,s\neq 0} \frac{\bar{\psi}_{k+s}\bar{\psi}_{k+p+q}\psi_{k+p}\psi_{k+s}\psi_{k+q}}{(pq)^2} \\ &- \frac{\varkappa^2}{K^4}\sum_{p,q} \frac{\bar{\psi}_k\bar{\psi}_{k+p+q}\psi_{k+p}\psi_{k+q}\psi_k}{(pq)^2}. \end{aligned} \tag{9}$$

The sums are taken to have nonzero denominators.

The approximate conservation of W_k is again checked by calculation of the evolution of W_k. The corrections eliminate terms to the next order of smallness. To demonstrate that the phase of S_k evolves as described by Eq. (7), one can calculate $i\bar{S}_k\partial_t S_k$ and check that the answer is conserved to the exactitude of the approximation. The calculation is rather lengthy. The next-order corrections to phase velocities of Eq. (6) are given by

$$v_k = k^2K^2 + 4\varkappa(I_W - W_k/2) - \frac{2\varkappa^2}{K^2}\sum_{p\neq 0} \frac{(W_{k+p} + 2W_k)W_{k+p}}{p^2}. \tag{10}$$

For the inversion formula, one follows by iterations. The first-order corrections are calculated with Eq. (8), and the ψ_m values true to the terms proportional to $\varkappa$ are obtained. These values are used in the $\varkappa$- and $\varkappa^2$-proportional terms of Eq. (9), which gives with known S_k the ψ values correct to ν^2.

4. EXAMPLES

In this section we demonstrate how the approach works for the situation with two strong orders ψ_0, ψ_1, and two weak secondary orders ψ_{-1} and ψ_2. The symmetric case

$\psi_0 = \psi_1$ for weak nonlinearity can be solved in elliptic functions. With Eqs. (4)–(8), the solution is straightforward.

We have to the first approximation [neglecting products that include weak orders in Eq. (8)] that

$$\psi_1(t) \approx S_1(t), \tag{11}$$

$$\psi_2(t) \approx S_2(t) - \varkappa/K^2\overline{S_0(t)}S_1(t)^2 \tag{12}$$

and have similar equations for ψ_0, ψ_{-1}. The amplitudes S_k have simple dependence on the propagation path given by Eqs. (6) and (7).

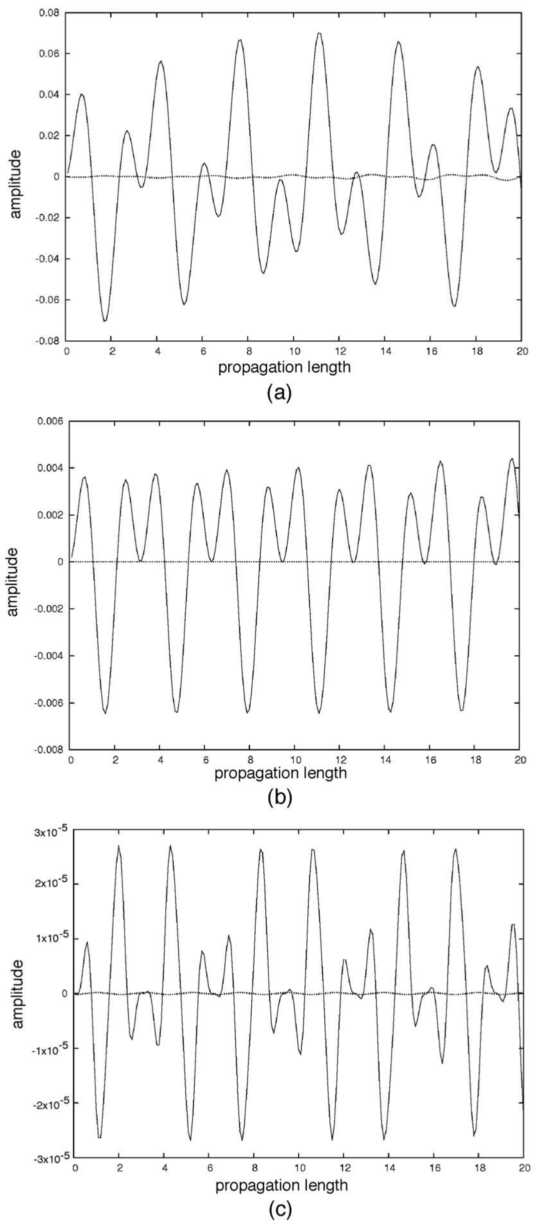

Fig. 1. Real parts of exact numerical solutions for complex amplitudes ψ_k (solid curves) and the difference between exact solutions and analytic approximations with Eqs. (9) and (10) (dotted curves close to zero). For all figures $\psi_0(0)=1$, $\psi_1(0)=0.8$, $K=1$, and other amplitudes are initially zero. (a) Solution for ψ_2 and $\varkappa=-0.05$, (b) solution for ψ_2 and $\varkappa=-0.005$, (c) solution for ψ_3 and $\varkappa=-0.005$. The relative error of reconstruction diminishes proportionally to $\varkappa$.

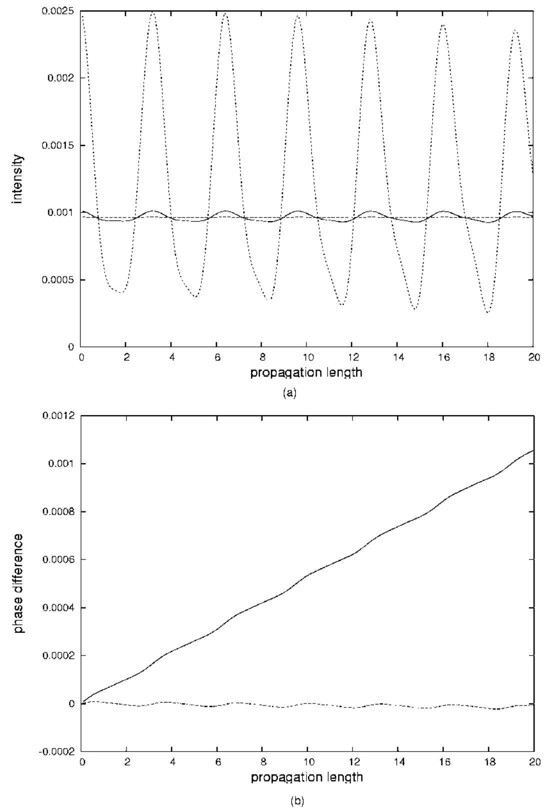

Fig. 2. Demonstration of conservation for nonlinear order intensity W_0 and linearity of phase for S_0 with propagation. $S_0(t)$ is calculated with a numerical solution for ψ using Eqs. (9) and (10) with initial conditions $\psi_0(0)=0.05$, $\psi_1(0)=1$, $\psi_2(0)=0.7$, $\psi_3(0)=0.3$, $K=1$. (a) W_0 with second approximation (nearly horizontal line), first approximation (solid curve), and the value of diffraction-order intensity $\bar{\psi}_0\psi_0$ (dotted curve) for $\varkappa=-0.02$. (b) The difference between the exact phase of S_0 and phase values obtained with the initial condition for the first approximation, Eq. (6) (upper trace), and the second approximation, Eq. (10), $\varkappa=-0.005$.

If the initial amplitude of secondary order $\psi_2(0)$ is such so that $S_2 \approx 0$, then

$$\psi_2(t) \approx -\varkappa/K^2\overline{\psi_0(t)}\psi_1(t)^2. \tag{13}$$

The same happens for the -1st order, and thus for $S_{-1} \approx S_2 \approx 0$ we have the propagation of the soliton type, for which the form of the wave is maintained with propagation. In fact, we obtain the approximation to a cnoidal wave.

For positive W_2, there are two possibilities. If $W_2 < \varkappa^2/K^4|\psi_0^2\psi_1^4|$, the phase of the second order with respect to the phase of $\bar{\psi}_0\psi_1^2$ is fixed within some limits, which means propagation similar to the solitonic one. For $W_2 > \varkappa^2/K^4|\psi_0^2\psi_1^4|$ the propagation is similar to the linear one, and the phase of the second order is not fixed with respect to the phases of strong orders. Nevertheless, even for propagation similar to linear propagation, the correlations exist between the order phase and intensity. These correlations are most pronounced for $W_2 \approx \varkappa^2/K^4|\psi_0^2\psi_1^4|$, which corresponds to zero initial amplitude ψ_2.

With the approximation to the next order, the solution can be refined. In Fig. 1 we present a comparison of exact numerical solutions and approximations with Eqs. (9) and

(10). It is seen that the approach gives good correspondence up to $\nu \approx 1$ for positive nonlinearity. (For comparison, the modulation instability onset corresponds to $\nu \approx 2.5$.) For negative nonlinearity, the validity is somewhat better. The relative exactitude of approximation grows for smaller ν, and it describes correctly additional orders ψ_{-2}, ψ_3 as well.

We also illustrate numerically the conservation of nonlinear order intensity and linearity of its phase for a situation of three strong orders in Fig. 2.

5. TWO-DIMENSIONAL CASE

The equations can be expanded to the two-dimensional (2D) periodic case. For the 2D case the Fourier components of $\psi(x,y)$ are numbered not by integers k but by vectors $\mathbf{k}$ that form (not necessarily rectangular) a 2D lattice. If we choose the first period in $\mathbf{k}$ space equal to K along the k_x axis and take the second period as a vector $K(a,b)$, then lattice vectors are written as $\mathbf{k}=(k_1+k_2a, k_2b)$ with integers k_1, k_2. The modification in Eq. (4) for the 2D case is having the sum over nonzero vectors $\mathbf{p},\mathbf{q}$ and taking the scalar product $\mathbf{pq}$ instead of pq. Then, similar to the 1D case, $\partial W_k/\partial t = O(\varkappa^2 L^4 I^3)$. It is seen that Eq. (4) can have a zero denominator if the scalar product $\mathbf{pq}=p_1q_1 + a(p_2q_1+q_2p_1)+(a^2+b^2)p_2q_2=0$; thus the nonlinear amplitudes can be introduced only if the lattice vectors do not form right angles. The exact equality can be avoided, for example, if $b \neq 0$ is rational and a is irrational, but for big enough $\mathbf{p},\mathbf{q}$ the denominator can be closer to zero than any given δ. This is different from the 1D case, where the absolute value of the denominator is never smaller than 1. Equations (4)–(10) are valid for the 2D case if sums are taken over lattice vectors and products pq, p^2 are changed to scalar products $\mathbf{pq},\mathbf{pp}$. Clearly, the equations describe the situation correctly only if denominators in involved terms are not close to zero.

Let us consider the situation with weak order $\psi_{(0,0)}$ and three strong orders $\psi_{(0,1)}, \psi_{(1,0)}, \psi_{(1,1)}$. Let the second vector of the grating be $K(a,1)$ (See Fig. 3.) For $\psi_{(0,0)}$ we have to the first approximation

$$\psi_{(0,0)}(t) \approx S_{0,0}(t) - 2\varkappa/K^2\overline{S_{(1,1)}(t)}S_{(1,0)}(t)S_{(0,1)}(t)/a. \tag{14}$$

For big enough a (nonrectangular lattice), the general character of propagation is similar to the 1D case. The initially weak secondary order remains weak upon propagation, and the movement is quasiperiodic. For small a,

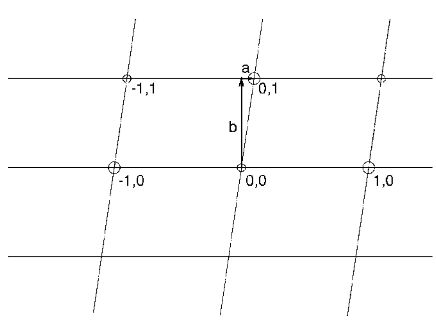

Fig. 3. Lattice configuration for 2D propagation.

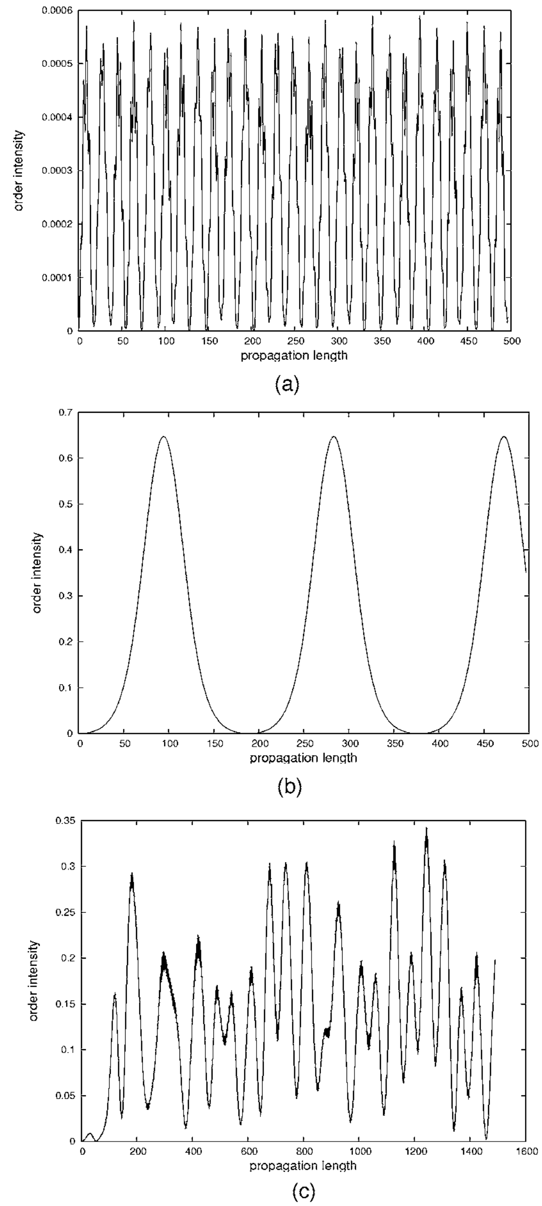

Fig. 4. Different long-distance propagation scenarios for the order $\psi_{-1,1}$ (the geometry of Fig. 3). (a) For a nonrectangular lattice the propagation character is similar to the 1D case, and energy transfer to the order is small, $a=0.2$, $\varkappa=0.01$, $K=b=1$, initially $\psi_{-1,0}=\psi_{0,1}=1$, $\psi_{0,0}=-0.1i$, $\psi_{1,0}=1+0.1i$. (b) A big energy transfer is possible for a nearly rectangular lattice [$a=0.02$, $\psi_{1,0}=0$, other parameters are the same as for the (a) case]. (c) Chaotic solution for the nearly rectangular lattice [the same parameters as for the (a) case, but $a=0.02$].

when the lattice becomes nearly rectangular, the variations of $\psi_{(0,0)}$ become larger, and the approximation breaks.

For small a, if $\psi_{(0,1)}, \psi_{(1,0)}$ are initially strong and $\psi_{(0,0)}, \psi_{(1,1)}$ are initially weak, the weak pair undergoes periodical exponential amplification for both signs of nonlinearity.

We illustrate numerically in Fig. 4 three possible types of propagation for two dimensions with the configuration of Fig. 3. For big enough a, the quasiperiodic regime is realized, and $\psi_{(1,1)}$ remains weak upon propagation. This is the situation of the validity of our approximation.

If a is small, for the situation with a zero third pump $\psi_{-1,0}$, there is a strong periodic energy transfer to the order (1, 1). Finally, if we add the third pump, the behavior becomes chaotic. Now the order (0, 0) can be amplified either by a pair $\psi_{(0,1)}, \psi_{(1,0)}$ or by a pair $\psi_{(0,1)}, \psi_{(-1,0)}$, producing strong order $\psi_{1,1}$ or $\psi_{(-1,1)}$, respectively. For a symmetric situation with $a=0$ and equal pumps, the unstable equilibrium exists. The Hamiltonian chaos is developed close to such points, and numerics demonstrate chaotic behavior for slightly nonsymmetrical conditions.

6. DISCUSSION AND CONCLUSIONS

The advantage of the proposed approximation is that it directly takes into account recurrence properties of a periodic pattern propagating in a nonlinear medium. For a small nonlinearity, initially strong diffraction orders propagate similarly to the linear case but have small quasiperiodic variations of intensities and phases. Different from the linear case, initially weak orders are not independent and can be linked to strong ones. This produces soliton-type propagation. For small nonlinearity the method gives characteristic phase velocities and variations of order intensities upon propagation, which are quite difficult to find with usual perturbation methods. Determination of phase velocities is especially important for understanding behavior for nonintegrable problems, for example, for propagation in a nonlinear medium with a grating.

Further approximation terms can be found (see Appendix A), but their practical usefulness seems quite limited because of the large number of products they include.

For the 2D case, the series for conserved quantities have small denominators, and the phase space has a complicated structure. Numerics indicate that for lattices that are not rectangular the proposed approximation can be useful as well. For rectangular lattices, strong energy transfer to initially weak orders and deterministic chaos are possible even for small nonlinearity.

Both in the 1D and the 2D cases, the approach permits conclusions on general long-term behavior of periodic patterns.

The paper discusses the behavior of a nonlinear periodic wave for a long propagation path. For experimental observation of recurrence effects (e.g., in photorefractive crystals), it is necessary to eliminate perturbations that are not periodic in transverse coordinates. By the nature of the approximation, the method is not good for small angles between beams giving big spatial periods. In practice, the noise waves with directions close to the direction of the strong pump are amplified upon propagation. This process is especially pronounced for positive nonlinearity, for which the amplification is initially exponential. In one dimension the noise will fill the spectral band around strong orders, which is permitted by conservation laws. For two dimensions the situation is worse because for two pumps the exponential amplification of noise is possible for both positive and negative nonlinearities and for directions far from the direction of strong beam propagation.

The unwanted noise can be subdued in nonlinear waveguides, where the spatial spectrum is discrete. Some experiments with planar waveguide nonlinear propagation, which eliminate transverse instability, are reported in the literature.[1] The mode structure in 1D and 2D waveguides can, in principle, demonstrate the discussed recurrence effects.

APPENDIX A

The mathematical basis for the approximation is complete integrability of the Hamiltonian problem for the NLS equation for one dimension. It is known that the equation has an infinite number of integrals of motion. The usual set includes intensity, momentum, etc.[3] In Ref. 5, another set was proposed that has an advantage of direct relation to diffraction orders; these integrals in Fourier harmonics to the second power in nonlinearity strength are expressed as

$$J_k = (-1)^k \left\{ 2 + 4\varkappa L^2 \psi_k \bar{\psi}_k + 16\varkappa^2 L^4 \left[\sum_{p,q\neq 0} \frac{\psi_k \psi_{k+p+q} \bar{\psi}_{k+p} \bar{\psi}_{k+q} + \text{c.c.}}{4\pi^2 pq} + \frac{\psi_k^2 \bar{\psi}_k^2}{12} - \left(\sum_{p\neq 0} \frac{\bar{\psi}_{k+p} \psi_{k+p}}{2\pi p} \right)^2 + \sum_{p\neq 0} \frac{\bar{\psi}_{k+p} \psi_{k+p} \bar{\psi}_k \psi_k}{2\pi^2 p^2} \right] \right\} + \cdots . \tag{A1}$$

Comparing J_k with the expression for W_k, which follows from Eqs. (4) and (5), to the exactitude of approximation

$$J_k/(-1)^k - 2 = 4\varkappa L^2 W_k + 16\varkappa^2 L^4 \left[W_k^2/12 - \left(\sum_{p\neq 0} \frac{W_{k+p}}{2\pi p} \right)^2 + \sum_{p\neq 0} \frac{W_{k+p} W_k}{2\pi^2 p^2} \right]; \tag{A2}$$

thus the two sets are equivalent.

The integrals J_k are traces of a monodromy matrix for specified spectral parameter values, and the series converge for any nonlinearity value, not only for small one. Another property of these integrals is that they are in involution; i.e., the Poisson bracket $[J_k, J_m]=0$.[3] If there is a sufficient number of integrals of motion in involution, it is possible to construct action-angle variables giving conserved actions W_k and linear with propagation distance angles $\phi_k = -v_k t$.

The general procedure to obtain W_k includes integration on surfaces J_k=const. in phase space.[7] For these surfaces the closed cycles are taken, and the action variables are obtained as $W_k = 1/(2\pi i) \int_k \bar{\psi} \mathrm{d}\psi$. The simplest integration path is just changing ψ_k only. The cut with all other ψ and J_k fixed produces for ψ_k a closed path in a complex plane. The scalar product is simplified to $\bar{\psi}_k \mathrm{d}\psi_k$, and the problem of integration is reduced to finding $\bar{\psi}_k$ as a function of ψ_k, $\mathbf{J}$ along the path. For the linear approximation (disregarding constants), $J_k = \bar{\psi}_k \psi_k$; thus $\bar{\psi}_k = J_k/\psi_k$, and the integration $1/(2\pi i) \int \mathrm{d}\psi_k J_k/\psi_k$ in a complex plane gives $W_k = J_k$.[7]

To manage the **J** set to the next approximation for a small ν parameter, we can use the first approximation $\bar{\psi}_m = J_m/\psi$ in terms of Eq. (A1) proportional to $\varkappa$. This eliminates extra terms in J_k and gives Eqs. (4) and (5) for W_k.

The procedure of integration itself is reduced to finding poles, but it is necessary to be careful with integration paths. For large nonlinearity, in particular, if modulation instability exists, the path can split in two with divergence in series for $\bar{\psi}_k(\psi_k)$. Our approximation breaks in this case, and more refined methods of complex analysis are needed.

The next approximation to the series for S_k is found by guess. The regular procedure of finding J_k to the next approximation, expressing $\bar{\psi}_k(\psi_k)$ etc., seems more complicated. Note that one can add to S_k, as given by Eq. (9), e.g., the term $|\psi_s\psi_q|^2\psi_k$, without affecting conservation of W_k and the expression for phase velocities. To the first approximation, similar terms $|\psi_s|^2\psi_k$ are managed by a regular procedure.

The Hamiltonian is a function of action variables only. If we take

$$H = \sum_s s^2K^2W_s + 2\varkappa\sum_{s\neq p} W_sW_p - \varkappa\sum_s W_sW_s - 2\varkappa^2/K^2\sum_{s\neq p} W_sW_p^2/(s-p)^2, \tag{A3}$$

Eq. (A3) becomes the canonic equation $\mathrm{d}\phi_k/\mathrm{d}t = -v_k = \partial H(W)/\partial W_k$.

The proposed method of solution, different from the traditional one, gives practical, meaningful approximations for cases when there is a small number of spatial harmonics involved. The traditional method is better for solutions organized around a small number of propagation frequencies (i.e., solitonlike propagation), but the number of Fourier harmonics can be large. Thus, the two methods are complementary.

For two dimensions, the number of exact integrals of motion is not sufficient for complete integrability; nevertheless, one can expect that the phase space of the NLS equation for small nonlinearity still has some invariant torus structure in the spirit of the Kolmogorov–Arnold–Moser theorem. Then the equations will approximate invariant tori. (Note that for the 2D case we still can guarantee the conservation of W_k for certain propagation distances because $\partial W_k/\partial t$ is close to zero, as for the 1D case.) Detailed analysis of the 2D case is not in the scope of this paper.

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N. Korneev, the corresponding author, can be reached by e-mail at korneev@inaoep.mx.

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